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Asymptotic Error Analysis for the Heat Radiation Boundary Integral Equation

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Abstract. In this paper, a rigorous convergence and error analysis of the Galerkin boundary element method for the heat radiation integral equation in convex and non-convex enclosure geometries is presented. The convergence of the approximation is shown and qausi-optimal error estimates are presented. Numerical results have shown to be consistent with available theoretical results.

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1. Introduction

The heat radiation integral equation is a mathematical model for the brightness of a collection of one or more surfaces when their reflectivity and emissivity are given. The equation [see 3, 8, for example]

$$q(x) = \varepsilon(x)\,\sigma\, T^4(x) + \rho(x)\int_\Gamma G(x,y)q(y)\,d\Gamma_y,\; x \in \Gamma. \tag{1}$$

with $q(x)$ is the "brightness" or radiosity of x, the first part of equation (1) corresponds to the Stefan-Boltzmann radiation law, with ε is the emissivity coefficient, σ is the Stefan-Boltzmann constant which has the value $5.669996 \times 10^{-8}\frac{W}{m^2K}$. The function $\rho(x)$ gives the reflectivity at $x \in \Gamma$ with $\rho(x) = 1 - \varepsilon(x)$. In deriving this equation the reflectivity is assumed to be independent of the angle of which the reflection takes place, that is, the surface is a Lambartian diffuse reflector. The kernel $G(x,y)$ denotes the view factor between the points x and y on Γ. From the above consideration and for general enclosure geometries, $G(x,y)$ is given through

$$G(x,y) = G^*(x,y)\beta(x,y) = \frac{[n(y)\cdot(y-x)]\cdot[n(x)\cdot(x-y)]}{c_0|x-y|^{d+1}}\beta(x,y) \tag{2}$$

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where $c_0 = 2$ for $d = 2$ and $c_0 = \pi$ for $d = 3$.
For convex enclosure geometries, $\beta(x, y) = 1$. If the enclosure is not convex, then we have to take into account the visibility function

$$\beta(x,y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ can see each other} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The Fredholm integral equation (1) can be expressed as

$$q = g + Kq \tag{4}$$

where $Kq = (1 - \varepsilon)\widetilde{K}q$ and

$$\widetilde{K}q(x) = \int_\Gamma G(x,y)\, q(y)\, d\Gamma_y \text{ for } x \in \Gamma\, , q \in L^\infty(\Gamma) \tag{5}$$

The properties of the integral operator (5) have been thoroughly investigated in [5, 7]. We have shown that in the case of a smooth surface Γ the kernel of the integral equation is weakly singular of type $|x-y|^{-2(1-\delta)}$ with $\delta \in [0,1]$ and hence the kernel is integrable. Furthermore, the mapping $\widetilde{K} : L^p(\Gamma) \to L^p(\Gamma)$ is compact for $1 \le p \le \infty$ with $\|\widetilde{K}\| = 1$ in $L^p(\Gamma)$ and for the spectral radius we get $\rho(\widetilde{K}) = 1$. The application of Banach's fixed point theorem leads to the proof of the existence and uniqueness of the solution $q \in L^p(\Gamma)$ of the radiosity equation (1). In the numerical solution of (1), the Galerkin method has been the predominant form of numerical solution with piecewise constant functions as the approximations [1, 5, 6, 7]. The paper is organized as follows: In section 2, we assume Γ to be a smooth curve and use the boundary element method based on the Galerkin discretization scheme leading to a system of algebraic equations. In section 3, we establish some theoretical error estimates for the Galerkin method. In section 4, some numerical results are presented which confirm the theoretical results.

