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Traveling Wave Solutions for Some Nonlinear Partial Differential Equations by Using Modified $\left(\frac{w}{g}\right)$- Expansion Method

A.R. Shehata[1,*], Safaa Abu-Amra[2]

[1] *Department of Mathematics, Faculty of Science, Minia University, Minia, Egypt*
[2] *Department of Mathematics, Faculty of Science, Omar Al-Mukhtar University, Al-Bayda, Libya*

Abstract. In this paper, we use the modified $(\frac{w}{g})$- expansion method to find the traveling wave solutions for some nonlinear partial differential equations in mathematical physics namely the Zakharov Kuznetsov BBM (ZKBBM) equation and the Boussinesq equation . When w and g are taken some special choices, some families of direct expansion methods are obtained. we further give three forms of expansions methods via the modified $(\frac{g'}{g^n})$-expansion method, modified g' -expansion method and modified $(\frac{w}{g})$- expansion function method when w and g satisfy decoupled differential equations $w' = \mu\, g$, $g' = \lambda\, w$, where μ and λ are arbitrary constants. When the parameters are taken some special values the solitary wave is derived from the traveling waves. This method is reliable, simple, and gives many new exact solutions.

Key Words and Phrases: Modified $(\frac{g'}{g^n})$-expansion method, Modified g' -expansion method, Modified $(\frac{w}{g})$- expansion function method, Traveling Wave solutions; The Zakharov Kuznetsov BBM (ZKBBM) equation; The Boussinesq equation.

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1. Introduction

Nonlinear phenomena can be seen in a broad variety of scientific applications such as plasma of physics, hydrodynamics, fluid mechanics and optics fibers, solid state, acoustics and other discipline. There are many effective methods for finding the analytic approximate solutions and exact solutions of NPDEs among of this methods see [1–23, 25–35, 37, 38, 40–45]. Recently, Gepreel [24] have used the modified $(\frac{w}{g})$- expansion method to obtain the exact solutions for the integral member of nonlinear Kadomtsev- Petviashvili hierarchy equation. In the present paper, we use the modified $(\frac{w}{g})$- expansion method,

*Corresponding author.

Email addresses: `shehata1433@yahoo.com` (A.R. Shehata), `safahasoon@yahoo.com` (Safaa Abu-Amra)

where w and g are arbitrary function to study the traveling wave solutions for the following partial differential equations :-

(i) The Zakharov Kuznetsov BBM (ZKBBM) equation [36]

$$u_t + u_x - 2\,a\,u\,u_x - b\,u_{x\,x\,t} = 0 \tag{1}$$

(ii) The Boussinesq equation [39]

$$u_{t\,t} - u_{x\,x} - \left(u^2\right)_{x\,x} + u_{x\,x\,x\,x} = 0 \tag{2}$$

2. Description of the modified $\left(\frac{w}{g}\right)$- expansion method for NPDEs

In this section, we illustrate the main idea of the modified $\left(\frac{w}{g}\right)$- expansion method which discussed in [24, 41, 44]. For a given nonlinear partial differential equation

$$P(\,u\,, u_t\,, u_x\,, u_{t\,t}\,, u_{x\,x}\,, u_{x\,t}\,, \ldots\,) = 0\,, \tag{3}$$

where $u = u\,(x,t)$ is an unknown function, P is a polynomial in $u = u\,(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 1. We use the travelling wave transformation:

$$u = U\,(\xi) \quad , \xi = x - k\,t\,, \tag{4}$$

where k is a nonzero constant. The transformation (4) permits us to convert Eq.(3) to the following ordinary differential equation (ODE)

$$Q(U\,, U'\,, U''\,, \ldots) = 0\,. \tag{5}$$

Step 2. Suppose the solution of Eq. (5) can be expressed by a polynomial in a finite form of $\left(\frac{w}{g}\right)$ as follows :

$$U\,(\xi) = \sum_{i=0}^{m} a_i\,\left(\frac{w\,(\xi)}{g\,(\xi)}\right)^i \tag{6}$$

where $a_i\,(i = 1,\, 2,\, \ldots,\, m)$ are arbitrary constants to be determined later,and $w\,(\xi)\,,\, g\,(\xi)$ satisfy the following nonlinear first order differential equation:-

$$\left(\frac{w\,(\xi)}{g\,(\xi)}\right)' = a\, + b\,\left(\frac{w\,(\xi)}{g\,(\xi)}\right) + c\,\left(\frac{w\,(\xi)}{g\,(\xi)}\right)^2, \tag{7}$$

Or

$$w'\,g - w\,g' = a\,g^2 + b\,w\,g + c\,w^2 \tag{8}$$

where $a,\, b,\, c$ are arbitrary constants.

