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Integral Type Operators on Some Classes of Holomorphic Functions with Complex Order

Sunil K. Sharma[1], Kuldip Raj[2], and Ajay K. Sharma[2,*]

[1] *Department of Mathematics, Model Institute of Engineering and Technology, Kot Bhalwal-181122, J& K, India.*
[2] *School of Mathematics, Shri Mata Vaishno Devi University, Kakryal, Katra-182320, J& K, India*

Abstract. In this paper, we obtain some sufficient conditions for integral type operators on some classes of holomorphic functions with complex order.

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1. Introduction and Preliminaries

Let $\mathscr{A}$ be the class of all functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are holomorphic in the open unit disk $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function f in $\mathscr{A}$ is said to be starlike function of complex order $\gamma(\gamma \in \mathbb{C} \setminus \{0\})$ and type α, $(0 \leq \alpha < 1)$ if and only if

$$Re\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \alpha \quad (z \in \mathscr{U}). \tag{1}$$

We denote by $S^*_\alpha(\gamma)$ the class of all such functions. Also a function f in $\mathscr{A}$ is said to be convex of complex order $\gamma(\gamma \in \mathbb{C} \setminus \{0\})$ and type α, $(0 \leq \alpha < 1)$ if and only if

$$Re\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathscr{U}). \tag{2}$$

*Corresponding author.

Email addresses: sunilksharma42@yahoo.co.in (S. Sharma) kuldeepraj68@hotmail.com (K. Raj) aksju_76@yahoo.com (A. Sharma)

We denote by $C_\alpha(\gamma)$ the class of all such functions. The classes $S_0^*(\gamma)$ and $C_0(\gamma)$ were introduced by Nasr and Aouf and Wiatrowski respectively, in [8, 15]. While the classes $S_\alpha^*(\gamma)$ and $C_\alpha(\gamma)$ were defined and studied by Frasin in [6]. Note that the classes $S_\alpha^*(\gamma)$ and $C_\alpha(\gamma)$ are generalization of the classes λ-spiral like functions of order α which was introduced by Libera [7]. The class $S_\alpha^*(\cos\lambda e^{-i\lambda})$ ($|\lambda| < \frac{\pi}{2}$, $0 \le \alpha < 1$) of λ-Roberton functions of order α was introduced by Chichra in [4]. In this paper we introduced integral type operators:

For $i \in \{1, 2, \cdots, n\}$, let $\alpha_i \in \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers and $f, \varphi \in \mathscr{A}$ such that $\varphi(\mathscr{U}) \subset \mathscr{U}$. We define integral type operators as follows:

$$F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z) = \big((f_1 \circ \varphi)(z)\big)^{\alpha_1}\big((f_2 \circ \varphi)(z)\big)^{\alpha_2} \cdots \big((f_n \circ \varphi)(z)\big)^{\alpha_n}, \tag{3}$$

$$G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z) = \Big(\frac{(f_1 \circ \varphi)(z)}{z}\Big)^{\alpha_1}\Big(\frac{(f_2 \circ \varphi)(z)}{z}\Big)^{\alpha_2} \cdots \Big(\frac{(f_n \circ \varphi)(z)}{z}\Big)^{\alpha_n}, \tag{4}$$

$$H_{n,\varphi,g,\alpha_1,\alpha_2,\cdots,\alpha_n}(z) = \int_0^z \big((f_1 \circ \varphi)(\zeta)\big)^{\alpha_1} \cdots \big((f_n \circ \varphi)(\zeta)\big)^{\alpha_n} g'(\zeta)d\zeta \tag{5}$$

and

$$I_{n,\varphi,g,\alpha_1,\alpha_2,\cdots,\alpha_n}(z) = \int_0^z \big((f_1 \circ \varphi)'(\zeta)\big)^{\alpha_1} \cdots \big((f_n \circ \varphi)'(\zeta)\big)^{\alpha_n}(\zeta)g(\zeta). \tag{6}$$

