EUROPEAN JOURNAL OF MATHEMATICAL SCIENCES
Vol. 2, No. 2, 2013, 140-151
www.ejmathsci.com

Ordered Filters of Heyting Almost Distributive Lattices

M. Sambasiva Rao[1,*], G.C. Rao[2]

[1] *Department of Mathematics, M.V.G.R. College of Engineering, Vizianagaram, Andhra Pradesh, India-535004*
[2] *Department of Mathematics, Andhra University, Visakhapatnam, Andhra Pradesh, India-530003*

Abstract. The notion of ordered filters is introduced in Heyting almost distributive lattices and the properties of these filters are then studied. A set of characterization theorems of ordered filters are proved. Some necessary conditions of ordered filters are derived. Some congruences are introduced in terms of ordered filters and a relation is established among these congruences.

2010 Mathematics Subject Classifications: 06D99, 06D15

Key Words and Phrases: Heyting almost distributive lattice, ordered filter, homomorphism, F-identity element, congruence.

---

1. Introduction

The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [5] as a common abstraction to most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. In [4], Rao, Berhanu and Ratna Mani introduced the concept of Heyting almost distributive lattices as a generalization of Heyting algebra in the class of ADLs. Later they introduced the class of implicative filters [3] and studied their properties. In [2], Chan and Shum introduced the notions of ordered filters in implicative semigroups and studied the homomorphic properties of these filters.

In this paper, the notion of ordered filters is introduced in Heyting almost distributive lattices (HADLs). Some necessary and sufficient conditions are derived for a non-empty set of a HADL to become an ordered filter. Some properties of these ordered filters are then studied under homomorphisms.

The concept of F-identity elements is introduced and their properties are studied. Three congruences are introduced respectively in terms of ordered filters, F-identity elements and ideals of F-identity elements. A relation is then established among these congruences.

*Corresponding author.

Email addresses: mssraomaths35@rediffmail.com (M. Rao), gcraomaths@yahoo.co.in (G. Rao)

2. Preliminaries

In this section, certain definitions and important results are collected and presented from [1, 4, 5], those will be required in the text of the paper.

Definition 1 ([5]). *An Almost Distributive Lattice(ADL) with zero is an algebra* $(L, \vee, \wedge, 0)$ *of type (2,2,0) satisfying the following properties:*

1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

3) $(x \vee y) \wedge y = y$

4) $(x \vee y) \wedge x = x$

5) $x \vee (x \wedge y) = x$

6) $0 \wedge x = 0$ *for any* $x, y, z \in L$

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L.

Definition 2 ([5]). *Let* L *be a non-empty set. Fix* $x_0 \in L$. *For any* $x, y \in L$, *define* $x \wedge y = y$, $x \vee y = x$ *if* $x \neq x_0$, $x_0 \wedge y = x_0$ *and* $x_0 \vee y = y$. *Then* $(L, \vee, \wedge, x_0)$ *is an ADL and it is called a Discrete ADL with* x_0 *as its* 0. *Alternatively, Discrete ADL is defined as an ADL in which every non-zero element is maximal in the poset* $(L, \leq)$.

Theorem 1. *[5] Let* L *be an ADL. Then for any* $a, b, c \in L$, *we have the following:*

1) $a \vee b = a \Leftrightarrow a \wedge b = b$

2) $a \vee b = b \Leftrightarrow a \wedge b = a$

3) $a \wedge b = b \wedge a$ *whenever* $a \leq b$

4) $\wedge$ *is associative in* L

5) $a \wedge b \wedge c = b \wedge a \wedge c$

6) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

7) $a \wedge a = a$ *and* $a \vee a = a$

8) $0 \vee a = a$ *and* $a \wedge 0 = 0$

9) $a \vee b) \wedge c = (b \vee a) \wedge c$

Definition 3 ([3]). *A non-empty subset* F *of a ADL* L *is called a filter if it satisfies the following conditions:*

(F_1) $x, y \in F$ *implies* $x \wedge y \in F$

(F_2) $x \in L, a \in F$ *imply* $x \vee a \in F$

Theorem 2 ([5]). *Let F be a filter in L and $x, y \in L$. Then $x \vee y \in F \Leftrightarrow y \vee x \in F$.*