Step 3. Determine the positive integer m in the formula (6) by balancing the nonlinear term(s) and the highest order derivative in Eq. (5).

Step 4. Substituting Eq. (6) into (5) along with (7), and collect all terms with the same powers of $\left(\frac{w}{g}\right)^i$, $(i = 0, 1, ..., m)$. Setting each coefficient of $\left(\frac{w}{g}\right)^i$, $(i = 0, 1, ..., m)$ to be zero, we yield a set of algebraic equations for $a_i\,(i = 0, 1, ..., m)\ and\, k$.

Step 5. Solving these over-determined system of algebraic equations with the help of Maple software package to determine $a_i\ (i = 0, 1, ..., m)\ and\, k$.

Step 6. Using the results which obtained in the above steps to derive a series of fundamental solutions of nonlinear partial differential equation (3).

Remark 1. Furthermore, if we put $g = 1\,, b = 0\,, and\, c = 1$. In this case the solution (6) takes the form

$$U(\xi) = \sum_{i=0}^{m} a_i\, w^i\,, \tag{9}$$

where $a_i\ (i = 0, 1, ..., m)$ are arbitrary constants, and w satisfy the following relation

$$w' = a + w^2\,. \tag{10}$$

In this case the $\left(\frac{w}{g}\right)$- expansion method is equivalent to the tanh- function method. Also, when $g = 1$, and a, b, c are nonzero constants the $\left(\frac{w}{g}\right)$-expansion method is equivalent to the Riccati expansion function method [5, 45].

Remark 2. [24, 41, 44] If we choose $w = g'\,, a = -\mu, b = -\lambda, and\, c = -1$. In this case the solution (6) takes the form

$$U(\xi) = \sum_{i=0}^{m} a_i\,\left(\frac{g'}{g}\right)^i \tag{11}$$

where $a_i\ (i = 0, 1, ..., m)$ are arbitrary constants, and g satisfy the following second order differential equation:-

$$g'' + \lambda\, g' + \mu\, g = 0 \tag{12}$$

In this case the $\left(\frac{w}{g}\right)$-expansion method is equivalent to the $\left(\frac{G'}{G}\right)$- expansion method proposed by Wang et al [38] .

Remark 3. If we put $w = \frac{g'}{g^{n-1}}\,, b = 0$. We have a new form of exact solution takes the form

$$U(\xi) = \sum_{i=0}^{m} a_i\,\left(\frac{g'}{g^n}\right)^i, \tag{13}$$

where $a_i\ (i = 0, 1, ..., m)$ are arbitrary constants, and g satisfy the following nonlinear second order differential equation

$$g^n\, g'' - n\, g^{n-1}\,\left(g'\right)^2 = a\, g^{2n} + c\,\left(g'\right)^2\,, \tag{14}$$

which is called $\left(\frac{g'}{g^n}\right)$ - expansion method and when $n = 2\,,3\,,...$,we have the same result proposed in [24, 41, 44].

Remark 4. If we put $w = g\,g'$. We have a new form of exact solution takes the form

$$U\,(\xi) = \sum_{i=0}^{m} a_i\,\left(g'\right)^i \tag{15}$$

where a_i $(i = 0,\,1,\,...,\,m)$ are arbitrary constants, and g satisfy the nonlinear second order differential equation

$$g'' = a\, + b\,g' + c\,\left(g'\right)^2 \tag{16}$$

which is called (g')- expansion method and proposed in [24, 41, 44].

Remark 5. If we have a new form of exact solution takes the form

$$U\,(\xi) = \sum_{i=0}^{m} a_i\,\left(\frac{w}{g}\right)^i\,, \tag{17}$$

where a_i $(i = 0,\,1,\,...,\,m)$ are arbitrary constants, and $w\,and\,g$ satisfy the first order differential equations

$$w' = \lambda\,g\,, g' = \mu\,w\,, \tag{18}$$

where $\lambda\,and\,\mu$ are arbitrary nonzero constants, which is called $\left(\frac{w}{g}\right)$- expansion function method.

3. Applications of the modified $\left(\frac{w}{g}\right)$- expansion method for NPDEs

Here, we use the three Eq.(17) expansions methods namely the $\left(\frac{g'}{g^n}\right)$ -expansion method, (g')-expansion method and $\left(\frac{w}{g}\right)$-expansion method to construct the traveling wave solutions for nonlinear evolution equations in mathematical physics namely, the Zakharov Kuznetsov BBM (ZKBBM) equation and the Boussinesq equation in mathematical physics which are very important in the mathematical science and have been a great attention by many researcher in physics and engineering.