The integral type operators $F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}$, $G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}$, $H_{n,\varphi,g,\alpha_1,\alpha_2,\cdots,\alpha_n}$ and $I_{n,\varphi,g,\alpha_1,\alpha_2,\cdots,\alpha_n}$ are generalizations of some well known operators, defined respectively, as

$$F_{1,\varphi,\alpha}(z) = \big((f_1 \circ \varphi)(z)\big)^{\alpha}, \qquad G_{1,\varphi,\alpha}(z) = \Big(\frac{(f_1 \circ \varphi)(z)}{z}\Big)^{\alpha}.$$

$$H_{1,\varphi,g,\alpha}(z) = \int_0^z \big((f_1 \circ \varphi)(\zeta)\big)^{\alpha} g'(\zeta)d\zeta, \qquad I_{1,\varphi,g,\alpha}(z) = \int_0^z \big((f_1 \circ \varphi)'(\zeta)\big)^{\alpha} g(\zeta)d\zeta.$$

The operator $F_\varphi = F_{1,\varphi,1}$ is known as composition operator defined as

$$F_\varphi(z) = f_1 \circ \varphi, \quad f_1 \in \mathscr{A}.$$

The operator $H_{1,\varphi,g,1}$ induced by g and φ defined as

$$H_{1,\varphi,g,1}(z) = \int_0^z f_1(\varphi(\zeta))dg(\zeta) = \int_0^z f_1(\varphi(\zeta))g'(\zeta)d\zeta = \int_0^1 f_1(\varphi(tz))\, z\, g'(tz)dt$$

can be viewed as a generalization of the Riemann-Stieltjes operator T_g induced by g, defined by

$$T_g f(z) = \int_0^z f(\zeta)dg(\zeta) = \int_0^1 f(tz)zg'(tz)dt, \quad z \in \mathbb{D}.$$

C. Pommerenke [9] initiated the study of Riemann-Stieltjes operator on the Hardy space H^2, where he showed that T_g is bounded on H^2 if and only if g is in $BMOA$. This was extended to other Hardy spaces H^p, $1 \le p < \infty$, in [1] and [2], where compactness of T_g on H^p and

Schatten class membership of T_g on H^2 was also completely characterized in terms of the symbol g. The operator $I_{1,\varphi,g,1}$ induced by g and φ, defined as

$$I_{1,\varphi,g,1}(z) = \int_0^z f_1'(\varphi(\zeta))\varphi'(\zeta)g(\zeta)d\zeta, \quad z \in \mathbb{D}.$$

The operator $I_{1,\varphi,g,1}$ is the generalization of the operator J_g, recently defined by Yoneda in [18] as

$$J_g f(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta, \quad z \in \mathbb{D}.$$

Also, if $g(z) = \varphi'(z)$, then $I_{1,\varphi,g,1}$ reduces to difference of the composition operator F_φ and the point evaluation map, defined as

$$I_{1,\varphi,\varphi',1} f_1(z) = (f_1 \circ \varphi)(z) - f_1(\varphi(0)).$$

These operators have gained increasing attention during the last three decades, mainly due to the fact that they provide ways and means to link classical function theory to functional analysis and operator theory. For general background on composition operators, we refer to [5] and [11] and the references therein. Recently, several authors have studied Riemann-Stieltjes type operators on different spaces of holomorphic functions. For example, one can refer to [3, 10, 12, 13, 14, 16, 17, 18] and the related references therein for the study of these operators on different spaces of holomorphic functions. In this paper we obtain some sufficient conditions for the integral type operators $F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)$, $G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)$, $H_{n,\varphi,g,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)$ and $I_{n,\varphi,g,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)$ to be in the classes $S^*_\alpha(\gamma)$ and $C_\alpha(\gamma)$.

2. Main Results

Theorem 1. *Let $\alpha \in \mathbb{R}^+$ and $\gamma \in \mathbb{C} \setminus \{0\}$ such that $0 \leq \alpha Re\left\{\frac{1}{\gamma} - 1\right\} < 1$. Suppose that $f \circ \varphi$ and $f' \circ \varphi \in S^*_0(\gamma)$ and $\varphi \in C_0(\gamma)$. Then $F_{1,\varphi,\alpha} \in C_\eta(\gamma)$, where $\eta = \alpha Re\left\{\frac{1}{\gamma} - 1\right\}$.*