Theorem 3 ([5]). *Let m be a maximal element of the partially ordered set $(L, \leq)$. Then the following hold.*

1) $m \vee x = m$ *for all* $x \in L$

2) $m \wedge x = x$ *for all* $x \in L$

3) $(m] = L$

Definition 4 ([4]). *Let $(L, \vee, \wedge, 0, m)$ be an ADL with 0 and a maximal element m. Suppose $\rightarrow$ is a binary operation on L satisfying the following conditions:*

1) $a \rightarrow a = m$

2) $(a \rightarrow b) \wedge b = b$

3) $a \wedge (a \rightarrow b) = a \wedge b \wedge m$

4) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$

5) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$ *for all* $a, b, c \in L$.

Then $(L, \vee, \wedge, \rightarrow, 0, m)$ is called a Heyting ADL or simply HADL. Throughout this article L stands for a Heyting ADL (HADL), unless otherwise mentioned.

Lemma 1 ([5]). *Let m be a maximal element in L. Then for any $x, y \in L$, the following conditions hold.*

1) $x \leq y \Rightarrow x \rightarrow y = m$

2) $m \rightarrow x = x \wedge m$

3) $x \rightarrow m = m$

Lemma 2 ([4]). *Let m be a maximal element in L. Then for any $a, x, y \in L$ with $x \leq y$, the following conditions hold.*

1) $a \rightarrow x \leq a \rightarrow y$

2) $y \rightarrow a \leq x \rightarrow a$

Theorem 4 ([4]). *Let $(L, \vee, \wedge, \rightarrow, 0, m)$ be a HADL. Then for $x, y \in L$, the following conditions are equivalent:*

1) $x \rightarrow y = m$

2) $x \wedge m \leq y \wedge m$

3) $x \rightarrow (y \wedge m) = m$

4) $y \wedge x = x$

Lemma 3 ([4]). *Let m be a maximal element in L. Then for any $a, b, c \in L$, the following conditions hold.*

1) $b \wedge m \leq (a \rightarrow b) \wedge m$

2) $a \rightarrow (a \wedge c) = a \rightarrow c$

3) $a \wedge b \wedge m = a \wedge c \wedge m \Leftrightarrow (a \rightarrow b) \wedge m = (a \rightarrow c) \wedge m$

4) $a \wedge m \leq b \wedge m \Leftrightarrow (a \rightarrow b) \wedge m = m$

5) $a \wedge c \wedge m \leq b \wedge m \Leftrightarrow c \wedge m \leq (a \rightarrow b) \wedge m$

6) $a \wedge m \leq ((a \rightarrow b) \rightarrow b) \wedge m$

7) $a \wedge m \leq (b \rightarrow c) \wedge m \Leftrightarrow b \wedge m \leq (a \rightarrow c) \wedge m$

8) $(a \rightarrow (b \rightarrow c)) \wedge m = ((a \wedge b) \rightarrow c) \wedge m$

9) $((a \wedge b) \rightarrow c) \wedge m = ((b \wedge a) \rightarrow c) \wedge m$

10) $(a \rightarrow (b \rightarrow c)) \wedge m = (b \rightarrow (a \rightarrow c)) \wedge m$

Definition 5 ([4]). *Let $(L, \vee, \wedge, \rightarrow, 0, m)$ and $(L', \vee, \wedge, \rightarrow, 0', m')$ be two HADLs. Then the mapping $f : L \longrightarrow L'$ is called a homomorphism of L into L' if for any $x, y \in L$, the following conditions hold*

1) $f(x \wedge y) = f(x) \wedge f(y)$

2) $f(x \vee y) = f(x) \vee f(y)$

3) $f(x \rightarrow y) = f(x) \rightarrow f(y)$

4) $f(0) = 0'$

Theorem 5 ([5]). *An equivalence relation θ on a HADL L is a congruence if and only if $(a, b) \in \theta$ implies $(a \wedge c, b \wedge c), (a \vee c, b \vee c) \in \theta$ for all $a, b, c \in L$.*

3. Ordered Filters of HADLs

In this section, the concept of ordered filters is introduced in HADLs and these filters are then characterized. A necessary and sufficient condition is derived for any non-empty set to become an ordered filter. The homomorphic images of ordered filters are studied.