3.1. Traveling wave solutions of the Zakharov Kuznetsov BBM (ZKBBM) equation

Here, we discuss the traveling wave solution of the nonlinear the Zakharov Kuznetsov BBM (ZKBBM) equation (1) where a,b are nonzero constants. We use the transformation

$$u\,(x,t) = u\,(\xi)\ ,\ \xi = x - V\,t\,, \tag{19}$$

where V is a constant to be determined later. Converting Eq.(1) into an ODE for $u\,(x,t)$ by using Eq.(19),we have

$$(1 + V)\ u' - 2\,a\,u\,u' - b\,V\,u''' = 0\,. \tag{20}$$

Integrating Eq.(20) with respect to ξ once, we obtain

$$(1+V)\,u - a\,u^2 - bVu'' + C = 0\,, \tag{21}$$

where C is the integration constant to be determined later.

3.1.1. Modified $\left(\frac{g'}{g^n}\right)$- expansion method for the Zakharov Kuznetsov BBM (ZKBBM) equation

Here, we use the direct method $\left(\frac{g'}{g^n}\right)$-expansion method when $n = 2$ to find the traveling wave solutions of Eq.(21) and by balancing the order of $u''\,and\,u^2$, we get $m = 2$. Let us assume the solution of Eq.(21) has the following form:

$$u\,(\xi) = a_0 + a_1\left(\frac{g'}{g^2}\right) + a_2\left(\frac{g'}{g^2}\right)^2, \tag{22}$$

Noting $\left(\frac{g'}{g^2}\right)' = a + c\left(\frac{g'}{g^2}\right)^2$, we have the concrete form of u', $u''\,and\,u^2$, then substitute them into Eq.(21), collect all terms with same order of $\left(\frac{g'}{g^2}\right)$, and set the coefficients of all powers of $\left(\frac{w}{g}\right)^m$ to zeros. We will get a system of algebraic equations for a_0, a_1, a_2, $C\,and\,V$. and after some algebraic calculations we have

$$C = \frac{1}{4}\,\frac{-V^2 - 2\,V - 1 + 16\,b^2\,V^2\,c^2\,a^2}{a}\,, a_0 = -\frac{1}{2}\,\frac{V - 1 + 8\,b\,V\,c\,a}{a}\,,\ a_1 = 0\,, a_2 = \frac{-6\,b\,V\,c^2}{a} \tag{23}$$

Substituting Eq.(23) and the general solution of Eq.(14) into Eq.(22), then we find the general solution of Eq(1) has the following families:

Family 1. If. $c\,a > 0$,

$$g\,(\xi) = \frac{2\,c}{\ln\left[\frac{c}{a}\,\left(A_1\,sin\,(\sqrt{a\,c}\,\xi) - A_2\,cos\,(\sqrt{ac}\,\xi)\right)^2\right]}\,, \tag{24}$$

and

$$\frac{g'}{g^2} = \sqrt{\frac{a}{c}}\left[\frac{A_1\,cos\,(\sqrt{a\,c}\,\xi) + A_2\,sin\,(\sqrt{a\,c}\,\xi)}{A_1\,sin\,(\sqrt{a\,c}\,\xi) - A_2\,cos\,(\sqrt{ac}\,\xi)}\right]. \tag{25}$$

,then the solution of Eq. (21) takes the form:

$$u\,(\xi) = -\frac{1}{2}\,\frac{V - 1 + 8\,b\,V\,c\,a}{a} + \frac{-6\,b\,V\,c^2}{a}\,\frac{a}{c}\left(\frac{A_1\,cos\,(\sqrt{a\,c}\,\xi) + A_2\,sin\,(\sqrt{a\,c}\,\xi)}{A_1\,sin\,(\sqrt{a\,c}\,\xi) - A_2\,cos\,(\sqrt{a\,c}\,\xi)}\right)^2, \tag{26}$$

where $\xi = x + V\,t$. Consequently, the traveling wave solution for the nonlinear the Zakharov Kuznetsov BBM (ZKBBM) equation (1) take the form:

$$u_1(x,t) = \frac{V + 1 - 8\,b\,V\,c\,a}{2\,a} - 6\,b\,V\,c\left(\frac{A_1\,cos\,(\sqrt{a\,c}\,\,(x + V\,t)) + A_2\,sin\,(\sqrt{a\,c}\,\,(x + V\,t))}{A_1\,sin\,(\sqrt{a\,c}\,\,(x + V\,t)) - A_2\,cos\,(\sqrt{a\,c}\,\,(x + V\,t))}\right)^2 \tag{27}$$

