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An Alternative Forward Measure Approach to Hedging Defaultable Contingent Claims

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Abstract. In this paper, we present a *forward measure approach* to hedge defaultable contingent claims under stochastic interest rates. The relation of hedging strategies between the risk neutral measure and forward measure is deduced. Under the invariance martingale property and reduced-form model for default risk, to hedge a defaultable contingent claim depending on the forward price, one has to invest the same amount of this contingent claim value in the defaultable zero-coupon, and use defaultfree zero-coupon bond to hedge interest risk which extends the result of Blanchet-Scalliet, Jeanblane [4].

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1. Introduction

The traditional *Black-Scholes option pricing* formula [3] is derived under the assumption that there is no default risk of the option issuer. In recent years, *OTC options* have become increasingly popular, and hence the default risk of the option issuer should be considered in the pricing. Default risk is the risk that the agents cannot fulfill their obligations in the contracts. Modeling default risk is one of the fundamental problems of interest in finance. In general, default risk models have two main categories: *structural* and *reduced form*. Reduced form approach considers the default to be an exogenously specified jump process, derives the default probability as the instantaneous likelihood of default. The default time is a totally inaccessible stopping time, which is usually defined as the first jump time of a cox process with a given intensity. For reduced form model, interested readers can refer to Jarrow and Turnbull [8], Duffie and Singleton [5], Lando [10]. Bielecki and Rutkowski [2] described the pricing and hedging of defaultable contingent claims for details.

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The *forward measure pricing* methodology was introduced by Jamshidian [7]. It has been widely used in pricing securities when interest rates are stochastic. The formal definition of a forward probability measure was explicitly introduced in Geman [6]. In particular, Geman [6] observed that the forward price of any financial asset follows a (local) martingale under the forward neutral probability associated with the settlement date of a forward contract. Recently, the forward price is reinterpreted using the forward measure. It is shown that the forward price is an expectation of the terminal payoff of the underlying asset price, like the futures price which is a risk neutral expectation, except that the probability measure under which the expectation is taken is different from the risk neutral measure by an adjustment term.

The idea of our paper originates from Blanchet-Scalliet, Jeanblane [4], Liao and Huang [12]. The former one gives the hedging porfolio of vulnerable contingent claims using defaultable zero-coupon and default-free assets, when the default-free market is complete, and the second one applies the forward risk-neutral pricing approach to derive the closed-form valuation of the vulnerable option under stochastic interests rates. Therefore, we try to hedge defaultable contingent claims under stochastic interest rates using forward measure and forward prices. Our result claims that there *exists* connection of hedging strategies between the risk neutral measure and forward measure. Under the invariance martingale property and reduced-form model for default risk, to hedge a defaultable contingent claim depending on the forward price, one has to invest the same amount of this contingent claim value in the defaultable zero-coupon, and use default-free zero-coupon bond to hedge interest risk which extends the result of Blanchet-Scalliet, Jeanblane [4].

The paper is organized as follows: In Section 2, we first introduce the basic knowledge of forward measure, then we present the valuation framework under forward measure and reduced-form models for defaultable contingent claims. Correspondingly, the hedging strategy is derived; In Section 3, we quickly summarize the main results of the article.

2. Valuation and Hedging under Forward Measure

2.1. Forward Measure and Forward Price

In this section, we introduce the forward measure approach which distinguishes the no-arbitrage valuation within the classical Black-Scholes framework. Our aim is to price contingent claims under stochastic interest rates.

Given a complete probability space $(\Omega, \mathscr{F}, P^*)$, T is a strictly positive real number. Let $(\mathscr{F}_t)_{t\geq 0}$ be the σ-algebra at time t, for any $0 \leq t \leq T$, $\mathscr{F}_t \subset \mathscr{F}$, P^* is the risk-neutral measure. There are three financial assets in the market: the riskless asset β_t(i.e., money market account), a risky asset S_t(i.e., stock) and the defaultfree zero coupon bond $B(t,T)$. The process β_t satisfies $d\beta_t = \beta_t r_t dt$, $\beta_0 = 1$, and S_t, $B(t,T)$ satisfy respectively

$$dS_t = S_t(r_t dt + \sigma_t dW_S(t)), \quad S_0 > 0, \tag{1}$$

$$dB(t,T) = B(t,T)(r_t dt + b(t,T)dW_B(t)), B(T,T) = 1, \tag{2}$$

where r_t stands for the instantaneous, continuously compounded interest rate, $W_S(t)$ and

