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Stability of Evolutionary Functional Homomorphisms

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Abstract. In this article, we consider a strongly continuous evolution family of operators on an arbitrary Banach space X, which can be thought of as a propagator for the differential equation

$$\frac{du}{dt} = A(t)u(t), t \in R \text{ on } X,$$

with generally unbounded operators A(t). We associate a C_0 - functional homomorphism with this evolution family and study its stability properties.

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1. Introduction

In literature, the stability properties concerning the solutions of differential equations have witnessed a significant development. The study of stability of solutions of the non - autonomous abstract Cauchy problem (NCP)

$$\frac{du}{dt} = A(t)u(t), u(s) = x, t \geq s \geq 0,$$

requires a family of operators creating an evolution family of operators on super function spaces $L_p(R_+, X)$ or $C_{00}(R, X)$. The concrete operators like translations, multiplications and weighted composition operators play a significant role. An interaction of these concrete operators and stability properties of the evolution system is presented in this article by associating an evolutionary homomorphism. This homomorphism helps to study the qualitative properties of evolution families which are associated with the NCP.

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2. Preliminaries

Let X be a Banach space and $B(X)$ be the Banach algebra of all bounded linear operators on X. The norms on X and $B(X)$ will be denoted by $\|\cdot\|$.

For $\infty > p \geq 1$, let

$$E_p = L_p(R_+, X) = \{f / f : R_+ \to X, \int_0^\infty \|f(t)\|^p dt < \infty\}$$

with L_p norm and

$$E_\infty = C_{00}(R_+, X) = \{f / f \in C(R_+, X) \text{ and } f \text{ vanishes at 0 and at } \infty\}$$

with supremum norm.

Here $C(R_+, B(X))$ denotes the bounded strongly continuous functions $a : R_+ \to B(X)$. For $a \in C(R_+, B(X))$, set $\|a\| = Sup_{s \in R} \|a(s)\|_{B(X)}$ and let M_a denote the multiplication operator on $L_p(R_+, X)$ defined by $M_a f(s) = a(s)f(s)$, $(f \in L_p(R, X))$. Then the mapping $a \to M_a$ is an isometry from $C(R_+, B(X))$ to $B(L_p(R_+, X))$. So the operator M_a will be denoted simply by a.

If $\phi : R_+ \to R_+$ be a continuous map define C_ϕ on E_p, $\infty \geq p \geq 1$ by $C_\phi f = f \circ \phi$. This map is a linear transformation. It is known that C_ϕ is a bounded operator if and only if $\mu\phi^{-1}(E) \leq b\mu(E)$ for some $b > 0$ and for every measurable set E of R_+ [3]. C_ϕ is called a composition operator on E_p, $\infty \geq p \geq 1$.

An operator of the type $M_a C_\phi$ is known as a weighted composition operator on $L_p(R_+, X)$ and is denoted by aC_ϕ. These operators have been studied on different function spaces [3].

3. Evolutionary Functional Homomorphisms

Definition 1. *Let X be a Banach space. A continuous function $F : R_+ \to B(X)$ is called a functional homomorphism if $F(s+t) = F(s)F(t)$ for $s, t \in R_+$.*

We consider the functional homomorphisms satisfying $F(0) = I$, the identity of $B(X)$. If F is continuous with respect to the strong operator topology on $B(X)$, then it is called a C_0 - functional homomorphism [4]. The infinitesimal generator B of $F : R_+ \to B(X)$ is defined as

$$Bx = \lim_{t \to 0^+} \frac{F(t)x - x}{t},$$

whenever the limit exists. The operator B need not be bounded [2].

Consider the NCP,

$$\frac{du}{dt} = A(t)u(t), u(s) = x, t \geq s \geq 0.$$

We assume that this equation is well - posed. This means that there exists an evolution family $\{U(t,s)\}_{t \geq s \geq 0}$ of bounded linear operators on X for this equation i.e.,
$x(t) = U(t,s)x(s), t \geq s$.

Definition 2. *Let X be a Banach space. A family $\{U(t,s)\}_{t\geq s\geq 0} \subset B(X)$ of operators $U(t,s)$ is called a strongly continuous evolution family if the following conditions are satisfied*

(i) $U(t,s) = U(t,r)U(r,s), t \geq r \geq s \geq 0.$

(ii) $U(t,t)x = x$ *for all* $t \geq 0$ *and* $x \in X$.

(iii) *For each* $x \in X$*, the function* $(t,s) \to U(t,s)x$ *is continuous on the set* $\{(t,s)/t \geq s \geq 0\}$.