Definition 6. *Let L be a HADL with a maximal element m. A non-empty subset F of L is called an ordered filter if it satisfies the following conditions for all $x, y \in L$:*

(O_1) $x, y \in F$ *implies* $x \wedge y \in F$

(O_2) $x \in F$ *and* $x \wedge m \leq y \wedge m$ *imply* $y \in F$

Example 1. *Let L be a discrete ADL with 0 and with at least two elements. Fix $m(\neq 0) \in L$ and define for any $x, y \in L$.*

$$x \to y = \begin{cases} 0 & \text{if } x \neq 0, y = 0 \\ m & \text{otherwise} \end{cases}$$

Then clearly $(L, \vee, \wedge, \to, 0, m)$ is an HADL and $\{m\}$ is an ordered filter in L.

Let F be an ordered filter of an HADL L. Then choose $x \in F$. Since $x \wedge m \leq m = m \wedge m$, by the condition O_2, we get that $m \in F$. The following Lemma can help us to understand the relation between a filter and an ordered filter of a HADL.

Lemma 4. *Let m be a maximal element of a HADL L. Then every filter of L is an ordered filter.*

Proof. Let F be a filter of L. Clearly $m \in F$. Let $x, y \in F$. Since F is a filter in L, we get that $x \wedge y \in F$. Assume that $x \in F$ and $x \wedge m \leq y \wedge m$. Then clearly $x \wedge m \in F$. Now $y \wedge x \wedge m = x \wedge y \wedge m = x \wedge y \wedge m \wedge m = x \wedge m \wedge y \wedge m = x \wedge m \in F$. Since F is a filter in L, it yields that $y = y \vee (y \wedge x \wedge m) \in F$. Therefore F is an ordered filter of L.

The set M_0 of all maximal elements of L is a filter and hence an ordered filter in L. If L has more than one maximal elements, then for any maximal element m, the set $\{m\}$ is an ordered filter but not a filter in L. In the following, a set of equivalent conditions are derived for every non-empty subset of a HADL to become an ordered filter.

Theorem 6. *Let m be a maximal element of L and F a non-empty subset of L. Then F is an ordered filter of L if and only if it satisfies the following properties.*

(O_3) $m \in F$

(O_4) $x \in F$, $x \to y \in F$ *imply* $y \in F$ *for all* $x, y \in L$

Proof. Assume that F is an ordered filter of L. Then clearly $m \in F$. Let $x \in F$ and $x \to y \in F$. Then $x \wedge y \wedge m = x \wedge (x \to y) \in F$. We have always $x \wedge y \wedge m \wedge m = x \wedge y \wedge m \leq y \wedge m$. Since $x \wedge y \wedge m \in F$ and F is an ordered filter, we get that $y \in F$. Conversely, assume that F satisfies the conditions O_3 and O_4. Now, let $x, y \in F$. Then we have the following consequence:

$$y \wedge m \leq (x \to y) \wedge m$$

$$\begin{aligned} &= (x \to y) \wedge m \wedge m \\ &= m \wedge (x \to y) \wedge m \\ &= (x \to x) \wedge (x \to y) \wedge m \\ &= (x \to (x \wedge y)) \wedge m \end{aligned}$$

Thus by Theorem 4, we get that $y \to [x \to (x \wedge y)] = m \in F$. By the repeated application of the property O_4, we get that $x \wedge y \in F$. Again, let $x \in L$ and $x \wedge m \leq y \wedge m$. Then $x \to y = m \in F$. By property O_4, we get $y \in F$. Hence F is an ordered filter of L.

Theorem 7. *Let m be a maximal element of L and F a non-empty subset of L. Then F is an ordered filter if and only if it satisfies the following property.*

(O_5) $x \wedge m \leq (y \to z) \wedge m$ implies $z \in F$ for all $x, y \in F$ and $z \in L$

Proof. Assume that F is an ordered filter of L. Let m be a maximal element in L. Let $x \wedge m \leq (y \to z) \wedge m$ for all $x, y \in F$ and $z \in L$. Then we get that $x \to (y \to z) = m \in F$. Since $x, y \in F$, it follows from O_4 that $z \in F$.