2. Galerkin Boundary Element Method

For the numerical simulation of the integral equation (1), we use the boundary element method. We consider a Galerkin-Bubnov formulation and choose the basis trial function $\phi_k(t)$ with local support $\Gamma_k \subset \Gamma$. The approximation solution has the general form

$$q_h(t) = \sum_{k=1}^{n} q_k \phi_{k,n}(t). \tag{6}$$

We let

$$\langle u, w \rangle_\Gamma = \int_0^1 u(t) w(t) |\dot{x}(t)| dt \tag{7}$$

Inserting the ansatz function (6) into (4) gives

$$\sum_{k=1}^{n} q_k \langle \phi_{k,n}, \phi_{l,n} \rangle_\Gamma = \langle g, \phi_{l,n} \rangle_\Gamma + \sum_{k=1}^{n} q_k \langle K\phi_{k,n}, \phi_{l,n} \rangle_\Gamma. \tag{8}$$

By introducing the vectors $a = (q_k)_{k=1,2,3,...n}$ and $b = \langle g, \phi_{l,n} \rangle_\Gamma, l = 1, 2, 3, \ldots n$ the matrices $M = (M_{l,k})_{k=1,2,3,...n}$ with

$$M_{l,k} = \langle \phi_{k,n}, \phi_{l,n} \rangle_\Gamma = \int_0^1 \phi_{l,n}(t)\phi_{k,n}(t)|\acute{x}(t)|dt \tag{9}$$

and $S = (S_{l,k})_{k=1,2,3,...n}$ with

$$S_{l,k} = \langle K\phi_{k,n}, \phi_{l,n} \rangle_\Gamma = \int_0^1 \int_0^1 (1-\varepsilon)\phi_{l,n}(t)G(t,\tau)\phi_{k,n}(\tau)|\acute{x}(t)||\acute{x}(\tau)|dt\,d\tau. \tag{10}$$

then (8) can be written as

$$(M_n - S_n)a_n = b_n. \tag{11}$$

The mass matrix M in (11) is symmetric, positive definite, and diagonally dominant. Hence it is invertible. Consequently (11) can always be written in the form

$$(I - M_n^{-1}S_n)a_n = M_n^{-1}b_n. \tag{12}$$

To express the fact that the discrete equation (12) corresponds to the continuous equation (4), we write (12) as

$$q_n = g_n + K_n q_n. \tag{13}$$

where $q_n = a_n, g_n = M_n^{-1}b_n$ and $K_n = M_n^{-1}S_n$. Some solution methods for the discrete equation (13), for example, cg- method with or without preconditioning, direct solvers, multigrid methods have been compared in [5, 7]. In three-dimensional case, the conjugate gradient algorithm with preconditioning has been applied and turned out to be the most efficient method [see 6, for more details]. The ansatz function (6) satisfy the following properties:

Approximation property:
Let $\sigma \le \tau \le d$ and $\sigma < r + \frac{1}{2}$. Then there exists a constant c such that for every $v \in H^\tau(\Gamma)$ there exists a sequence $\chi_h \in S_h^{d,r}$ providing

$$||v - \chi_h||_{H^\sigma(\Gamma)} \le ch^{\tau-\sigma}||v||_{H^\tau(\Gamma)}. \tag{14}$$

Inverse property:
For $\sigma < r + \frac{1}{2}$ there exists a constant M such that for all $\chi_h \in S_h^{d,r}$,

$$||\chi_h||_{H^\tau(\Gamma)} \le Mh^{\sigma-\tau}||\chi_h||_{H^\sigma(\Gamma)} \tag{15}$$

3. The Asymptotic Error Analysis

3.1. Theoretical Error Analysis

In this section we establish error estimates for the Galerkin discretized equation (8). Most of the asymptotic error estimates $\|q - q_h\|_{L^2(\Gamma)}$ are formulated in Sobolev spaces. It holds the following lemma:

Lemma 1. *The integral operator $A = I - K$ is $L^2 - elliptic$. Furthermore A is a positive definite operator which satisfies the Gärding inequality on Γ .*

Proof. From Lemma 3 in [5] it follows that

$$\|Kq\|_{L^2(\Gamma)} \leq (1-\varepsilon)\|q\|_{L^2(\Gamma)}. \tag{16}$$

Moreover, K satisfies the inequality

$$\langle Kq, q\rangle_{L^2(\Gamma)} \leq (1-\varepsilon)\langle q, q\rangle_{L^2(\Gamma)}. \tag{17}$$