Family 2. If. $c\,a < 0$,

$$g\,(\xi) = -\frac{2\,c}{2\,c\,\sqrt{|a\,c|}\,\xi - \ln\left[\frac{c}{4\,a}\left(A_1\,e^{2\,\sqrt{|a\,c|}\,\xi} - A_2\right)^2\right]} \tag{28}$$

and

$$\frac{g'}{g^2} = \frac{1}{2c}\left[2c\,\sqrt{|a\,c|} - \frac{4\sqrt{|a\,c|}\,A_1\,e^{2\,\sqrt{|a\,c|}\,\xi}}{A_1\,e^{2\,\sqrt{|a\,c|}\,\xi} - A_2}\right] \tag{29}$$

,then the solution of Eq.(21) takes the form:

$$u\,(\xi) = -\frac{1}{2}\,\frac{V - 1 + 8\,b\,V\,c\,a}{a} + \frac{-6\,b\,V\,c^2}{a}\,\frac{1}{4\,c^2}\left(\left[2c\,\sqrt{|a\,c|} - \frac{4\sqrt{|a\,c|}\,A_1\,e^{2\,\sqrt{|a\,c|}\,\xi}}{A_1\,e^{2\,\sqrt{|a\,c|}\,\xi} - A_2}\right]\right)^2 \tag{30}$$

where $\xi = x + V\,t$. Consequently, the traveling wave solution for the nonlinear the Zakharov Kuznetsov BBM (ZKBBM) equation (1) take the form:

$$u_2(x,t) = \frac{V + 1 - 8\,b\,V\,c\,a}{2\,a} - \frac{3\,b\,V\,c^2}{2\,a}\left(\left[2c\,\sqrt{|a\,c|} - \frac{4\sqrt{|a\,c|}\,A_1\,e^{2\,\sqrt{|a\,c|}\,(x+V\,t)}}{A_1\,e^{2\,\sqrt{|a\,c|}\,(x+V\,t)} - A_2}\right]\right)^2 \tag{31}$$

3.1.2. Modified (g')- expansion method for the Zakharov Kuznetsov BBM (ZKBBM) equation

Here, we use the (g')- expansion method to find the traveling wave solutions of Eq.(21) and by balancing the order of $u''\,and\,u^2$, we get $m = 2$. Let us assume the solution of Eq.(21) has the following form:

$$u\,(\xi) = a_0 + a_1\,\left(g'\right) + a_2\,\left(g'\right)^2\,, \tag{32}$$

Similarly, noting $(g')' = a + b\,g' + c\,(g')^2$, one substitutes the new form of $u'\,, u''\,and\,u^2$ into Eq.(21), collect all terms with same order of (g') ,and set the coefficients of all powers of $\left(\frac{w}{g}\right)^m$ to zeros. We will get a system of algebraic equations for $a_0\,, a_1\,, a_2\,, C\,and\,V$. and after some algebraic calculations we have

$$C = \frac{1}{4}\,\frac{-1 - 2\,V - V^2 + b^6\,V^2 - 8\,b^4\,V^2\,c\,a + 16\,b^2\,V^2\,c^2\,a^2}{a}\,,\; a_0 = -\frac{1}{2}\,\frac{-1 - V + b^3\,V + 8\,b\,V\,c\,a}{a},$$
$$a_1 = -\frac{6\,b^2\,V\,c}{a}\,, a_2 = -\frac{6\,b\,V\,c^2}{a}\,, \tag{33}$$

Substituting Eq.(33) and the general solution of Eq.(16) into Eq.(32), then we find the general solution of Eq(1) has the following families:

Family 3. If. $\Delta = 4\,a\,c - b^2 > 0$,

$$g\,(\xi) = \frac{1}{2\,c}\left[\ln\left(1 + tan^2(\frac{1}{2}\sqrt{\Delta}\,\xi)\right) - b\,\xi\right] \tag{34}$$

and

$$g'(\xi) = \frac{1}{2\,c}\left[\sqrt{\Delta}\,tan\left(\frac{1}{2}\sqrt{\Delta}\,\xi\right) - b\right] \tag{35}$$