Conversely, assume that F satisfies the condition O_5. Now for any $x \in F$, we have $x \wedge m \leq m = m \wedge m = (x \to m) \wedge m$. Thus from the condition O_5, we get $m \in F$. Let $x \in F$ and $x \to y \in F$. Since $x \wedge m \leq ((x \to y) \to y) \wedge m$, by the assumed condition O_5 it yields that $y \in F$. Thus F is an ordered filter of L.

The following Corollary is a direct consequence of above Theorem.

Corollary 1. *Let m be a maximal element of an HADL L and F a non-empty subset of L. Then F is an ordered filter if and only if it satisfies the following:*

(O_6) $x \to (y \to z) = m$ implies $z \in F$ for all $x, y \in F$ and $z \in L$

Lemma 5. *Let F be an ordered filter of L. Then for all $x, y, z \in L$, the following condition holds.*

$$(x \to y) \to z \in F \text{ implies } x \to (y \to z) \in F$$

Proof. Let $x, y, z \in L$ and m a maximal element of L. Suppose $(x \to y) \to z \in F$. Then we have the following consequence:

$$\begin{aligned} y \wedge m \leq (x \to y) \wedge m \quad &\Rightarrow \quad ((x \to y) \wedge m) \to z \leq (y \wedge m) \to z \\ &\Rightarrow \quad (((x \to y) \wedge m) \to z) \wedge m \leq ((y \wedge m) \to z) \wedge m \\ &\Rightarrow \quad ((x \to y) \to (m \to z)) \wedge m \leq (y \to (m \to z)) \wedge m \\ &\Rightarrow \quad ((x \to y) \to (z \wedge m)) \wedge m \leq (y \to (z \wedge m)) \wedge m \\ &\Rightarrow \quad (((x \to y) \to z) \wedge m) \wedge m \leq ((y \to z) \wedge m) \wedge m \\ &\Rightarrow \quad ((x \to y) \to z) \wedge m \leq (y \to z) \wedge m \\ &\Rightarrow \quad ((x \to y) \to z) \to (y \to z) = m \in F \\ &\Rightarrow \quad y \to z \in F \text{ since } (x \to y) \to z \in F \end{aligned}$$

Since $(y \to z) \wedge m \leq (x \to (y \to z)) \wedge m$, we get $(y \to z) \to (x \to (y \to z)) = m \in F$. Since $y \to z \in F$, we get $x \to (y \to z) \in F$. This completes the proof.

Theorem 8. *Let F be an ordered filter of a HADL L with a maximal element m. Then for all $x, y \in L$, it satisfies the following condition.*

(O_7) $(\forall\, x, y \in L)(\forall\, a \in F)$, $(x \to a) \to y \in F$ *implies* $x \to y \in F$

Proof. Suppose F is an ordered filter of L. Let $x, y \in L$ and $a \in F$ be such that $(x \to a) \to y \in F$. Now we have the following

$$\begin{aligned} a \wedge m \leq (x \to a) \wedge m \quad &\Rightarrow \quad a \to (x \to a) = m \in F \\ &\Rightarrow \quad x \to a \in F \text{ since } a \in F) \\ &\Rightarrow \quad y \in F \text{ since } (x \to a) \to y \in F) \end{aligned}$$

Since $y \wedge m \leq (x \to y) \wedge m$, we get $y \to (x \to y) = m \in F$. Since $y \in F$, we get $x \to y \in F$.

In the following Theorem, the result of Chan and Shum [3] on implicative homomorphisms is generalized to the case of HADLs.

Theorem 9. *Let $(L, \vee, \wedge, \to, 0, m)$ and $(L', \vee, \wedge, \to, 0', m')$ be two HADLs. Let α be a mapping from L into L' such that $\alpha(x \to y) = \alpha(x) \to \alpha(y)$ for all $x, y \in L$. Let $\mathscr{F} = \{x \in L \mid \alpha(x) \wedge m' = m'\}$. Then the following properties hold.*

1) $\alpha(m) = m'$

2) $x \leq y$ *implies* $\alpha(x) \wedge m' \leq \alpha(y) \wedge m'$

3) If $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$ *for all* $x, y \in L$, *then* $\mathscr{F}$ *is an ordered filter of* L

4) If $\mathscr{F} = \{m\}$, *then the following implication holds:*

$$\alpha(x) = \alpha(y) \text{ implies } x \to y = y \to x \text{ for all } x, y \in L$$

Proof. 1). By Definition 4, we get $\alpha(m) = \alpha(m \to m) = \alpha(m) \to \alpha(m) = m'$.