From (17) together with the definition $A = I - K$ leads to

$$\varepsilon\langle q, q\rangle_{L^2(\Gamma)} \leq \langle Aq, q\rangle_{L^2(\Gamma)} \leq (2-\varepsilon)\langle q, q\rangle_{L^2(\Gamma)}. \tag{18}$$

Furthermore, A satisfies the Gärding inequality, i.e., for all $q \epsilon L^2\Gamma$ and $\varepsilon \geq 0$ the following holds

$$Re\langle Aq, q\rangle_{L^2(\Gamma)} = Re\int_\Gamma qAq\, d\Gamma_x \geq \varepsilon\|q\|^2_{L^2(\Gamma)}.$$

□

We now let $q \epsilon H_h \subset L^2(\Gamma)$ and then define

$$q(t) = \sum_{i=1}^{n} q^{(i)}\phi_i(t) \tag{19}$$

Substituting (19) into (18) leads to

$$\varepsilon\|\sum_{i=1}^{n} q^{(i)}\phi_i\|^2_{L^2(\Gamma)} \leq \sum_{i,j=1}^{n} q^{(i)}q^{(i)}\langle A\phi, \phi_j\rangle_{L^2(\Gamma)} \leq (2-\varepsilon)\|\sum_{i=1}^{n} q^{(i)}\phi_i\|^2_{L^2(\Gamma)} \tag{20}$$

For the consistency of the Galerkin approximation we require the approximation property

$$\lim_{h\to 0}\inf\|w - q_h\|_{L^2(\Gamma)} = 0. \tag{21}$$

As it is well known for linear problems, the convergence $q \to q_h$ can only be established if the approximation equation (8) are stable, which can be formulated in terms of the Ladyzenkaya-Babuska -Brezzi condition, in short LBB-condition [2]: There exists $\gamma > 0$ such that for all $w_h \in H_h$ and the whole family H_h with $h \to 0$

$$\sup_{0\neq w_h \in H_h} |\langle Aq_h, wh\rangle_{L^2(\Gamma)}|\frac{1}{\|w_h\|_{L^2(\Gamma)}} \geq \gamma\|q_h\|_{L^2(\Gamma)}. \tag{22}$$

It holds the following lemma:

Lemma 2 (Cea's Lemma)**.** *The integral operator $A = I - K$ is a pseudo-differential operator of order zero, and the LBB-condition holds. Then $\forall q \in L^2(\Gamma)$ we have the quasi error estimate*

$$\|q - q_h\|_{L^2(\Gamma)} \le c \inf_{w_h \in H_h} \|q - w_h\|_{L^2(\Gamma)} \tag{23}$$

where the constant c is independent of q and h .

Proof.

(i) Following the tools used in[9, 10], since $Aq_h = 0$ implies with (22) also $q_h = 0$, and since (8) is a system of linear equations, the uniqueness implies solvability.

(ii) Due to the previous arguments, the solution q_h of (8) exists and satisfying

$$\langle Aq_h, w_h \rangle_{L^2(\Gamma)} = \langle Aq, w_h \rangle_{L^2(\Gamma)} \text{ for all } w_h \in H_h. \tag{24}$$

Hence the mapping $q \mapsto q_h = G_h q$, the Galerkin projection G_h exists for every h. Moreover, for every $q \in L^2(\Gamma)$ we find G_h is a projection

$$G_h q_h = q_h \text{ for } q_h \in H_h \text{ i.e., } G_h|_{H_h} = I|_{H_h}. \tag{25}$$

Furthermore, for every $q \in L^2(\Gamma)$ we have with (22)

$$\|G_h q\|_{L^2(\Gamma)} = \|q_h\|_{L^2(\Gamma)} \le \frac{1}{\gamma} |\langle Aq_h, w_h^* \rangle_{L^2(\Gamma)}|$$

with a specific w_h^*, $\|w_h^*\|_{L^2(\Gamma)} = 1$ since on the finite dimensional unit- sphere the supremum (22) becomes maximum. Inserting (24) into (25) and using the continuity of $L^2 - duality$ and of A, we obtain