Evolution families are also called evolution systems, evolution operators, evolution process, propagators or fundamental solutions in literature. In contrast to semi - groups, it is possible that the mapping $t \to U(t,s)x$ is differentiable only for $x = 0$. This happens if $U(t,s) = \frac{p(t)}{p(s)}$ on $X = C$ and $p : R \to [1,2]$ is continuous and nowhere differentiable. Thus an evolution family need not be generated by an operator family $B(\cdot)$.

Definition 3. *The system $\{U(t,s)\}_{t\geq s\geq 0}$ is exponentially bounded if there exists $w \in R$, $k \geq 1$ such that $\|U(t,s)\| \leq ke^{w(t-s)}$.*

We now associate a C_0 - functional homomorphism with an evolution family.

Theorem 1. *The strongly continuous and exponentially bounded evolution family $\{U(t,s)\}_{t\geq s\geq 0} \subset B(E_p)$, $1 \leq p \leq \infty$ generates a C_0 - functional homomorphism $F_p(t)$ on E_p given by*

$$\begin{aligned}(F_p(t)f)(s) &= U(s,s-t)f(s-t) \text{ for } s \geq t\\ &= 0 \text{ for } s < t\end{aligned}$$

Moreover the infinitesimal generator B_p of F_p is given by the formula $B_p = -\frac{d}{dx} + A$ where M_A is the multiplication operator on E_p induced by A, which is the generator of the C_0 - functional homomorphism $a : R_+ \to B(E_p)$ given by $(a(t))(s) = U(s,s-t)_{s\geq t\geq 0}$.

Proof. Let $s \geq t_1 \geq 0$, $s \geq t_2 \geq 0$ and $s \geq t_1 + t_2 \geq 0$. For $\infty \geq p \geq 1$ we have

$$\begin{aligned}(F_p(t_1+t_2)f)(s) &= U(s,s-t_1-t_2)f(s-t_1-t_2)\\ &= U(s,s-t_1)U(s-t_1,s-t_1-t_2)f(s-t_1-t_2)\\ &= F_p(t_1)U(s,s-t_2)f(s-t_2)\\ &= (F_p(t_1)F_p(t_2)f)(s)\end{aligned}$$

showing that $F_p(t_1+t_2) = F_p(t_1)F_p(t_2)$. Obviously $F_p(0) = I$.

We now prove the strong continuity of $F_p(t)$. For $t \in R$, define $\theta_t(s) = U(s,s-t)_{s\geq t\geq 0}$. Then M_{θ_t} is a multiplication operator on E_p and is denoted by θ_t. Further $a : R_+ \to B(E_p)$ defined by $a(t) = \theta_t$ is a C_0 - functional homomorphism.

Let $\phi_t : R_+ \to R_+$ be a map defined by $\phi_t(s) = s - t$ for $s \geq t$. Then C_{ϕ_t} is an isometry on E_p for every $t \in R$. Further $S : R_+ \to B(E_p)$ defined by $S(t) = C_{\phi_t}$ is also a C_0 - functional homomorphism. Thus the mapping $F_p : R_+ \to B(E_p)$ is given by $F_p(t) = \theta_t C_{\phi_t}$ and is also a C_0 - functional homomorphism.

Let $f \in L_p(R_+, X)$ and $\{t_n\}$ be a sequence in R_+ converging to $t \in R_+$. Then

$$\begin{aligned}
\|F_p(t_n)f - F_p(t)f\| &= \|\theta_{t_n} C_{\phi_{t_n}} f - \theta_t C_{\phi_t} f\| \\
&\leq \|\theta_{t_n} C_{\phi_{t_n}} f - \theta_{t_n} C_{\phi_t} f\| + \|\theta_{t_n} C_{\phi_t} f - \theta_t C_{\phi_t} f\| \\
&\leq \|U(s, s-t_n)\| \|C_{\phi_{t_n}} f - C_{\phi_t} f\| + \|U(s, s-t_n) - U(s, s-t)\| \|C_{\phi_t} f\|
\end{aligned}$$

Using uniform boundedness principle for $\{U(s, s - t_n)\}$ it can be concluded that $F_p(t)$ is strongly continuous. Further

$$\begin{aligned}
(B_p f)(x) &= lim_{t\to 0^+} \frac{F_p(t)f(x) - f(x)}{t} \\
&= lim_{t\to 0^+} \frac{U(x, x-t)f(x-t) - f(x)}{t} \\
&= lim_{t\to 0^+} \frac{U(x, x-t)f(x-t) - U(x, x-t)f(x)}{t} + lim_{t\to 0^+} \frac{U(x, x-t)f(x) - f(x)}{t} \\
&= -\frac{d}{dx} f(x) + A(x)f(x)
\end{aligned}$$

as desired.