2). Suppose that $x \leq y$. Then we get $\alpha(x) \to \alpha(y) = \alpha(x \to y) = \alpha(m) = m'$. From the Theorem 4, it can be concluded that $\alpha(x) \wedge m' \leq \alpha(y) \wedge m'$.

3). Clearly $\alpha(m) \wedge m' = m' \wedge m' = m'$. Hence $m \in \mathscr{F}$. Let $x, y \in \mathscr{F}$. Then $\alpha(x) \wedge m' = \alpha(y) \wedge m' = m'$ and hence

$$\alpha(x \wedge y) \wedge m' = \alpha(x) \wedge \alpha(y) \wedge m' \wedge m' = \alpha(x) \wedge m' \wedge \alpha(y) \wedge m' = m' \wedge m' = m'.$$

Therefore $x \wedge y \in \mathscr{F}$. Again, let $x \in \mathscr{F}$ and $x \wedge m \leq y \wedge m$. Then by condition 2), we get $\alpha(x) \wedge m' = m'$ and $\alpha(x \wedge m) \wedge m' \leq \alpha(y \wedge m) \wedge m'$. Thus

$$m' = \alpha(x) \wedge m' = \alpha(x) \wedge \alpha(m) \wedge m' = \alpha(x \wedge m) \wedge m' \leq \alpha(y \wedge m) \wedge m' = \alpha(y) \wedge m' \wedge m' = \alpha(y) \wedge m'.$$

Hence $\alpha(y) \wedge m' = m'$ and thus $y \in \mathscr{F}$. Therefore $\mathscr{F}$ is an ordered filter of L.

4). Assume that $\mathscr{F} = \{m\}$. Lat $x, y \in L$ be such that $\alpha(x) = \alpha(y)$. Then $\alpha(x \to y) \wedge m' = (\alpha(x) \to \alpha(y)) \wedge m' = (\alpha(x) \to \alpha(x)) \wedge m' = m' \wedge m' = m'$. This means that $x \to y \in \mathscr{F} = \{m\}$. Therefore $x \to y = m$. By using the similar argument, we can get $y \to x = m$. Therefore $x \to y = y \to x$.

Theorem 10. *Let $(L, \vee, \wedge, \to, 0, m)$ and $(L', \vee, \wedge, \to, 0', m')$ be two HADLs and $\alpha : L \longrightarrow L'$ an onto mapping such that $\alpha(x \to y) = \alpha(x) \to \alpha(y)$ for all $x, y \in L$. Then the following properties hold.*

1) If F is an ordered filter of L, then $\alpha(F)$ is an ordered filter of L'

2) If F' is an ordered filter of L', then $\alpha^{-1}(F')$ is an ordered filter of L

Proof. 1). Let F be an ordered filter of L. Since $m \in F$, we get $m' = \alpha(m) \in \alpha(F)$. Let $\alpha(x) \in \alpha(F)$ and $\alpha(x) \to y' \in \alpha(F)$ for some $y' \in L'$. Since α is onto, there exists $y \in L$ such that $\alpha(y) = y'$. Now $\alpha(x \to y) = \alpha(x) \to \alpha(y) = \alpha(x) \to y' \in \alpha(F)$. Hence $x \to y \in F$. Since F is an ordered filter of L, we get that $y \in F$. Therefore $y' = \alpha(y) \in \alpha(F)$. Thus we can conclude that $\alpha(F)$ is an ordered filter of L'.

2). Let F' be an ordered filter of L'. It is clear that $m = \alpha^{-1}(m') \in \alpha^{-1}(F')$. Let $x \in \alpha^{-1}(F')$ and $x \to y \in \alpha^{-1}(F')$. Then $\alpha(x) \in F'$ and $\alpha(x) \to \alpha(y) = \alpha(x \to y) \in F'$. Since F' is an ordered filter, it can be concluded that $\alpha(y) \in F'$. Hence $y \in \alpha^{-1}(F')$. Therefore $\alpha^{-1}(F')$ is an ordered filter of L.