$$\begin{aligned} \|G_h q\|_{L^2(\Gamma)} &= \|q_h\|_{L^2(\Gamma)} \le \frac{1}{\gamma} |\langle Aq_h, w_h^* \rangle_{L^2(\Gamma)}| \\ &= \frac{1}{\gamma} |\langle Aq_h, w_h^* \rangle_{L^2(\Gamma)}| \le c \|Aq\|_{L^2(\Gamma)} \|w_h^* q\|_{L^2(\Gamma)} \le \widetilde{c} \|q\|_{L^2(\Gamma)} \end{aligned} \tag{26}$$

where the constant $\widetilde{c}$ is independent of q and h. Hence the LBB-condition (22) indeed implies stability. For (23) we use the inequality

$$\begin{aligned} \|q - q_h\|_{L^2(\Gamma)} =& \|q - w_h + G_h w_h - G_h q\|_{L^2(\Gamma)} \\ &\le (1 + \widetilde{c}) \|q - w_h\|_{L^2(\Gamma)} \text{ for every } w_h \in H_h. \end{aligned}$$

Theorem 1. *Let the integral operator A be a strongly elliptic operator of order α, and in addition $\alpha < 2r + 1$. Let $\alpha - d \le \sigma \le \frac{\alpha}{2} \le \tau \le d$ be satisfied. Then it holds for the solution q_h of the Galerkin equation $\langle Aq_h, w_h \rangle_{L^2(\Gamma)} = \langle g, w_h \rangle_{L^2(\Gamma)}$ for all $w_h \in H_h$.*
with $q \in H_h$ the asymptotic error estimate

$$\|q_h - q\|_{H^\sigma(\Gamma} \le c h^{\tau - \sigma} \|q\|_{H^\tau(\Gamma)}. \tag{27}$$

Lemma 3. *Let the ansatz function be piecewise linear. Moreover, $I-K$ is a pseudo-differential operator of order $\alpha = 0$ then follows from (27) the error estimates*

$$\|q_h - q\|_{L^2(\Gamma)} \leq ch^2 \|q\|_{H^2(\Gamma)}. \tag{28}$$

The accuracy of the numerical integration of the Stiffness matrix $S = (S_{ij})_{i,j=1,\dots n}$ must be discussed in relation to the asymptotic error estimation. Then it is very necessary to consider the following Lemma from Strang [4].

Lemma 4 (Strang Lemma)**.** *Suppose that the bilinear form $a_h(.,.)$ is uniformly $H_h - elliptic$. Then there exists a constant c independent of q and h such that*

$$\|q - q_h\|_{L^2(\Gamma)} \leq c \left(\inf_{w_h \in H_h} \left\{ \|q - w_h\|_{L^2(\Gamma)} + \sup_{w_h \in H_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_{L^2(\Gamma)}} \right\} + \sup_{w_h \in H_h} \frac{|g(w_h) - g_h(w_h)|}{\|w_h\|_{L^2(\Gamma)}} \right) \tag{29}$$

where the terms $a(v_h, w_h)$, $g(w_h)$, $g_h(w_h)$ and $a_h(v_h, w_h)$ in (29) are defined as follows:

$$\begin{aligned} a_h(v_h, w_h) &= \langle (I-K)v_h, w_h \rangle_{L^2(\Gamma)} = \langle Av_h, w_h \rangle_{L^2(\Gamma)} \\ g(w_h) &= \langle g, w_h \rangle_{L^2(\Gamma)} \\ g_h(w_h) &= \langle g_h, w_h \rangle_{L^2(\Gamma)} \text{ and} \\ a_h(v_h, w_h) &= \langle a_h, w_h \rangle_{L^2(\Gamma)}. \end{aligned}$$

The approximation $a_h(v_h, w_h)$ has the form

$$\begin{aligned} a_h(v_h, w_h) &= \int_\Gamma (I-K)\, v_h\, w_h \, d\Gamma_x \\ &= \int_\Gamma v_h(x)\, w_h(x)\, d\Gamma_x - \int_\Gamma \int_\Gamma (1 - \varepsilon(x))\, G(x,y)\, v_h(x)\, w_h(y)\, d\Gamma_x\, d\Gamma_y \end{aligned}$$