,then the solution of Eq.(21) takes the form:

$$u(\xi) = -\frac{1}{2}\frac{-1 - V + b^3\,V + 8\,b\,V\,c\,a}{a} - \frac{6\,b^2\,V\,c}{a}\,\frac{1}{2\,c}\left[\sqrt{4\,a\,c - b^2}\,tan\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\xi\right) - b\right]\ldots$$
$$-\frac{6\,b\,V\,c^2}{a}\,\frac{1}{4\,c^2}\left(\left[\sqrt{4\,a\,c - b^2}\,tan\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\xi\right) - b\right]\right)^2, \tag{36}$$

where $\xi = x + V\,t$. Consequently, the traveling wave solution for the nonlinear the Zakharov Kuznetsov BBM (ZKBBM) equation (1) take the form:

$$u_3(x,t) = \frac{1 + V - b^3\,V - 8\,b\,V\,c\,a}{2\,a} - \frac{3\,b^2\,V}{a}\left[\sqrt{4\,a\,c - b^2}\,tan\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,(x + V\,t)\right) - b\right]\ldots$$
$$-\frac{6\,b\,V\,c^2}{a}\,\frac{1}{4\,c^2}\left(\left[\sqrt{4\,a\,c - b^2}\,tan\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,(x + V\,t)\right) - b\right]\right)^2, \tag{37}$$

Family 4. If. $\Delta = 4\,a\,c - b^2 < 0$,

$$g(\xi) = \frac{1}{2\,c}\left[\ln\left(tanh^2\left(\frac{1}{2}\sqrt{-\Delta}\,\xi\right) - 1\right) - b\,\xi\right] \tag{38}$$

and

$$g'(\xi) = \frac{1}{2\,c}\left[\sqrt{-\Delta}\,tanh\left(\frac{1}{2}\sqrt{-\Delta}\,\xi\right) + b\right] \tag{39}$$

,then the solution of Eq.(21) takes the form:

$$u(\xi) = -\frac{1}{2}\frac{-1 - V + b^3\,V + 8\,b\,V\,c\,a}{a} - \frac{6\,b^2\,V\,c}{a}\,\frac{1}{2\,c}\left[\sqrt{-(4\,a\,c - b^2)}\,tanh\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\xi\right) + b\right]\ldots$$
$$-\frac{6\,b\,V\,c^2}{a}\,\frac{1}{4\,c^2}\left(\left[\sqrt{-(4\,a\,c - b^2)}\,tanh\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\xi\right) + b\right]\right)^2, \tag{40}$$

where $\xi = x + V\,t$. Consequently, the traveling wave solution for the nonlinear the Zakharov Kuznetsov BBM (ZKBBM) equation (1) take the form:

$$u_4(x,t) = \frac{1 + V - b^3\,V - 8\,b\,V\,c\,a}{2\,a} - \frac{3\,b^2\,V}{a}\left[\sqrt{-(4\,a\,c - b^2)}\,tanh\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,(x + V\,t)\right) + b\right]\ldots$$
$$-\frac{6\,b\,V\,c^2}{a}\,\frac{1}{4\,c^2}\left(\left[\sqrt{-(4\,a\,c - b^2)}\,tanh\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,(x + V\,t)\right) + b\right]\right)^2, \tag{41}$$

3.1.3. Modified $\left(\frac{w}{g}\right)$- expansion method for the Zakharov Kuznetsov BBM (ZKBBM) equation

Here, we use the direct method $\left(\frac{w}{g}\right)$-expansion method to find the traveling wave solutions of Eq.(21) and by balancing the order of $u''\, and\, u^2$, we get $m = 2$. Let us assume the solution of Eq.(21) has the following form:

$$u(\xi) = a_0 + a_1\left(\frac{w}{g}\right) + a_2\left(\frac{w}{g}\right)^2, \tag{42}$$

we have the concrete form of $u',u''\, and\, u^2$,then substitute them into Eq.(21), collect all terms with same order of $\left(\frac{w}{g}\right)$, and set the coefficients of all powers of $\left(\frac{w}{g}\right)^m$ to zeros. We will get a system of algebraic equations for a_0, a_1, a_2, $C\, and\, V$ and after some algebraic calculations we have

$$C = \frac{1}{4}\,\frac{-V^2 - 2V - 1 + 16\,b^2\,V^2\,\mu^2\,\lambda^2}{a}, a_0 = \frac{1}{2}\frac{V + 1 + 8\,b\,V\,\mu\lambda}{a}, a_1 = 0, a_2 = \frac{-6\,b\,V\,\mu^2}{a}, \tag{43}$$

Substituting Eq.(43) and the general solution of Eq.(18) into Eq.(42), then we find the general solution of Eq(1) has the following families:

Family 5. If. $\lambda > 0$ *and* $\mu > 0$, in this family the solution of the coupled ordinary differential equation (18) have the following:

$$\left(\frac{w}{g}\right) = \frac{\sqrt{\lambda}}{\sqrt{\mu}}\left(\frac{A_1\sqrt{\mu}\,cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{\lambda}\,sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right)}{A_1\sqrt{\mu}\,sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{\lambda}\,cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right)}\right) \tag{44}$$

,then the solution of Eq.(21) takes the form:

$$u(\xi) = \frac{1}{2}\frac{V + 1 + 8\,b\,V\,\mu\lambda}{a} + \frac{-6\,b\,V\,\mu^2}{a}\,\frac{\lambda}{\mu}\left(\frac{A_1\sqrt{\mu}\,cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{\lambda}\,sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right)}{A_1\sqrt{\mu}\,sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{\lambda}\,cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right)}\right)^2, \tag{45}$$

where $\xi = x + V\,t$. Consequently, the solitary wave solution of Eq.(1) takes the following from:

$$\begin{aligned} u_5(x,t) &= \frac{V + 1 + 8\,b\,V\,\mu\,\lambda}{2\,a}\dots \\ &+ \frac{-6\,b\,V\,\mu\,\lambda}{a}\left(\frac{A_1\sqrt{\mu}\,cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,(x + V\,t)\right) + A_2\sqrt{\lambda}\,sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,(x + V\,t)\right)}{A_1\sqrt{\mu}\,sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,(x + V\,t)\right) + A_2\sqrt{\lambda}\,cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,(x + V\,t)\right)}\right)^2, \end{aligned} \tag{46}$$

Family 6. If. $\lambda < 0$ *and* $\mu < 0$, in this family the solution of the coupled system ordinary (18) have the following:

$$\left(\frac{w}{g}\right) = \frac{\sqrt{-\lambda}}{\sqrt{-\mu}} \left(\frac{A_1\sqrt{-\mu}\, cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right) - A_2\sqrt{-\lambda}\, sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right)}{A_1\sqrt{-\mu}\, sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right) + A_2\sqrt{-\lambda}\, cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right)}\right), \tag{47}$$

,then the solution of Eq.(21) takes the form:

$$u\left(\xi\right) = \frac{1}{2}\frac{V+1+8\,b\,V\,\mu\,\lambda}{a} + \frac{-6\,b\,V\,\mu^2}{a}\frac{\lambda}{\mu}\left(\frac{A_1\sqrt{-\mu}\, cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right) - A_2\sqrt{-\lambda}\, sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right)}{A_1\sqrt{-\mu}\, sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right) + A_2\sqrt{-\lambda}\, cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right)}\right)^2, \tag{48}$$

where $\xi = x + V\,t$. Consequently, the solitary wave solution of Eq.(1) takes the following from:

$$u_6(x,t) = \frac{V+1+8\,b\,V\,\mu\,\lambda}{2\,a}\dots$$
$$+\frac{-6\,b\,V\,\mu\,\lambda}{a}\left(\frac{A_1\sqrt{-\mu}\, cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\left(x+V\,t\right)\right) - A_2\sqrt{-\lambda}\, sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\left(x+V\,t\right)\right)}{A_1\sqrt{-\mu}\, sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\left(x+V\,t\right)\right) + A_2\sqrt{-\lambda}\, cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\left(x+V\,t\right)\right)}\right)^2, \tag{49}$$

Family 7. If. $\lambda > 0$ *and* $\mu < 0$, in this family the solution of the coupled system ordinary (18) have the following:

$$\left(\frac{w}{g}\right) = \sqrt{\frac{\lambda}{-\mu}}\left(\frac{A_1\sqrt{-\mu}\, cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right) + A_2\sqrt{\lambda}\, sin\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right)}{-A_1\sqrt{-\mu}\, sin\left(\sqrt{\lambda}\,-\mu\,\xi\right) + A_2\sqrt{\lambda}\, cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right)}\right), \tag{50}$$

,then the solution of Eq.(21) takes the form:

$$u\left(\xi\right) = \frac{1}{2}\frac{V+1+8\,b\,V\,\mu\,\lambda}{a} + \frac{-6\,b\,V\,\mu^2}{a}\frac{\lambda}{-\mu}\left(\frac{A_1\sqrt{-\mu}\, cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right) + A_2\sqrt{\lambda}\, sin\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right)}{-A_1\sqrt{-\mu}\, sin\left(\sqrt{\lambda}\,-\mu\,\xi\right) + A_2\sqrt{\lambda}\, cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right)}\right)^2, \tag{51}$$

where $\xi = x + V\,t$. Consequently, the periodic wave solution of Eq.(1) takes the following from:

$$u_7(x,t) = \frac{V+1+8\,b\,V\,\mu\,\lambda}{2\,a}\dots$$
$$+\frac{6\,b\,V\,\mu\,\lambda}{a}\left(\frac{A_1\sqrt{-\mu}\, cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\left(x+V\,t\right)\right) + A_2\sqrt{\lambda}\, sin\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\left(x+V\,t\right)\right)}{-A_1\sqrt{-\mu}\, sin\left(\sqrt{\lambda}\,-\mu\,\left(x+V\,t\right)\right) + A_2\sqrt{\lambda}\, cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\left(x+V\,t\right)\right)}\right)^2, \tag{52}$$