Proof. We have

$$F'_{1,\varphi,\alpha}(z) = \alpha\big(f(\varphi(z))\big)^{\alpha-1} f'(\varphi(z))\varphi'(z) \tag{7}$$

and

$$\begin{aligned} F''_{1,\varphi,\alpha}(z) \quad &= \quad \alpha(\alpha-1)\big(f(\varphi(z))\big)^{\alpha-2}\big(f'(\varphi(z)\varphi'(z)\big)^2 \\ &+ \quad \alpha\big(f(\varphi(z))\big)^{\alpha-1}\big(f''(\varphi(z))(\varphi'(z))^2 + f'(\varphi(z))\varphi''(z)\big). \end{aligned} \tag{8}$$

Dividing (8) by (7), we have

$$\frac{F''_{1,\varphi,\alpha}(z)}{F'_{1,\varphi,\alpha}(z)} = (\alpha-1)\frac{f'(\varphi(z))\varphi'(z)}{f(\varphi(z))} + \frac{f''(\varphi(z))\varphi'(z)}{f'(\varphi(z))} + \frac{\varphi''(z)}{\varphi'(z)}. \tag{9}$$

Multiplying (9) by z, we have

$$\frac{zF''_{1,\varphi,\alpha}(z)}{F'_{1,\varphi,\alpha}(z)} = (\alpha-1)\frac{z(f\circ\varphi)'(z)}{f(\varphi(z))} + \frac{z(f'\circ\varphi)'(z)}{f'(\varphi(z))} + \frac{z\varphi''(z)}{\varphi'(z)}. \tag{10}$$

Multiplying (10) with $1/\gamma$, we have

$$\begin{aligned}\frac{1}{\gamma}\frac{zF''_{1,\varphi,\alpha}(z)}{F'_{1,\varphi,\alpha}(z)} &= (\alpha-1)\frac{1}{\gamma}\frac{z(f\circ\varphi)'(z)}{f(\varphi(z))} + \frac{1}{\gamma}\frac{z(f'\circ\varphi)'(z)}{f'(\varphi(z))} + \frac{1}{\gamma}\frac{z\varphi''(z)}{\varphi'(z)}\\ &= (\alpha-1)\frac{1}{\gamma}\Big(\frac{z(f\circ\varphi)'(z)}{f(\varphi(z))}-1\Big) + \frac{1}{\gamma}\Big(\frac{z(f'\circ\varphi)'(z)}{f'(\varphi(z))}-1\Big)\\ &+ \frac{1}{\gamma}\frac{z\varphi''(z)}{\varphi'(z)} + \frac{\alpha}{\gamma}.\end{aligned}$$

The above relation is equivalent to

$$\begin{aligned}1+\frac{1}{\gamma}\frac{zF''_{1,\varphi,\alpha}(z)}{F'_{1,\varphi,\alpha}(z)} &= (\alpha-1)\Big\{1+\frac{1}{\gamma}\Big(\frac{z(f\circ\varphi)'(z)}{f(\varphi(z))}-1\Big)\Big\} + \alpha\Big(\frac{1}{\gamma}-1\Big)\\ &+ \Big\{1+\frac{1}{\gamma}\Big(\frac{z(f'\circ\varphi)'(z)}{f'(\varphi(z))}-1\Big)\Big\} + \Big\{1+\frac{1}{\gamma}\frac{z\varphi''(z)}{\varphi'(z)}\Big\}.\end{aligned} \tag{11}$$

Equating real parts of (11), we obtain

$$\begin{aligned}Re\Big\{1+\frac{1}{\gamma}\frac{zF''_{1,\varphi,\alpha}(z)}{F'_{1,\varphi,\alpha}(z)}\Big\} &= (\alpha-1)Re\Big\{1+\frac{1}{\gamma}\Big(\frac{z(f\circ\varphi)'(z)}{f(\varphi(z))}-1\Big)\Big\}\\ &+ Re\Big\{1+\frac{1}{\gamma}\Big(\frac{z(f'\circ\varphi)'(z)}{f'(\varphi(z))}-1\Big)\Big\}\\ &+ Re\Big\{1+\frac{1}{\gamma}\frac{z\varphi''(z)}{\varphi'(z)}\Big\} + \alpha Re\Big\{\frac{1}{\gamma}-1\Big\}.\end{aligned}$$

Since $f\circ\varphi$ and $f'\circ\varphi \in S_0^*(\gamma)$ and $\varphi \in C_0(\gamma)$, we have $F_{1,\varphi,\alpha} \in C_\eta(\gamma)$, where $\eta = \alpha Re\Big\{\frac{1}{\gamma}-1\Big\}$.