The coefficients $M_{k,l}$ of the mass matrix M (without the quadrature error) are

$$M_{k,l} = a(\phi_k, \phi_l) = \sum_{k=1}^{n} \left\{ \int_\Gamma \phi_k(x)\, \phi_l(x)\, d\Gamma_x - \int_\Gamma \int_\Gamma (1-\varepsilon(x))\, G(x,y)\, \phi_k(x)\, \phi_l(y)\, d\Gamma_x\, d\Gamma_y \right\}.$$

Replacing the above integration by the Gaussian quadrature, yields the approximation formula:

$$\widetilde{M_{k,l}} = a(\phi_k, \phi_l) = \sum_{k=1}^{m} W_i F_{k,l}(x_i) + \sum_{i=1}^{m} \sum_{j=1}^{m} W_i W_j E_k(x_i, y_j),$$

where $F_{k,l}$ and $E_{k,l}$ are given by $F_{k,l} = \phi_k(x)\, \phi_l(x)$, and

$$E_{k,l}(x,y) = (1 - \varepsilon(x))\, G(x,y) \phi_k(x)\, \phi_l(y).$$

Here m denotes the order of quadrature and the coefficients W_i and W_j are the weights of the quadrature form. The ellipticity of a_h follows directly from [6]. It holds

$$\varepsilon \|q\|^2_{L^2(\Gamma)} \leq \langle Aq, q\rangle_{L^2(\Gamma)} \leq (2-\varepsilon)\|q\|^2_{L^2(\Gamma)} \tag{30}$$

Assume that the approximation operator A_h satisfies the approximation inequality

$$\|(A - A_h)q\|^2_{L^2(\Gamma)} \leq c h_l^\tau \|q\|_{H^\tau(\Gamma)} \tag{31}$$

with τ defined as in (14). Further, let q_h be the assigned ansatz function, then follows

$$\varepsilon \|q_h\|^2_{L^2(\Gamma)} \leq \langle Aq_h, q_h\rangle_{L^2(\Gamma)} + c\, h_l^\tau \|q_h\|_{H^\tau(\Gamma)} \cdot \|q_h\|_{L^2(\Gamma)}. \tag{32}$$

In virtue of the inverse inequality (15) we get

$$\varepsilon \|q_h\|^2_{L^2(\Gamma)} \leq \langle Aq_h, q_h\rangle_{L^2(\Gamma)} + c_1^* \left(\frac{h_l}{h_{l-1}}\right)^\tau \|q_h\|^2_{L^2(\Gamma)}. \tag{33}$$

Finally we obtain

$$\left(\varepsilon - c_1^* \left(\frac{h_l}{h_{l-1}}\right)^\tau\right) \|q_h\|^2_{L^2(\Gamma)} \leq \langle A_h q_h, q_h\rangle_{L^2(\Gamma)}. \tag{34}$$

Under the assumption $c_2^* \leq \left(\frac{h_l}{h_{l-1}}\right) \leq c_3^*$ one obtains for the case $\tau = 1$

$$\langle A_h q_h, q_h\rangle_{L^2(\Gamma)} \geq \frac{1}{2}\varepsilon \|q_h\|^2_{L^2(\Gamma)}. \tag{35}$$

Hence ellipticity is proved. This shows how exact the numerical quadrature error must be.

4. Numerical Examples for the Error Estimation

In his section, we present numerical results confirming the theoretical results established for the boundary element Galerkin method and exhibit expected rates of convergence in the $L_2 - norm$.