4. Congruences and Ordered Filters

In this section, three different congruence relations are introduced in terms of ordered filters of a HADL and then their properties are studied. Finally, a relation is established among these congruences.

Definition 7. *Let m be a maximal element of L. For any ordered filter F of L and $a \in L$, the set F^a is defined as follows:*

$$F^a = \{x \in L \mid (a \to x) \wedge m \in F\}$$

It is obvious that $F^0 = L$ and $F^m = F$.

Proposition 1. *Let m be a maximal element of L. If F is an ordered filter of L and $a \in L$, then F^a is an ordered filter of L containing F.*

Proof. Clearly $m \in F^a$. Let $x, y \in F^a$. Then $(a \to x) \wedge m \in F$ and $(a \to y) \wedge m \in F$. Then $(a \to (x \wedge y)) \wedge m = (a \to x) \wedge (a \to y) \wedge m = (a \to x) \wedge m \wedge (y \to a) \wedge m \in F$. Hence $x \wedge y \in F^a$. Let $x \in F^a$ and $x \wedge m \leq y \wedge m$. Then $(a \to x) \wedge m \in F$ and

$$(a \to x) \wedge m = (a \to x) \wedge (a \to m) = a \to (x \wedge m) \leq a \to (y \wedge m) = (a \to y) \wedge (a \to m) = (a \to y) \wedge m.$$

Since F is an ordered filter and $(a \to x) \wedge m \in F$, we get $(a \to y) \wedge m \in F$. Hence $y \in F^a$. Therefore F^a is an ordered filter of L. Let $x \in F$. Since $x \wedge m \leq (a \to x) \wedge m$, we can get $(a \to x) \wedge m \in F$. Hence $x \in F^a$. Therefore F^a is an ordered filter containing F.

Lemma 6. *Let F, G be two ordered filters of L. Then for any $a, b \in L$, we have the following properties:*

1) $a \leq b$ *implies* $F^b \subseteq F^a$

2) $F \subseteq G$ *implies* $F^a \subseteq G^a$

3) $F^a \cap G^a = (F \cap G)^a$

4) $F^{a \vee b} = F^a \cap F^b = F^{b \vee a}$

5) $F^{a \wedge b} = F^{b \wedge a}$

Proof. 1). Let $a \leq b$. If $x \in F^b$, then $(b \to x) \wedge m \in F$. Since $(b \to x) \wedge m \leq (a \to x) \wedge m$, we get $(a \to x) \wedge m \in F$. Thus $x \in F^a$. Hence $F^b \subseteq F^a$.

2). Suppose that $F \subseteq G$. Let $x \in F^a$. Then we get $(a \to x) \wedge m \in F \subseteq G$. Hence $x \in G^a$. Therefore $F^a \subseteq G^a$.

3). It is trivial that $(F \cap G)^a \subseteq F^a \cap G^a$. Conversely, let $x \in F^a \cap G^a$. Then we get $a \to x \in F \cap G$. Therefore $x \in (F \cap G)^a$.

4). Clearly $F^{a \vee b} \subseteq F^a \cap F^b$. Conversely, let $x \in F^a \cap F^b$. Then $(a \to x) \wedge m, (b \to x) \wedge m \in F$. Since F is an ordered filter, we get

$$((a \vee b) \to x) \wedge m = (a \to x) \wedge (b \to x) \wedge m = (a \to x) \wedge m \wedge (b \to x) \wedge m \in F.$$

Hence $x \in F^{a \vee b}$. Therefore $F^a \cap F^b = F^{a \vee b}$. The remaining part is trivial.

5). For any $x \in L$, from Lemma 3(9), we get that

$$x \in F^{a \wedge b} \Longleftrightarrow ((a \wedge b) \to x) \wedge m \in F \Longleftrightarrow (b \wedge a) \to x) \wedge m \in F \Longleftrightarrow x \in F^{b \wedge a}.$$

Therefore it yields that $F^{a \wedge b} = F^{b \wedge a}$.

For any ordered filter F of a HADL L, $\widehat{F}$ denotes the set of all ordered filters of the form $F^x, x \in L$. Then we have the following result.