Corollary 1. *Let* $\gamma \in \mathbb{C}\setminus\{0\}$ *such that* $0 \le Re\Big\{\frac{1}{\gamma}-1\Big\} < 1$. *Suppose that* $f'\circ\varphi \in S_0^*(\gamma)$ *and* $\varphi \in C_0(\gamma)$. *Then* $F_\varphi \in C_\eta(\gamma)$, *where* $\eta = Re\Big\{\frac{1}{\gamma}-1\Big\}$.

Proof. Putting $\alpha = 1$ in Theorem 1, we obtain the result.

Corollary 2. *Let* $\alpha \in \mathbb{R}^+$ *and* $\gamma \in \mathbb{C}\setminus\{0\}$ *such that* $0 \le \alpha Re\Big\{\frac{1}{\gamma}-1\Big\} < 1$. *Suppose that* $f\circ\varphi$ *and* $f'\circ\varphi \in S_0^*(\cos\lambda e^{-i\lambda}) = S_0^\lambda(|\lambda| < \frac{\pi}{2})$ *and* $\varphi \in C_0(\cos\lambda e^{-i\lambda}) = C_0^\lambda(|\lambda| < \frac{\pi}{2})$. *Then* $F_{1,\varphi,\alpha} \in C_\eta(\cos\lambda e^{-i\lambda})$, *where* $\eta = \alpha Re\Big\{\frac{1}{\gamma}-1\Big\}$.

Theorem 2. *Let* $\alpha_1, \alpha_2, \cdots \alpha_n \in \mathbb{R}^+$ *and* $\gamma \in \mathbb{C} \setminus \{0\}$ *such that* $0 \le 1 - Re\left\{\frac{1}{\gamma}\right\} - \sum_{i=1}^{n} \alpha_i < 1$. *Suppose that* $f_i \circ \varphi \in S_0^*(\gamma)$*. Then* $G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n} \in S_\eta^*(\gamma)$*, where* $\eta = 1 - Re\left\{\frac{1}{\gamma}\right\} - \sum_{i=1}^{n} \alpha_i$.

Proof. From (4), we have

$$G'_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z) = \sum_{i=1}^{n} \alpha_i \Big(\frac{z f_i'(\varphi(z))\varphi'(z) - f_i(\varphi(z))}{z f_i(\varphi(z))} \Big) G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z).$$

Thus we have

$$\begin{aligned} \frac{G'_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)}{G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)} &= \sum_{i=1}^{n} \alpha_i \Big(\frac{z f_i'(\varphi(z))\varphi'(z) - f_i(\varphi(z))}{z f_i(\varphi(z))} \Big) \\ &= \sum_{i=1}^{n} \alpha_i \Big(\frac{f_i'(\varphi(z))\varphi'(z)}{f_i(\varphi(z))} - \frac{1}{z} \Big). \end{aligned}$$

Multiplying the above relation by z, we have

$$\frac{z G'_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)}{G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)} - 1 = -1 + \sum_{i=1}^{n} \alpha_i \Big(\frac{z f_i'(\varphi(z))\varphi'(z)}{f_i(\varphi(z))} - 1 \Big). \tag{12}$$

Multiplying the relation in (12) with $1/\gamma$, we obtained

$$\begin{aligned} 1 + \frac{1}{\gamma} \Big(\frac{z G'_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)}{G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)} - 1 \Big) = 1 - \frac{1}{\gamma} - \sum_{i=1}^{n} \alpha_i \\ + \sum_{i=1}^{n} \alpha_i \Big(1 + \frac{1}{\gamma} \Big(\frac{z f_i'(\varphi(z))\varphi'(z)}{f_i(\varphi(z))} - 1 \Big) \Big). \end{aligned} \tag{13}$$

Equating real parts of (13), we have

$$\begin{aligned} Re\Big\{ 1 + \frac{1}{\gamma} \Big(\frac{z G'_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)}{G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)} - 1 \Big) \Big\} = 1 - Re\Big\{\frac{1}{\gamma}\Big\} - \sum_{i=1}^{n} \alpha_i \\ + \sum_{i=1}^{n} \alpha_i \Big(1 + \frac{1}{\gamma} \Big(\frac{z f_i'(\varphi(z))\varphi'(z)}{f_i(\varphi(z))} - 1 \Big) \Big). \end{aligned}$$