4.1. Convex boundary

Let Γ describes the boundary of a unit square and suppose that

$$q(t) = \begin{cases} 4t & \text{for } t \in [0, 0.25) \\ 1 & \text{for } t \in [0.25, 0.5) \\ 3 - 4t & \text{for } t \in [0.5, 0.75) \\ 0 & \text{for } t \in [0.75, 1) \end{cases} \tag{36}$$

is a given exact solution of the radiosity integral equation

$$q(t) = g(t) + (1-\varepsilon)\int_0^1 G^*(t,\tau)q(\tau)|x'(\tau)|d\tau. \tag{37}$$

Then the exact $g(t)$ for the given exact $q(t)$ can be calculated as follows:
For $t \geq 0$ and $t < 0.25$ we have

$$\begin{aligned} g_1(t) &= 4t - 4(1-\varepsilon(t))\left\{\int_0^{0.25} G^*_{11}(t,\tau).4\tau d\tau + \int_{0.25}^{0.5} G^*_{12}(t,\tau).1d\tau \right.\\ &+ \left.\int_{0.5}^{0.75} G^*_{13}(t,\tau).(3-4\tau)d\tau + \int_{0.75}^{1} G^*_{14}(t,\tau).0\tau d\tau\right\} \end{aligned} \tag{38}$$

where $G^*_{11} = 0, G^*_{12} = G^*_{21} = \frac{(1-4t)(4\tau-1)}{2[(1-4t)^2+(4\tau-1)^2]^{2/3}}, G^*_{13} = G^*_{31} = \frac{1}{2[16(t-\frac{3}{4}+\tau)^2+1]^{2/3}}$.
For $t \geq 0.25$ and $t < 0.5$ we have

$$\begin{aligned} g_2(t) &= 1.0 - 4(1-\varepsilon(t))\left\{\int_0^{0.25} G^*_{21}(t,\tau).4\tau d\tau + \int_{0.25}^{0.5} G^*_{22}(t,\tau).1d\tau \right.\\ &+ \left.\int_{0.5}^{0.75} G^*_{23}(t,\tau).(3-4\tau)d\tau + \int_{0.75}^{1} G^*_{24}(t,\tau).0\tau d\tau\right\} \end{aligned} \tag{39}$$

where $G^*_{22} = 0, G^*_{23} = G^*_{32} = \frac{(1-2t)(2\tau-1)}{2[(2t-1)^2+(2\tau-1)^2]^{2/3}}, G^*_{24} = G^*_{42} = \frac{1}{2[16(t-\frac{5}{4}+\tau)^2+1]^{2/3}}$.
For $t \geq 0.5$ and $t < 0.75$ we have

$$\begin{aligned} g_3(t) &= (3-4t) - 4(1-\varepsilon(t))\left\{\int_0^{0.25} G^*_{31}(t,\tau).4\tau d\tau + \int_{0.25}^{0.5} G^*_{32}(t,\tau).1d\tau \right.\\ &+ \left.\int_{0.5}^{0.75} G^*_{33}(t,\tau).(3-4\tau)d\tau + \int_{0.75}^{1} G^*_{34}(t,\tau).0\tau d\tau\right\} \end{aligned} \tag{40}$$

with $G^*_{33} = 0$ and $G^*_{34} = G^*_{43} = \frac{(3-4t)(4\tau-3)}{2[(3-4t)^2+(4\tau-3)^2]^{2/3}}$.
For $t \geq 0.75$ and $t < 1.0$ holds

$$\begin{aligned} g_4(t) &= -4(1-\varepsilon(t))\left\{\int_0^{0.25} G^*_{41}(t,\tau).4\tau d\tau + \int_{0.25}^{0.5} G^*_{42}(t,\tau).1d\tau \right.\\ &+ \left.\int_{0.5}^{0.75} G^*_{43}(t,\tau).(3-4\tau)d\tau + \int_{0.75}^{1} G^*_{44}(t,\tau).0\tau d\tau\right\} \end{aligned} \tag{41}$$

with $G^*_{44} = 0$.
The exact $g(t)$ in (37), (38), (39), and (40) have been explicitly calculated. The estimated order of convergence was computed as $eoc = \log_2\left(\frac{e_G(h_l)}{e_G(h_{l+1})}\right)$, where l refers to the refinement level, and $h_l = 2h_{l+1}$. Table 1 contains the numerical results for the Galerkin scheme in the case of convex boundary. Clearly the results support the notion that the asymptotic order of convergence is quadratic with respect to the mesh size h of the boundary discretization.