Family 8. If. $\lambda < 0 \; and \; \mu > 0$, in this family the solution of the coupled system ordinary (18) have the following:

$$\left(\frac{w}{g}\right) = \frac{\sqrt{-\lambda}}{\sqrt{\mu}}\left(\frac{A_1\sqrt{\mu}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right) - A_2\sqrt{-\lambda}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right)}{A_1\sqrt{\mu}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{-\lambda}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right)}\right), \tag{53}$$

,then the solution of Eq.(21) takes the form:

$$u\,(\xi) = \frac{1}{2}\frac{V+1+8\,b\,V\,\mu\,\lambda}{a} + \frac{-6\,b\,V\,\mu^2}{a}\frac{-\lambda}{\mu}\frac{\sqrt{-\lambda}}{\sqrt{\mu}}\left(\frac{A_1\sqrt{\mu}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right) - A_2\sqrt{-\lambda}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right)}{A_1\sqrt{\mu}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{-\lambda}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right)}\right), \tag{54}$$

where $\xi = x + V\,t$. Consequently, the periodic wave solution of Eq.(1) takes the following from:

$$u_8\,(x,t) = \frac{V+1+8\,b\,V\,\mu\,\lambda}{2\,a}\dots$$
$$+\frac{6\,b\,V\,\mu\,\lambda}{a}\frac{\sqrt{-\lambda}}{\sqrt{\mu}}\left(\frac{A_1\sqrt{\mu}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,(x+V\,t)\right) - A_2\sqrt{-\lambda}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,(x+V\,t)\right)}{A_1\sqrt{\mu}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,(x+V\,t)\right) + A_2\sqrt{-\lambda}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,(x+V\,t)\right)}\right), \tag{55}$$

3.1.4. Numerical solutions for the exact solutions for the Zakharov Kuznetsov BBM (ZKBBM) equation

Here, we study the behavior of the traveling wave solutions which are given above subsections, according to some figures. Hence , we take some special values of the parameters to show the behavior of extended rational $\left(\frac{g'}{g^n}\right)$- expansion method, (g')- expansion method and $\left(\frac{w}{g}\right)$- expansion method for the Zakharov Kuznetsov BBM (ZKBBM) equation

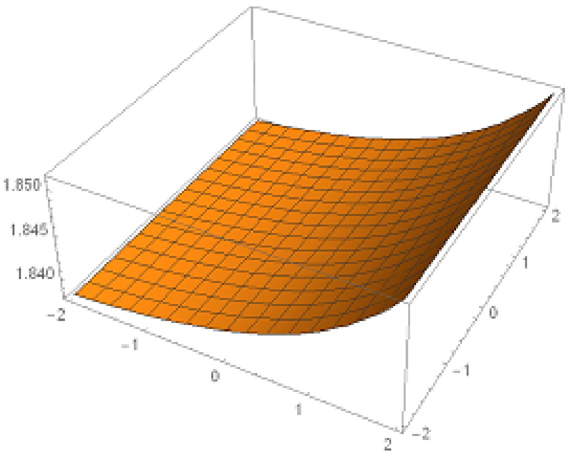

Figure 1: The plot of the solution(27)when $a = 0.3\,, c = 0.1\,, A_1 = 0.3\,, A_2 = 0.4\,, V = 0.1\,, b = -0.1$

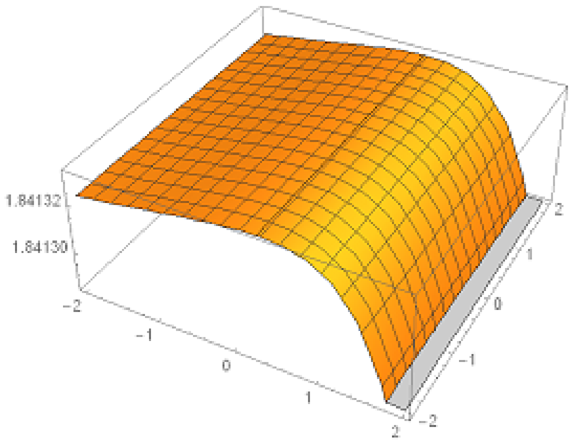

Figure 2: The plot of the solution(31)when $a = 0.3, c = -0.2, A_1 = 0.1, A_2 = 0.7, V = 0.1, b = 0.1$