Since $0 \le 1 - Re\left\{\frac{1}{\gamma}\right\} - \sum_{i=1}^{n} \alpha_i < 1$ and $f_i \circ \varphi \in S_0^*(\gamma)$, we have $G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n} \in S_\eta^*(\gamma)$, where $\eta = 1 - Re\left\{\frac{1}{\gamma}\right\} - \sum_{i=1}^{n} \alpha_i$.

Corollary 3. *Let* $\gamma \in \mathbb{C} \setminus \{0\}$ *such that* $0 \le 1 - Re\left\{\frac{1}{\gamma}\right\} - \alpha < 1$*. Suppose that* $f \circ \varphi \in S_0^*(\gamma)$*. Then* $G_{1,\varphi,\alpha} \in S_\eta^*(\gamma)$*, where* $\eta = 1 - Re\left\{\frac{1}{\gamma}\right\} - \alpha$.

Corollary 4. *Let* $\alpha_1, \alpha_2, \cdots \alpha_n \in \mathbb{R}^+$ *and* $\gamma \in \mathbb{C} \setminus \{0\}$ *such that* $0 \leq 1 - Re\left\{\frac{1}{\gamma}\right\} - \sum_{i=1}^{n} \alpha_i < 1$. *Suppose that* $f_i \circ \varphi \in S_0^*(\cos\lambda e^{-i\lambda}) = S_0^{\lambda}(|\lambda| < \frac{\pi}{2})$. *Then* $G_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n} \in S_\eta^*(\cos\lambda e^{-i\lambda}) = S_0^{\lambda}(|\lambda| < \frac{\pi}{2})$, *where* $\eta = 1 - Re\left\{\frac{1}{\gamma}\right\} - \sum_{i=1}^{n} \alpha_i$.

Theorem 3. *Let* $\alpha_1, \alpha_2, \cdots \alpha_n \in \mathbb{R}^+$ *and* $\gamma \in \mathbb{C} \setminus \{0\}$ *such that* $0 \leq Re\left\{\left(\frac{1}{\gamma} - 1\right)\sum_{i=1}^{n} \alpha_i\right\} < 1$. *Suppose that* $f_i \circ \varphi \in S_0^*(\gamma)$. *Then* $F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n} \in S_\eta^*(\gamma)$, *where* $\eta = Re\left\{\left(\frac{1}{\gamma} - 1\right)\sum_{i=1}^{n} \alpha_i\right\}$.

Proof. From (3), we have

$$F'_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n} = \sum_{i=1}^{n} \alpha_i \frac{f_i'(\varphi(z))\varphi'(z)}{(f_i \circ \varphi)(z)} F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z). \tag{14}$$

Thus we have

$$\frac{F'_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)}{F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \frac{f_i'(\varphi(z))\varphi'(z)}{(f_i \circ \varphi)(z)}.$$

Multiplying the above relation by z, we have

$$\frac{zF'_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)}{F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)} - 1 = \sum_{i=1}^{n} \alpha_i \Big(\frac{zf_i'(\varphi(z))\varphi'(z)}{(f_i \circ \varphi)(z)} - 1\Big) + \sum_{i=1}^{n} \alpha_i. \tag{15}$$

Multiplying the relation in (15) with $1/\gamma$, we obtained

$$\frac{1}{\gamma}\Big(\frac{zF'_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)}{F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)} - 1\Big) = \sum_{i=1}^{n} \alpha_i \frac{1}{\gamma}\Big(\frac{zf_i'(\varphi(z))\varphi'(z)}{(f_i \circ \varphi)(z)} - 1\Big) + \frac{1}{\gamma}\sum_{i=1}^{n} \alpha_i. \tag{16}$$