$W_B(t)$ are standard Brownian motion defined on $(\Omega, \mathscr{F}, P^*)$ with the correlation coefficient ρ. By $It\hat{o}$ formula, the discount stock prices under risk neutral measure become

$$d\tilde{S}_t = d(S_t\beta_t^{-1}) = \tilde{S}_t\sigma_t dW_S(t).$$

Definition 1. *A probability measure P^T is called the* forward martingale measure *(or* forward measure*) for the settlement date T, with the Radon-Nikodým derivative given by*

$$\frac{dP^T}{dP^*} = \frac{\beta_T^{-1}}{\mathbb{E}_{P^*}[\beta_T^{-1}]} = \frac{1}{\beta_T B(0,T)}.$$

Notice that for every $t \in [0,T]$, when restricted to the F_t,

$$\eta_t = \frac{dP^T}{dP^*}\bigg|_{\mathscr{F}_t} = \mathbb{E}_{P^*}\left[\frac{1}{\beta_T B(0,T)}\bigg|\mathscr{F}_t\right] = \frac{B(t,T)}{\beta_t B(0,T)}.$$

When the bond price is governed by (2), an explicit representation for η_t is available, and $d\eta_t = -\eta_t b(t,T)dt$.

The forward price of a European contingent claim X which settles at time T can be expressed as the conditional expectation under the forward measure P^T, that is, $F_X(t,T) = X_t/B(t,T) = \mathbb{E}_{P^T}[X|\mathscr{F}_t]$, for all $t \in [0,T]$. Moreover, the relative price of any traded security (which pays no coupons or dividends) follows a local martingale under the forward probability measure P^T, provided that the price of a bond which matures at time T is taken as a numeraire. For example, the forward price of S_t follows $F_S(t,T) = S_t/B(t,T) = \mathbb{E}_{P^T}[S_T|\mathscr{F}_t]$, for all $t \in [0,T]$. For simplicity, we write $b(t,T) = b_t$. By $It\hat{o}$ formula,

$$\begin{aligned} dF_S(t,T) =& B(t,T)^{-1}dS_t + S_t dB(t,T)^{-1} + d\langle S_t, B(t,T)^{-1}\rangle \\ =& F_S(t,T)(r_t dt + \sigma_t dW_S(t) - r_t dt - b_t dW_B(t) + b^2(t,T)dt - \rho\sigma_t b_t dt) \\ =& F_S(t,T)(\sigma_t dW_S(t) - b_t dW_B(t) + (b_t^2 - \rho\sigma_t b_t)dt). \end{aligned}$$

Let $\sigma_{F_S} dW_t = \sigma_t dW_S(t) - b_t dW_B(t)$, and $\sigma_{F_S}^2 = \sigma_t^2 - 2\sigma_t b_t \rho + b_t^2$, then

$$\begin{aligned} dF_S(t,T) =& F_S(t,T)(\sigma_{F_S} dW_t + (b_t^2 - \rho\sigma_t b_t)dt) \\ =& F_S(t,T)\sigma_{F_S} dW_{F_S}^{P^T}, \end{aligned}$$

where $dW_{F_S}^{P^T} = dW_t + \psi_t dt$, $\psi_t = (b_t^2 - \rho\sigma_t b_t)/\sigma_{F_S}$.

2.2. The Valuation Framework with Default Risk

In this section, the forward measure approach of the Black-Scholes valuation under stochastic interest rates and default risk are employed.