Theorem 11. *Let F be an ordered filter of a HADL L. Then the set $\widehat{F} = \{F^x \mid x \in L\}$ forms a distributive lattice.*

Proof. For any F^a, F^b, it is clear from Lemma 6(4) that $F^{a \vee b}$ is the infemum for F^a and F^b. It is remaining to prove that $F^{a \wedge b}$ is the supremum of F^a and F^b. Clearly $F^a, F^b \subseteq F^{a \wedge b}$. Let F^c be an upper bound for both F^a and F^b. Let $x \in F^{a \wedge b}$. Then

$$(a \to (b \to x)) \wedge m = ((a \wedge b) \to x) \wedge m \in F.$$

Hence $b \to x \in F^a \subseteq F^c$. Then $(b \to (c \to x)) \wedge m = (c \to (b \to x)) \wedge m \in F$. Hence $c \to x \in F^b \subseteq F^c$. Therefore $(c \to x) \wedge m = ((c \wedge c) \to x) \wedge m = (c \to (c \to x)) \wedge m \in F$. Thus

$x \in F^c$. Hence $F^{a\wedge b} \subseteq F^c$. Now consider $F^{a\wedge b} = F^a \sqcup F^b$. Then it can be easily verified that $(\widehat{F}, \sqcup, \cap)$ is a distributive lattice.

In the following, we introduce a congruence relation on L in terms of ordered filters of the form $F^x, x \in L$. Throughout this section, a congruence on an a HADL means an equivalence relation satisfying the property of Theorem 5.

Definition 8. *Let F be an ordered filter of L. Define a relation θ as follows:*

$$(a,b) \in \theta \textit{ if and only if } F^a = F^b \textit{ for all } a,b \in L$$

Theorem 12. *For any ordered filter F of a HADL L, the relation θ defined above is a congruence on L.*

Proof. Clearly θ is an equivalence relation on L. Let $(a,b) \in \theta$. Then $F^a = F^b$. Let $x \in F^{a\wedge c}$. Then $(a \to (c \to x)) \wedge m = ((a \wedge c) \to x) \wedge m \in F$. Hence $c \to x \in F^a = F^b$. Thus $((b\wedge c) \to x) \wedge m = (b \to (c \to x)) \wedge m \in F$. Hence $x \in F^{b\wedge c}$. Therefore $F^{a\wedge c} \subseteq F^{b\wedge c}$. Similarly, we can get $F^{a\wedge c} \subseteq F^{b\wedge c}$. Therefore $F^{a\wedge c} = F^{b\wedge c}$. Thus $(a \wedge c, b \wedge c) \in \theta$. It is also clear that $F^{a\vee c} = F^a \cap F^c = F^b \cap F^c = F^{b\vee c}$. Hence $(a \vee c, b \vee c) \in \theta$. Therefore θ is a congruence on L.

Definition 9. *An element $a \in L$ is called F-identity if $F^a = L$.*

Denote the set of all F-identity elements by D_F. Then we have the following:

Lemma 7. *Let F, G be two ordered filters of L and $a \in L$. Then we have the following:*

1) $F^0 = L$

2) D_F *is a lattice ideal of* L

3) D_F *is a sublattice of* L

4) If $F \cap D_F \neq \emptyset$*, then* $F = D_F = L$

Proof. 1). It is clear.

2). Clearly $0 \in D_F$. Let $x, y \in D_F$. Then $F^{x\vee y} = F^x \cap F^y = L$. Hence $x \vee y \in D_F$. Let $x \in D_F$, $y \leq x$. Then $L = F^x \subseteq F^y$. Therefore $y \in D_F$.

3). Let $x, y \in D_F$. Since $x \wedge y \leq y$, it can be concluded that $L = F^y \subseteq F^{x\wedge y}$.

4). Let $x \in F \cap D_F$. Then we get $F = F^x = L$. Therefore $F = D_F = L$.

The following Theorem is a routine verification.

Theorem 13. *For any $a, b \in L$, define a relation Ψ_{D_F} as $(a,b) \in \Psi_{D_F}$ if and only if $a \vee d = b \vee d$ for some $d \in D_F$. Then Ψ_{D_F} is a congruence relation on L.*

Definition 10. *Let F be an ordered filter of a HADL L. For any $a \in L$, define* $(a, D_F) = \{ x \mid x \wedge a \in D_F \}$.