Equating the real parts of (16), we have

$$\begin{aligned} Re\Big\{1 + \frac{1}{\gamma}\Big(\frac{zF'_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)}{F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n}(z)} - 1\Big)\Big\} &= Re\Big\{\Big(\frac{1}{\gamma} - 1\Big)\sum_{i=1}^{n} \alpha_i\Big\} \\ &\quad + \sum_{i=1}^{n} \alpha_i \Big\{1 + \frac{1}{\gamma}\Big(\frac{zf_i'(\varphi(z))\varphi'(z)}{(f_i \circ \varphi)(z)} - 1\Big)\Big\}. \end{aligned} \tag{17}$$

Since $0 \leq Re\left\{\left(\frac{1}{\gamma} - 1\right)\sum_{i=1}^{n} \alpha_i\right\} < 1$ and $f_i \circ \varphi \in S_0^*(\gamma)$, we have $F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n} \in S_\eta^*(\gamma)$, where $\eta = Re\left\{\left(\frac{1}{\gamma} - 1\right)\sum_{i=1}^{n} \alpha_i\right\}$.

Corollary 5. *Let* $\gamma \in \mathbb{C} \setminus \{0\}$ *such that* $0 \leq Re\left\{\left(\frac{1}{\gamma} - 1\right)\alpha\right\} < 1$. *Suppose that* $f \circ \varphi \in S_0^*(\gamma)$. *Then* $F_{1,\varphi,\alpha} \in S_\eta^*(\gamma)$, *where* $\eta = \alpha Re\left\{\frac{1}{\gamma} - 1\right\}$.

Corollary 6. *Let $\alpha_1, \alpha_2, \cdots \alpha_n \in \mathbb{R}^+$ and $\gamma \in \mathbb{C} \setminus \{0\}$ such that $0 \leq Re\Big\{\Big(\frac{1}{\gamma} - 1\Big) \sum_{i=1}^{n} \alpha_i\Big\} < 1$. Suppose that $f_i \circ \varphi \in S_0^*(\cos\lambda e^{-i\lambda}) = S_0^{\lambda}(|\lambda| < \frac{\pi}{2})$. Then $F_{n,\varphi,\alpha_1,\alpha_2,\cdots,\alpha_n} \in S_{\eta}^*(\cos\lambda e^{-i\lambda}) = S_0^{\lambda}(|\lambda| < \frac{\pi}{2})$, where $\eta = Re\Big\{\Big(\frac{1}{\gamma} - 1\Big) \sum_{i=1}^{n} \alpha_i\Big\}$.*

Theorem 4. *Let $\alpha_1, \alpha_2, \cdots \alpha_n \in \mathbb{R}^+$ and $\gamma \in \mathbb{C} \setminus \{0\}$ such that $0 \leq \alpha Re\Big\{\frac{1}{\gamma} - 1\Big\} < 1$. Suppose that $f_i \circ \varphi \in S_0^*(\gamma)$ and $g \in C_0(\gamma)$. Then $H_{n,\varphi,g,\alpha_1,\cdots,\alpha_n} \in C_{\eta}(\gamma)$, where $\eta = Re\Big\{\frac{1}{\gamma}\big(\sum_{i=1}^{n} \alpha_i - 1\big)\Big\}$.*

Proof. We have

$$H'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z) = (f_1 \circ \varphi)^{\alpha_1}(z), \cdots, (f_n \circ \varphi)^{\alpha_n}(z) g'(z).$$

Thus

$$\begin{aligned} H''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z) = \sum_{i=1}^{n} \frac{\alpha_i (f_i \circ \varphi)'(z)}{(f_i \circ \varphi)(z)} H'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z) \\ + (f_i \circ \varphi)^{\alpha}(z) \cdots (f_n \circ \varphi)^{\alpha}(z) g''(z). \end{aligned}$$

Therefore,

$$\frac{H''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}{H'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \frac{(f_i \circ \varphi)'(z)}{(f_i \circ \varphi)(z)} + \frac{g''(z)}{g'(z)}.$$

Multiplying both side by z, we have

$$\frac{zH''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}{H'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \Big(\frac{z(f_i \circ \varphi)'(z)}{(f_i \circ \varphi)(z)} - 1\Big) + \sum_{i=1}^{n} \alpha_i + \frac{zg''(z)}{g'(z)}. \tag{18}$$