A default event occurs at a random time τ(i.e., a non-negative random variable). The default process is defined as $N_t \triangleq 1_{\{\tau \le t\}}$, and $\mathscr{D}_t = \sigma(N_s, s \le t)$, the filtration $\mathscr{D}$ is used to describe the information about default time, where $\mathscr{D} = \{\mathscr{D}_t, 0 \le t \le T\}$. At any time t, the agent's information on the securities prices and default time is $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{D}_t$ and the agent knows whether or not the default has appeared. Hence, the default time τ is a $\mathscr{G}$ stopping time where $\mathscr{G} = \{\mathscr{G}_t, 0 \le t \le T\}$. In fact, $\mathscr{G}$ is the smallest filtration which contains $\mathscr{F}$. From Blanchet-Scalliet, Jeanblance [4] and Bielecki, Rutkowski [2], there *exists* an equivalent martingale measure denoted by Q^* in the enlarged market $\mathscr{G}$, and the invariance martingale property ((H) hypothesis) holds: Any $\mathscr{F}$-square integrable martingale is a $\mathscr{G}$-square integrable martingale.

Define $G_t \triangleq Q^*(\tau > t|\mathscr{F}_t)$, $\Gamma_t \triangleq -\ln G_t$, Γ_t is called $\mathscr{F}$ hazard process of τ. In general, (H) hypothesis is unstable when changing probability measure. By Proposition 1 of Blanchet-Scalliet and Jeanblane [4], if $\mathscr{F}$-market is complete and arbitrage-free, $\mathscr{G}$-market is arbitrage-free, then (H) hypothesis holds under any equivalent martingale measure.

To prove that $\mathscr{G}$-market is arbitrage-free, we need to prove the existence of a forward measure Q^T in $\mathscr{G}$-market. We know P^T is the forward measure for the $\mathscr{F}$-market and η_t is its Radon-Nikodym density, the process $F_S(t,T)\eta_t$ is $\mathscr{F}$ square integrable martingale under P^*, hence a $\mathscr{G}$ square integrable martingale under P^*. Then there exists at least one probability measure Q^T defined as $dQ^T|_{\mathscr{F}_t} = \eta_t dP^*|_{\mathscr{F}_t}$, and the $\mathscr{G}_t$-market is arbitrage-free.

Definition 2. *Let $F_t^T = Q^T(\tau \le t|\mathscr{F}_t)$, $G_t^T = 1 - F_t^T$, $F_t^T < 1$. Then $\mathscr{F}$ hazard process of τ under Q^T is denoted by $\Gamma_t^T = -\ln G_t^T = -\ln(1 - F_t^T)$.*

Furthermore, without loss of generality, suppose $dQ^T|_{\mathscr{G}_t} = \xi_t dQ^*|_{\mathscr{G}_t}$, the Randon-Nikodým derivatives of Q^T with respect to Q^* satisfies

$$d\xi_t = \xi_t(\psi_t dW_t + \theta_t dW_B(t) + \phi_t dM_t). \tag{3}$$

Then $dW_B^{P^T} = dW_B(t) + \theta_t dt$ are standard Brownian Motion under Q^T,
$dM_t^T = dM_t - \phi_t d\Gamma_t$, $L_t^T = 1_{\{\tau>t\}} e^{\Gamma_t^T}$ are $\mathscr{G}$-martingale under Q^T.

Since forward measure Q^T is also a risk-neutral equivalent martingale measure, from Bielecki, Rutkowski [2], the following properties hold: (F_t^T) is nonnegative bounded submartingale, Γ_t^T is increasing, and $L_t^T = 1_{\{\tau>t\}} e^{\Gamma_t^T}$ is martingale. Moreover, for $\mathscr{G}$ measurable random variable Y, we have

$$\mathbb{E}_{Q^T}\left[1_{\{\tau>t\}} Y \middle| \mathscr{G}_t\right] = 1_{\{\tau>t\}} e^{\Gamma_t^T} \mathbb{E}_{P^T}\left[Y \middle| \mathscr{F}_t\right].$$

We assume now that a defaultable zero-coupon bond $\rho(t,T)$ of maturity T is traded on the market, X_t is the defaultable contingent claim. We also assume that the market is arbitrage free, then under forward measure Q^T, the forward prices of $\rho(t,T)$ and X_t satisfy

$$F_\rho(t,T) = \frac{\rho(t,T)}{B(t,T)} = E_{Q^T}[1_{\{\tau>T\}}|\mathscr{G}_t] = 1_{\{\tau>t\}} e^{\Gamma_t^T} E_{P^T}[G_T^T|\mathscr{F}_t] \triangleq L_t^T Z_t, \tag{4}$$

$$F_X(t,T) = \frac{X_t}{B(t,T)} = E_{Q^T}[1_{\{\tau>T\}} X_T|\mathscr{G}_t] = 1_{\{\tau>t\}} e^{\Gamma_t^T} E_{P^T}[G_T^T X_T|\mathscr{F}_t] \triangleq L_t^T Z_t^X, \tag{5}$$

where $Z_t \triangleq E_{P^T}[G_T^T|\mathscr{F}_t]$, $Z_t^X \triangleq E_{P^T}[G_T^T X_T|\mathscr{F}_t]$.