Table 1: Errors and rate of convergence for the Galerkin method using the $L_2 - norm$

l	n_l	$L_2 - error$	eoc
2	4	4.457×10^{-2}	
3	8	9.483×10^{-3}	2.23
4	16	2.207×10^{-3}	2.10
5	32	5.490×10^{-4}	2.01
6	64	1.369×10^{-4}	2.01
7	128	3.414×10^{-5}	2.00
8	256	5.537×10^{-6}	2.00
9	512	2.139×10^{-6}	2.00

4.2. Non-Convex Boundary

Let

$$q(t) = 1 + \begin{cases} t^2(t-\frac{1}{4})^2 & \text{for } t \in [0, 0.25) \\ (t-\frac{1}{4})^2(t-\frac{1}{2})^2 & \text{for } t \in [0.25, 0.5) \\ (t-\frac{1}{2})^2(t-\frac{3}{4})^2 & \text{for } t \in [0.5, 0.75) \\ (t-\frac{3}{4})^2(t-1)^2 & \text{for } t \in [0.75, 1) \end{cases} \tag{42}$$

be the exact solution of the radiosity integral equation

$$q(t) = g(t) + (1-\varepsilon)\int_0^1 G(t,\tau)q(\tau)|x\acute{(}\tau)|d\tau.$$

with $G(t,\tau) = G^*(t,\tau)\beta(t,\tau)$ for the non-convex geometry shown in Figure 1. Then the exact

Figure 1: Non-Convex boundary

$g(t)$ can be calculated explicitly as follows :
For $t \geq 0$ and $t < 0.25$ we have

$$g_1(t) = q_1(t) - (1-\varepsilon(t))\left(\int_0^{0.25} G_1(t,\tau).q_1(\tau).4\pi d\tau\right) \tag{43}$$

where $q_1(t) = 1 + t^2(t - \frac{1}{4})^2$ and $G_1(t, \tau) = \frac{1}{4}|sin2\pi(t - \tau)|$.
For $t \geq 0.25$ and $t < 0.5$ we obtain

$$g_2(t) = q_2(t) - (1 - \varepsilon(t)) \left(\int_{0.25}^{0.5} G_2(t, \tau).q_2(\tau).4\pi d\tau \right) \tag{44}$$

where $q_2(t) = 1 + (t - \frac{1}{4})^2(t - \frac{1}{2})^2$ and $G_2(t, \tau) = \frac{1}{4}|sin2\pi(t - \tau)|$.
For $t \geq 0.5$ and $t < 0.75$ we have

$$g_3(t) = q_3(t) - (1 - \varepsilon(t)) \left(\int_{0.5}^{0.75} G_3(t, \tau).q_3(\tau).4\pi d\tau \right) \tag{45}$$

where $q_3(t) = 1 + (t - \frac{1}{2})^2(t - \frac{3}{4})^2$ and $G_3(t, \tau) = \frac{1}{4}|sin2\pi(t - \tau)|$.
For $t \geq 0.75$ and $t < 1.0$ we get

$$g_4(t) = q_4(t) - (1 - \varepsilon(t)) \left(\int_{1}^{0.75} G_4(t, \tau).q_4(\tau).12\pi d\tau \right) \tag{46}$$

where $q_4(t) = 1 + (t - \frac{3}{4})^2(t - 1)^2$ and $G_4(t, \tau) = \frac{1}{12}|sin2\pi(t - \tau)|$.
Finally, Table 2 contains the numerical results for the Galerkin scheme in the case of a non-convex boundary. Again we see clearly that the asymptotic order of convergence is quadratic in the $L_2 - norm$ which is in a good agreement with the theoretical estimates.

Table 2: Errors and rate of convergence for the Galerkin method using the $L_2 - norm$

l	n_l	$L_2 - error$	eoc
2	4	1.2345×10^{-1}	
3	8	2.6834×10^{-2}	2.16
4	16	6.3138×10^{-3}	2.09
5	32	1.5437×10^{-3}	2.03
6	64	3.8210×10^{-4}	2.01
7	128	9.5049×10^{-5}	2.01
8	256	2.3760×10^{-5}	2.00
9	512	5.9400×10^{-6}	2.00

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