4. Stability of Evolutionary Functional Homomorphisms

For an evolutionary C_0 - functional homomorphism $\{F_p(t)/t \geq 0\}$ with generator $B_p : D(B_p) \subseteq X \to X$, we state the following definitions of stability.

Definition 4. *A strongly continuous functional homomorphism $\{F_p(t)/t \geq 0\}$ is called*

(a) *Uniformly exponentially stable if there exists $\epsilon > 0$ such that $lim_{t\to\infty} e^{\epsilon t} \|F_p(t)\| = 0$.*

(b) *Uniformly stable if $lim_{t\to\infty} \|F_p(t)\| = 0$.*

(c) *Strongly stable if $lim_{t\to\infty} \|(F_p(t)f)(s)\| = 0$ for all $f(s) \in X$.*

Clearly (a) and (b) are equivalent. For an exponentially bounded evolution family $\{U(s,t)\}_{t\geq s\geq 0}$, we recall the definitions of stability.

Definition 5. *An exponentially bounded evolution family $\{U(s,t)\}_{t\geq s\geq 0}$ is called*

(a) *exponentially stable if there exists $N > 1, \nu > 0$ such that $\|U(t,s)\| \leq N e^{-\nu(t-s)}$ for $t \geq s \geq 0$.*

(b) *Uniformaly stable if $sup_{t\geq s\geq 0} \|U(t,s)\| < \infty$.*

(c) *Strongly stable if $lim_{t\to\infty} \|U(t,s)x\| = 0$ for all $x \in X, s \geq 0$.*

There is a one - one correspondence between evolutionary C_0 - homomorphism and exponentially bounded evolution families as described in the following theorems.

Theorem 2. *Let $\{U(t,s)\}_{t\geq s\geq 0} \subset B(E_p), 1 \leq p \leq \infty$ be a strongly continuous and exponentially bounded evolution family on X and let $F_p(t)$ be the associated C_0 - functional evolutionary homomorphism on E_p. Then the following are equivalent:*

(i) $\{U(t,s)\}_{t\geq s\geq 0}$ *is exponentially stable.*

(ii) $\{F_p(t)/t \geq 0\}$ *exponentially stable.*

(iii) $\sigma(B_p)$ *is a subset of* $C_- = \{z \in C/Rez < 0\}$.

Proof. We have $\|F_p(t)\| = \|\theta_t C_{\phi_t}\| \leq \|\theta_t\|$ and
$\|\theta_t\| = \|\theta_t C_{\phi_t} C_{\phi_{-t}}\| \leq \|\theta_t C_{\phi_t}\| = \|F_p(t)\|$ showing that $\|F_p(t)\| = \|\theta_t\|$.

Now $\|U(s,s-t)\| \leq ke^{-\omega t}$ is equivalent to $lim_{t\to\infty}\|\theta_t(s)\| = 0$ which is true if and only if $lim_{t\to\infty}\|F_p(t)\| = 0$, establishing the equivalence of (i) and (ii).

(ii) $\Rightarrow$ (iii) is obvious. Further (iii) $\Rightarrow$ (ii) is a consequence of the spectral mapping theorem [1].

Corollary 1. *Let $\{U(t,s)\}_{t\geq s\geq 0}$ be an exponentially bounded evolution family on a Banach space X and let $\{F_p(t)/t \geq 0\}$ be the evolutionary homomorphism associated with $\{U(t,s)\}_{t\geq s\geq 0}$ on E_p, $\infty \geq p \geq 1$. Then the family $\{U(t,s)\}_{t\geq s\geq 0}$ is uniformly stable if and only if $\{F_p(t)/t \geq 0\}$ is uniformly stable for all $\infty \geq p \geq 1$.*

Theorem 3. *Let $\{U(t,s)\}_{t\geq s\geq 0}$ be an exponentially bounded evolution family on a Banach space X and let $\{F_p(t)/t \geq 0\}$ be the evolutionary homomorphism associated with it on E_p, $\infty \geq p \geq 1$. Then $\{U(t,s)\}_{t\geq s\geq 0}$ is strongly stable if and only if $\{F_p(t)/t \geq 0\}$ is strongly stable for all $\infty \geq p \geq 1$.*

Proof. Using Theorem 1 $lim_{t\to\infty}\|F_p(t)f(s)\| = 0$ for all $f(s) \in X$ if and only if $lim_{t\to\infty}\|U(s,s-t)f(s-t)\| = 0$ and hence the proof.

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