Similar to the results under risk neutral measure in Blanchet-Scalliet, Jeanblance [4], we have Lemma 1, 2.

Lemma 1. *Suppose that (H) holds under Q^T and F^T is continuous.Then*

$$dF_\rho(t,T) = L^T_{t-}dZ_t - F_\rho(t-,T)dM^T_t.$$

Proof. The result follows from $F_\rho(t,T) = L^T_t Z_t$, $dL^T_t = -L^T_{t-}dM^T_t$ and $It\hat{o}$ fromula.

Lemma 2. *Suppose that (H) holds under Q^T and F^T is continuous, $Y \in \mathscr{F}_T$ is integrable. Then the $\mathscr{G}$-martingale $Y_t = \mathbb{E}_{Q^T}[Y 1_{\{\tau>T\}}|\mathscr{G}_t]$ admits the following decomposition*

$$Y_t = Z^Y_0 + \int_0^{t\wedge\tau} e^{\Gamma^T_u} dZ^Y_u - \int_0^t Y_{u-}dM^T_u,$$

where Z^Y is the $\mathscr{F}$-martingale, $Z^Y_t = \mathbb{E}_{P^T}[Y G^T_T|\mathscr{F}_t]$.

Since the default-free market is complete and $G^T_T \in \mathscr{F}$, $G^T_T X_T \in \mathscr{F}$, there exist predictable processes (U_t), (U^X_t), (V_t), (V^X_t) and the constants Z_0, Z^X_0 such that

$$Z_t = Z_0 + \int_0^t U_u dF_S(u,T) + \int_0^t V_u dW^{P^T}_B, \tag{6}$$

$$Z^X_t = Z^X_0 + \int_0^t U^X_u dF_S(u,T) + \int_0^t V^X_u dW^{P^T}_B. \tag{7}$$

Note that we add the special risk factor in zero coupon bond in equations (6), (7) directly in terms of $W^{P^T}_B$ here.

2.3. Hedging Strategies

In the following we give the relation of hazard processes Γ_t and Γ^T_t, and also the relation of predictable processes between risk-neutral probability measure and forward measure.

From Blanchet-Scalliet and Jeanblance [4], under risk-neutral measure,

$$\rho(t,T)\beta_t^{-1} = E^{Q^*}(1_{\{\tau>T\}}\beta_T^{-1}|\mathscr{G}_t) = 1_{\{\tau>t\}}e^{\Gamma_t}E^{P^*}(G_T\beta_T^{-1}|\mathscr{F}_t) \triangleq L_t\tilde{m}_t, \tag{8}$$

$$X_t\beta_t^{-1} = E^{Q^*}(1_{\{\tau>T\}}X_T\beta_T^{-1}\}|\mathscr{G}_t) = 1_{\{\tau>t\}}e^{\Gamma_t}E^{P^*}(G_T X_T\beta_T^{-1}|\mathscr{F}_t) \triangleq L_t\tilde{m}^X_t. \tag{9}$$

Assume that

$$\tilde{m}_t = \tilde{m}_0 + \int_0^t \mu_u d\tilde{S}_u, \quad \tilde{m}^X_t = \tilde{m}^X_0 + \int_0^t \mu^X_u d\tilde{S}_u,$$

where m_0 and m^X_0 are constants, μ_u and μ^X_u are predictable processes.