Multiplying both side of (18) by $1/\gamma$, we have

$$\frac{1}{\gamma} \frac{zH''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}{H'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \frac{1}{\gamma} \Big(\frac{z(f_i \circ \varphi)'(z)}{(f_i \circ \varphi)(z)} - 1\Big) + \frac{1}{\gamma} \sum_{i=1}^{n} \alpha_i + \frac{1}{\gamma} \frac{zg''(z)}{g'(z)}. \tag{19}$$

The above relation is equivalent to

$$\begin{aligned} \Big\{1 + \frac{1}{\gamma} \frac{zH''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}{H'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}\Big\} = \sum_{i=1}^{n} \alpha_i \Big\{1 + \frac{1}{\gamma} \Big(\frac{z(f_i \circ \varphi)'(z)}{(f_i \circ \varphi)(z)} - 1\Big)\Big\} \\ + \Big\{1 + \frac{1}{\gamma} \frac{zg''(z)}{g'(z)}\Big\} + \Big\{\Big(\frac{1}{\gamma} - 1\Big) \sum_{i=1}^{n} \alpha_i\Big\}. \end{aligned} \tag{20}$$

Equating real parts of (20), we obtain

$$Re\Big\{1 + \frac{1}{\gamma} \frac{zH''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}{H'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}\Big\} = \sum_{i=1}^{n} \alpha_i Re\Big\{1 + \frac{1}{\gamma} \Big(\frac{z(f_i \circ \varphi)'(z)}{(f_i \circ \varphi)(z)} - 1\Big)\Big\} \tag{21}$$

$$+Re\Big\{1+\frac{1}{\gamma}\frac{zg''(z)}{g'(z)}\Big\}+Re\Big\{\Big(\frac{1}{\gamma}-1\Big)\sum_{i=1}^{n}\alpha_i\Big\}.$$

Since $f_i\circ\varphi\in S_0^*(\gamma)$ and $g\in C_0(\gamma)$, we have $H_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}\in C_\eta(\gamma)$, where $\eta=Re\Big\{\Big(\frac{1}{\gamma}-1\Big)\sum_{i=1}^{n}\alpha_i\Big\}$.

Corollary 7. *Let $\gamma\in\mathbb{C}\setminus\{0\}$ such that $0\le\alpha Re\Big\{\frac{1}{\gamma}-1\Big\}<1$. Suppose that $f\circ\varphi\in S_0^*(\gamma)$ and $g\in C_0(\gamma)$. Then $H_{1,\varphi,g,\alpha}\in C_\eta(\gamma)$, where $\eta=\alpha Re\Big\{\frac{1}{\gamma}(-1)\Big\}$.*

Corollary 8. *Let $\alpha_1,\alpha_2,\cdots\alpha_n\in\mathbb{R}^+$ and $\gamma\in\mathbb{C}\setminus\{0\}$ such that $0\le\alpha Re\Big\{\frac{1}{\gamma}-1\Big\}<1$. Suppose that $f_i\circ\varphi\in S_0^*(\cos\lambda e^{-i\lambda})=S_0^\lambda(|\lambda|<\frac{\pi}{2})$ and $g\in C_0(\cos\lambda e^{-i\lambda})=C_0^\lambda(|\lambda|<\frac{\pi}{2})$. Then $H_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}\in C_\eta(\cos\lambda e^{-i\lambda})$, where $\eta=Re\Big\{\frac{1}{\gamma}\big(\sum_{i=1}^{n}\alpha_i-1\big)\Big\}$.*

Theorem 5. *Let $\alpha_1,\alpha_2,\cdots\alpha_n\in\mathbb{R}^+$ and $\gamma\in\mathbb{C}\setminus\{0\}$ such that $0\le Re\Big\{\frac{1}{\gamma}-\sum_{i=1}^{n}\alpha_i\Big\}<1$. Suppose that $f_i\circ\varphi\in C_0(\gamma)$ and $g\in S_0^*(\gamma)$. Then $I_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}\in C_\eta(\gamma)$, where $\eta=Re\Big\{\Big(\frac{1}{\gamma}-1\Big)\sum_{i=1}^{n}\alpha_i\Big\}$.*