Lemma 8. *Let F be an ordered filter of L. For any $a, b \in L$ we have*

1) (a, D_F) *is a lattice ideal of L*

2) $a \leq b$ *implies* $(b, D_F) \subseteq (a, D_F)$

3) $(a \vee b, D_F) = (a, D_F) \cap (b, D_F)$

Proof. 1). Clearly $0 \in (a, D_F)$. Let $x, y \in (a, D_F)$. Then $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) \in D_F$. Hence $x \vee y \in (a, D_F)$. Let $x \in (a, D_F)$ and $y \leq x$. Then $y \wedge a \leq x \wedge a \in D_F$. Hence $y \in (a, D_F)$. Therefore (a, D_F) is an ideal of L.

2). Suppose that $a \leq b$. Let $x \in (b, D_F)$. Then we get $b \wedge x \in D_F$. Hence $a \wedge x \leq b \wedge x \in D_F$. Thus it yields that $a \wedge x \in D_F$. Therefore $x \in (a, D_F)$.

3). Clearly $(a \vee b, D_F) \subseteq (a, D_F) \cap (b, D_F)$. Conversely, let $x \in (a, D_F) \cap (b, D_F)$. Hence $(a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) \in D_F$. Therefore it concludes $x \in (a \vee b, D)$.

Theorem 14. *For any $a, b \in L$, the relation Θ defined by $(a, b) \in \Theta$ if and only if $(a, D_F) = (b, D_F)$ is a lattice congruence on L.*

Proof. Clearly Θ is an equivalence relation. Suppose $(a, D_F) = (b, D_F)$. Then for any $c \in L$, we get

$$x \in (a \wedge c, D_F) \Leftrightarrow x \wedge a \wedge c \in D_F \Leftrightarrow x \wedge c \in (a, D_F) = (b, D_F) \Leftrightarrow c \wedge c \wedge b \in D_F \Leftrightarrow x \in (b \wedge c, D_F).$$

Hence $(a \wedge c, b \wedge c) \in \Theta$. Again $(a \vee c, D_F) = (a, D_F) \cap (c, D_F) = (b, D_F) \cap (c, D_F) = (a \vee c, D_F)$. Hence $(a \vee c, b \vee c) \in \Theta$.

Theorem 15. *Let F be an ordered filter of L. Then we have $\Psi_{D_F} \subseteq \theta \subseteq \Theta$.*

Proof. Let $a, b \in \Psi_{D_F}$. Then we can write $a \vee d = b \vee d$ for some $d \in D_F$. Now $F^a = F^a \cap L = F^a \cap F^d = F^{a \vee d} = F^{b \vee d} = F^b \cap F^d = F^b \cap L = F^b$. Therefore $(a, b) \in \theta$. Assume that $(a, b) \in \theta$. Then we have $F^a = F^b$. Now

$$\begin{aligned} x \in (a, D) &\Leftrightarrow a \wedge x \in D_F \Leftrightarrow F^{a \wedge x} = L \Leftrightarrow F^a \sqcup F^x \\ &= L \Leftrightarrow F^b \sqcup F^x = L \Leftrightarrow F^{b \wedge x} = L \Leftrightarrow b \wedge x \in D_F \Leftrightarrow x \in (b, D). \end{aligned}$$

Thus it yields that $(a, b) \in \Theta$. Therefore we can conclude that $\Psi_{D_F} \subseteq \theta \subseteq \Theta$.

References

[1] G. Birkhoff. *Lattice Theory*, Amer. Math. Soc. Colloq. XXV, Amer. Math. Soc., Providence, (1967), U.S.A.

[2] M.W. Chan and K.P. Shum. Homomorphisms of implicative semigroups, *Semigroup Forum*, **46** (1996), 7-15.

[3] G.C. Rao and B. Assaye. Implicative filters on Heyting almost distributive lattices, *Inter. J. Computational Cognition*, **8**, no. 4 (2010), 55-59.

[4] G.C. Rao, B. Assaye, and M.V.R. Mani. Heyting almost distributive lattices, *Inter. J. Computational Cognition*, **8**, no. 3 (2010), 85-89.

[5] U.M. Swamy and G.C. Rao, Almost distributive lattices, *Jour. Aust. Math. Soc*. (Series A), **31** (1981), 77-91.