Lemma 3. *The Randon-Nikodým derivatives of Q^T with respect to Q^* satisfies*

$$d\xi_t = \xi_t(\psi_t dW_t - b_t dW_B(t)).$$

The connections of the predictable processes between risk-neutral probability measure and forward measure are

$$U_t = \mu_t, \quad V_t = \frac{(\mu_t S_t - m_t)b_t}{B(t,T)},$$
$$U_t^X = \mu_t^X, \quad V_t^X = \frac{(\mu_t^X S_t - m_t^X)b_t}{B(t,T)}.$$

Proof. Combine (4) with (8), we have $Z_t = \frac{\tilde{m}_t \beta_t}{B(t,T)} e^{-\int_0^t \phi_s d\Gamma_s}$. By $It\hat{o}$ formula

$$\begin{aligned} d(B(t,T)^{-1}\tilde{m}_t\beta_t) =& B(t,T)^{-1}\left[(\beta_t\mu_t\sigma_t\tilde{S}_t dW_S(t) + \tilde{m}_t\beta_t r_t dt) + m_t(-r_t dt - b_t dW_B(t) + b_t^2 dt)\right. \\ & \left. - \mu_t\sigma_t S_t b_t \rho dt\right] \\ =& B(t,T)^{-1}[\mu_t S_t \sigma_t dW_S(t) - m_t b_t dW_B(t) + (m_t b_t^2 - \mu_t\sigma_t S_t b_t \rho)dt]. \end{aligned}$$

Because $\sigma_t dW_S(t) = \sigma_{F_S} dW_t + b_t dW_B(t)$, $\sigma_{F_S} dW_t = \sigma_{F_S} dW_{F_S}^{P^T} + (\sigma_t b_t \rho - b_t^2)dt$, then

$$\begin{aligned} & \mu_t S_t \sigma_t dW_S(t) - m_t b_t dW_B(t) + (m_t b_t^2 - \mu_t \sigma_t S_t b_t \rho)dt \\ =& \mu_t S_t \sigma_{F_S} dW_t + (\mu_t S_t - m_t) b_t dW_B(t) + (m_t b_t^2 - \mu_t \sigma_t S_t b_t \rho)dt \\ =& (\mu_t S_t \sigma_{F_S} dW_{F_S}^{P^T} + \mu_t S_t(\sigma_t b_t \rho - b_t^2)dt + (\mu_t S_t - m_t) b_t (dW_B^{P^T} - \theta_t dt) \\ & + (m_t b_t^2 - \mu_t \sigma_t S_t b_t \rho)dt \\ =& \mu_t S_t \sigma_{F_S} dW_{F_S}^{P^T} + (\mu_t S_t - m_t) b_t dW_B^{P^T} - ((\mu_t S_t - m_t) b_t^2 + (\mu_t S_t - m_t)\theta_t b_t)dt \\ =& \mu_t S_t \sigma_{F_S} dW_{F_S}^{P^T} + (\mu_t S_t - m_t) b_t dW_B^{P^T} - (\theta_t + b_t)(\mu_t S_t - m_t) b_t dt. \end{aligned}$$

Using $It\hat{o}$ formula again, we have

$$\begin{aligned} dZ_t =& \frac{\tilde{m}_t \beta_t}{B(t,T)} d(e^{-\int_0^t \phi_s d\Gamma_s}) + e^{-\int_0^t \phi_s d\Gamma_s} d\left(\frac{\tilde{m}_t \beta_t}{B(t,T)}\right) \\ =& e^{-\int_0^t \phi_s d\Gamma_s}\left(I_t + \frac{1}{B(t,T)}\left[\mu_t S_t \sigma_{F_S} dW_{F_S}^{P^T} + (\mu_t S_t - m_t) b_t dW_B^{P^T}\right]\right) \\ =& e^{-\int_0^t \phi_s d\Gamma_s}\left(I_t + \mu_t dF_S(t,T) + \frac{(\mu_t S_t - m_t) b_t}{B(t,T)} dW_B^{P^T}\right), \end{aligned}$$

where

$$I_t = \frac{-m_t \phi_t d\Gamma_t + (\theta_t + b_t)(\mu_t S_t - m_t) b_t dt}{B(t,T)}.$$

Because Z_t can be considered as the forward prices of G_T at time t, Z_t is martingale under forward measure P^T, then the term I_t equals zero, then we have $\phi_t = 0$, $\theta_t = -b_t$. Moreover, combine with (6), we conclude $U_t = \mu_t$ and $V_t = \frac{(\mu_t S_t - m_t) b_t}{B(t,T)}$. The proofs of equations about U_t^X and V_t^X are essentially the same as that of U_t and V_t. □

From Lemma 3, $\phi_t = 0$ means $\Gamma_t^T = \Gamma_t$, which implies that the hazard processes of default risk are the same under risk-neutral measure and forward measure.