Proof. We have

$$I'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)=\big((f_1\circ\varphi)'(z)\big)^{\alpha_1}\cdots\big((f_n\circ\varphi)(z)\big)^{\alpha_n}g(z).$$

$$I''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)=\sum_{i=1}^{n}\frac{\alpha_i(f_i(\varphi(z)))''}{(f_i'\circ\varphi)}I'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)\tag{22}$$
$$+\big((f_1\circ\varphi)'(z)\big)^{\alpha}\cdots\big((f_n\circ\varphi)'(z)\big)^{\alpha}g'(z)$$

$$\frac{I''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}{I'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}=\sum_{i=1}^{n}\alpha_i\frac{(f\circ\varphi)''(z)}{(f_i\circ\varphi)'(z)}+\frac{g'(z)}{g(z)}.$$

Multiplying both side by z, we have

$$\frac{zI''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}{I'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}=\sum_{i=1}^{n}\alpha_i\frac{z(f\circ\varphi)''(z)}{(f_i\circ\varphi)'(z)}+\Big(\frac{zg'(z)}{g(z)}-1\Big)+1.\tag{23}$$

Multiplying both side by $1/\gamma$, we have

$$\frac{1}{\gamma}\frac{zI''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}{I'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}=\sum_{i=1}^{n}\alpha_i\frac{1}{\gamma}\frac{z(f\circ\varphi)''(z)}{(f_i\circ\varphi)'(z)}+\frac{1}{\gamma}\Big(\frac{zg'(z)}{g(z)}-1\Big)+\frac{1}{\gamma}.\tag{24}$$

The above relation is equivalent to

$$1+\frac{1}{\gamma}\frac{zI''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}{I'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}=\sum_{i=1}^{n}\alpha_i\Big(1+\frac{z}{\gamma}\frac{(f\circ\varphi)''(z)}{(f_i\circ\varphi)'(z)}\Big)\tag{25}$$

$$+1+\frac{1}{\gamma}\Big(\frac{zg'(z)}{g(z)}-1\Big)+\frac{1}{\gamma}-\sum_{i=1}^{n}\alpha_i.$$

Equating real parts of (25), we obtain

$$1+Re\Big(\frac{1}{\gamma}\frac{zI''_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}{I'_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}(z)}\Big)=Re\sum_{i=1}^{n}\alpha_i\Big(1+\frac{z}{\gamma}\frac{(f\circ\varphi)''(z)}{(f_i\circ\varphi)'(z)}\Big) \tag{26}$$

$$+1+\frac{1}{\gamma}\Big(\frac{zg'(z)}{g(z)}-1\Big)+Re\{\frac{1}{\gamma}-\sum_{i=1}^{n}\alpha_i\}.$$

Since $f_i\circ\varphi\in C_0(\gamma)$ and $g\in S_0^*(\gamma)$, we have $I_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}\in C_\eta(\gamma)$, where $\eta=Re\Big\{\frac{1}{\gamma}-\sum_{i=1}^{n}\alpha_i\Big\}$.

Corollary 9. *Let $\gamma\in\mathbb{C}\setminus\{0\}$ such that $0\le Re\Big\{\frac{1}{\gamma}-\alpha\Big\}<1$. Suppose that $f_i\circ\varphi\in C_0(\gamma)$ and $g\in S_0^*(\gamma)$. Then $I_{1,\varphi,g,\alpha}\in C_\eta(\gamma)$, where $\eta=Re\Big\{\frac{1}{\gamma}-\alpha\Big\}\Big\}$.*

Corollary 10. *Let $\alpha_1,\alpha_2,\cdots\alpha_n\in\mathbb{R}^+$ and $\gamma\in\mathbb{C}\setminus\{0\}$ such that $0\le Re\Big\{\frac{1}{\gamma}-\sum_{i=1}^{n}\alpha_i\Big\}<1$. Suppose that $f_i\circ\varphi\in C_0(\cos\lambda e^{-i\lambda})=S_0^\lambda(|\lambda|<\frac{\pi}{2})$ and $g\in S_0^*(\cos\lambda e^{-i\lambda})=S_0^\lambda(|\lambda|<\frac{\pi}{2})$. Then $I_{n,\varphi,g,\alpha_1,\cdots,\alpha_n}\in C_\eta(\cos\lambda e^{-i\lambda})=C_0^\lambda(|\lambda|<\frac{\pi}{2})$, where $\eta=Re\Big\{\frac{1}{\gamma}-\sum_{i=1}^{n}\alpha_i\Big\}$.*

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