Theorem 1. *Suppose $\mathscr{F}$-market is complete and arbitrage free, the $\mathscr{G}$-market is arbitrage free, and $F_t = 1 - G_t = P^T(\tau \le t|\mathscr{F}_t)$ is continuous, $F_X(t,T)$ is the forward price of contingent claims $X_T 1_{\{\tau>T\}}$ at time t, that is, $F_X(t,T) = E^{Q^T}[1_{\{\tau>T\}})X_T|\mathscr{G}_t]$. When $\tau > t$, there exist self-financing hedging strategies $(\Phi_1(t), \Phi_2(t), \Phi_3(t), \Phi_4(t))$ such that*

$$\Phi_1(t)\beta_t + \Phi_2(t)S_t + \Phi_3(t)\rho(t,T) + \Phi_4(t)B(t,T) = X_t,$$

where

$$\begin{aligned}
\Phi_2(t) &= e^{\Gamma_t}\left(U_t^X - \frac{Z_t^X}{Z_t}U_t\right) = e^{\Gamma_t}\left(\mu_t^X - \frac{m_t^X}{m_t}\mu_t\right),\\
\Phi_3(t) &= \frac{Z_t^X}{Z_t} = \frac{m_t^X}{m_t},\\
\Phi_4(t) &= (b_t B(t,T))^{-1}e^{\Gamma_t}\left(V_t^X - \frac{Z_t^X}{Z_t}V_t\right) = B(t,T)^{-2}e^{\Gamma_t}\left(\mu_t^X - \frac{Z_t^X}{Z_t}\mu_t\right)S_t
\end{aligned}$$

Proof. By Lemma 2, we have

$$\begin{aligned}
F_X(t,T) =& Z_0^X + \int_0^{t\wedge\tau} e^{\Gamma_u} dZ_u^X - \int_0^t F_X(u-,T)dM_u^T\\
=& Z_0^X + \int_0^{t\wedge\tau} e^{\Gamma_u}U_u^X dF_S(u,T) + \int_0^{t\wedge\tau} e^{\Gamma_u}V_u^X dW_B^{P^T} - \int_0^t 1_{\{\tau>u\}}e^{\Gamma_{u^-}}Z_{u-}^X dM_u^T.
\end{aligned}$$

On the other hand, from Lemma 1,

$$dM_u^T = \frac{dF_\rho(u,T) - L_{u-}dZ_u}{F_\rho(u-,T)}, \quad F_\rho(t,T) = L_t^T Z_t,$$

then the above equation becomes

$$\begin{aligned}
& Z_0^X + \int_0^{t\wedge\tau} e^{\Gamma_u}U_u^X dF_s(u,T) + \int_0^{t\wedge\tau} e^{\Gamma_u}V_u^X dW_B^{P^T} + \int_0^{t\wedge\tau} \frac{Z_{u-}^X}{Z_u}dF_\rho(u,T)\\
& - \int_0^{t\wedge\tau} e^{\Gamma_u}\frac{Z_{u-}^X}{Z_u}U_u dF_S(u,T) - \int_0^{t\wedge\tau} e^{\Gamma_u}\frac{Z_{u-}^X}{Z_u}V_u dW_B^{P^T}\\
=& Z_0^X + \int_0^{t\wedge\tau} e^{\Gamma_u}\left(U_u^X - \frac{Z_u^X}{Z_u}U_u\right)dF_S(u,T) + \int_0^{t\wedge\tau} e^{\Gamma_u}\left(V_u^X - \frac{Z_u^X}{Z_u}V_u\right)dW_B^{P^T}\\
& + \int_0^{t\wedge\tau} \frac{Z_u^X}{Z_u}dF_\rho(u,T).
\end{aligned}$$

Therefore

$$\Phi_2(t) = e^{\Gamma_t}\left(U_t^X - \frac{Z_t^X}{Z_t}U_t\right), \quad \Phi_3(t) = \frac{Z_t^X}{Z_t}, \quad \tilde{\Phi}(t) \triangleq e^{\Gamma_t}\left(V_t^X - \frac{Z_t^X}{Z_t}V_t\right),$$

and

$$dF_X(t,T) = \Phi_2(t)dF_S(t,T) + \Phi_3(t)dF_\rho(t,T) + \tilde{\Phi}(t)dW_B^{P^T}.$$

By Lemma 3, $\Phi_2(t) = e^{\Gamma_t}\left(\mu_t^X - \frac{m_t^X}{m_t}\mu_t\right)$. From (4), (5), $F_X(t,T) = \frac{Z_t^X}{Z_t}F_\rho(t,T) = \Phi_3(t)F_\rho(t,T)$, then $\Phi_1(t)\beta_t + \Phi_2(t)S_t + \Phi_4(t)B(t,T) = 0$. From equation (2),

$$\begin{aligned}\Phi_4(t) =&(b_t B(t,T))^{-1}\tilde{\Phi}(t)\\ =&B(t,T)^{-2}e^{\Gamma_t}\left(\mu_t^X S_t - m_t^X - \frac{Z_t^X}{Z_t}(\mu_t S_t - m_t)\right)\\ =&B(t,T)^{-2}e^{\Gamma_t}\left(\mu_t^X - \frac{Z_t^X}{Z_t}\mu_t\right)S_t.\end{aligned}$$

Then $\Phi_1(t)$ can be calculated easily. □

2.4. Example

In this subsection, we look at a special case with $\rho = 1$, at which the risk factor that driven S_t and $B(t,T)$ is the same. Then $W_S(t) = W_B(t) \triangleq W_t$,

$$\sigma_{F_S} = \sigma_t - b_t, \quad \psi_t = (b_t^2 - \rho\sigma_t b_t)/\sigma_{F_S} = -b_t.$$

Let $dW_t^T \triangleq dW_{F_S}^{P^T} = dW_B^{P^T} = dW_t - b_t dt$, it is easy to get

$$dF_S(t,T) = F_S(t,T)(\sigma_t - b_t)dW_t^T. \tag{10}$$

Lemma 2 and subsequent equations (6), (7) degenerate to the following form:

$$Z_t = Z_0 + \int_0^t \tilde{U}_u dF_S(u,T), \quad Z_t^X = Z_0^X + \int_0^t \tilde{U}_u^X dF_S(u,T). \tag{11}$$

Lemma 4. *When $\rho = 1$, the Randon-Nikodým derivative of Q^T with respect to Q^* satisfies $d\xi_t = -\xi_t b_t dW(t)$. Moreover, the connections of the predictable processes between risk-neutral probability measure and forward measure are*

$$\tilde{U}_t = \frac{\mu_t\sigma_t S_t - m_t b_t}{S_t(\sigma_t - b_t)}, \quad \tilde{U}_t^X = \frac{\mu_t^X\sigma_t S_t - m_t^X b_t}{S_t(\sigma_t - b_t)}.$$

Proof. From the proof of Lemma 3, it follows that

$$\begin{aligned}dZ_t =&B(t,T)^{-1}\left[\mu_t S_t\sigma_{F_S}dW_{F_S}^{P^T} + (\mu_t S_t - m_t)b_t dW_B^{P^T}\right]\\ =&B(t,T)^{-1}\left[\mu_t S_t(\sigma_t - b_t)dW_t^T + (\mu_t S_t - m_t)b_t dW_t^T\right]\\ =&B(t,T)^{-1}(\mu_t S_t\sigma_t - m_t b_t)dW_t^T.\end{aligned}$$

Combine the above results with (10), the result follows easily. □

Theorem 2. *If the $\mathscr{F}$-market is complete and arbitrage free, the $\mathscr{G}$-market is arbitrage free, and $F_t^T = 1 - G_t^T = Q^T(\tau \le t|\mathscr{F}_t)$ is continuous, $F_X(t,T)$ is the forward price of contingent claims $X_T 1_{\{\tau>T\}}$ at time t, that is, $F_X(t,T) = \mathbb{E}_{Q^T}[1_{\{\tau>T\}}X_T|\mathscr{G}_t]$. When $\tau > t$, there exist self-financing hedging strategy $(\Phi)_t = (\Phi_1(t), \Phi_2(t), \Phi_3(t))$ such that*

$$\Phi_1(t)B(t,T) + \Phi_2(t)S_t + \Phi_3(t)\rho(t,T) = X_t,$$

where

$$\Phi_1(t)B(t,T) = -e^{\Gamma_t}\left(\tilde{U}_t^X - \frac{Z_t^X}{Z_t}\tilde{U}_t\right)S_t,$$

$$\Phi_2(t) = e^{\Gamma_t}\left(\tilde{U}_t^X - \frac{Z_t^X}{Z_t}\tilde{U}_t\right), \qquad \Phi_3(t) = \frac{Z_t^X}{Z_t}.$$

Proof. By use of the results of Lemma 2, we have

$$\begin{aligned}
F_X(t,T) =& Z_0^X + \int_0^{t\wedge\tau} e^{\Gamma_u} dZ_u^X - \int_0^t F_X(u-,T)dM_u^T \\
=& Z_0^X + \int_0^{t\wedge\tau} e^{\Gamma_u}\tilde{U}_u^X dF_S(u,T) - \int_0^t 1_{\{\tau>u\}}e^{\Gamma_{u-}}Z_{u-}^X dM_u^T \\
=& Z_0^X + \int_0^{t\wedge\tau} e^{\Gamma_u}\tilde{U}_u^X dF_S(u,T) + \int_0^t 1_{\{\tau>u\}}e^{\Gamma_{u-}}Z_{u-}^X \frac{dF_\rho(u,T) - L_{u-}^T dZ_u}{F_\rho(u-,T)} \\
=& Z_0^X + \int_0^{t\wedge\tau} e^{\Gamma_u}\tilde{U}_u^X dF_S(u,T) + \int_0^{t\wedge\tau} \frac{Z_{u-}^X}{Z_u} dF_\rho(u,T) - \int_0^{t\wedge\tau} e^{\Gamma_u}\frac{Z_{u-}^X}{Z_u}U_u dF_S(u,T) \\
=& Z_0^X + \int_0^{t\wedge\tau} e^{\Gamma_u}\left(\tilde{U}_u^X - \frac{Z_u^X}{Z_u}\tilde{U}_u\right)dF_S(u,T) + \int_0^{t\wedge\tau}\frac{Z_u^X}{Z_u}dF_\rho(u,T),
\end{aligned}$$

Therefore

$$\Phi_2(t) = e^{\Gamma_t}\left(\tilde{U}_t^X - \frac{Z_t^X}{Z_t}\tilde{U}_t\right), \qquad \Phi_3(t) = \frac{Z_t^X}{Z_t}.$$

and $X_t = \rho(t,T)\frac{Z_t^X}{Z_t} = \Phi_3(t)\rho(t,T)$, $\Phi_1(t)B(t,T) + \Phi_2(t)S_t = 0$, so

$$\Phi_1(t)B(t,T) = -e^{\Gamma_t}\left(\tilde{U}_t^X - \frac{Z_t^X}{Z_t}\tilde{U}_t\right)S_t.$$

□

3. Conclusions

In conclusion, when the interest rate is stochastic, under the invariance martingale property and reduced-form model for default risk, the hedging strategy is based on riskless asset, risky asset, defaultable zero-coupon bond and the defaultfree zero-coupon bond, which is different from the result in Blanchet-Scalliet and Jeanblane [4] under risk neutral measure. The amount one has to invest in the defaultable zero-coupon is equal to this contingent claim value, and also one has to use another security named *defaultfree zero-coupon bond* to hedge interest risk which extends the result of Blanchet-Scalliet, Jeanblane [4]. But when $\rho = 1$, the risky asset and the defaultfree zero-coupon bond have the same risk factor, it is possible to hedge the contingent claims using the riskless asset, risky asset, defaultable zero-coupon bond. Moreover, there exists connection of hedging strategies between the risk neutral measure and forward measure. In the future, we hope our results can be used to better understand [1, 13, 14, 9, 11, 15].

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