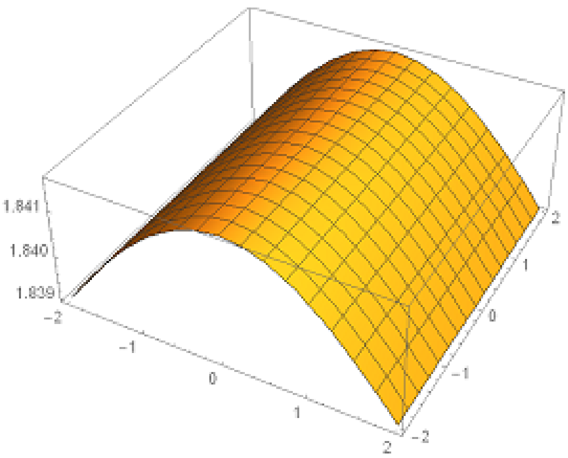

Figure 3: The plot of the solution(37)when $a = 0.3, c = -0.2, V = 0.1, b = 0.$

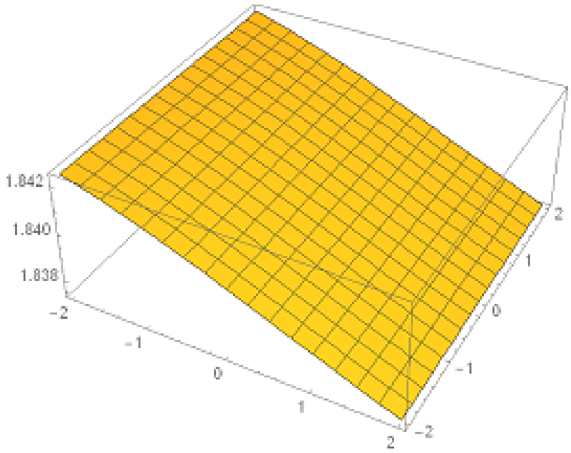

Figure 4: The plot of the solution(41)when $a = 0.3, c = -0.2, V = 0.1, b = 0.1.$

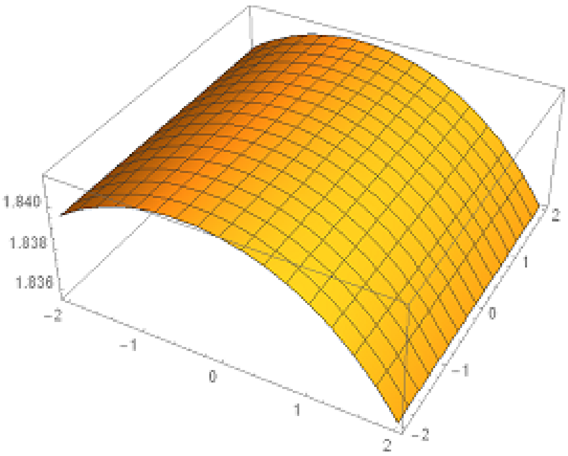

Figure 5: The plot of the solution(46)when $\lambda = 0.3, \mu = -0.2, A_1 = 0.1, A_2 = 0.7, V = 0.1, b = 0.1.$

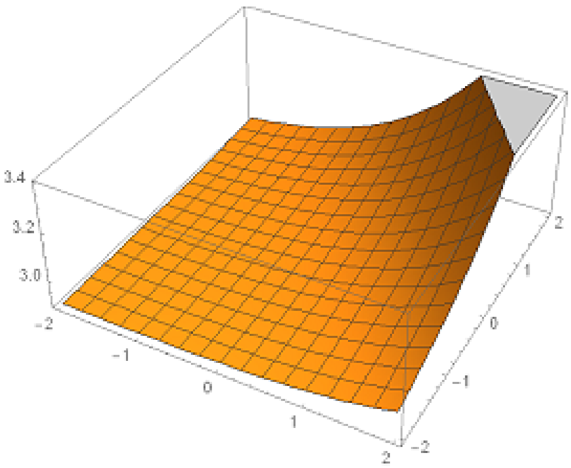

Figure 6: The plot of the solution(49)when $\lambda = -0.3, \mu = -0.2, A_1 = 0.9, A_2 = -0.9, V = 0.7, b = -0.1.$

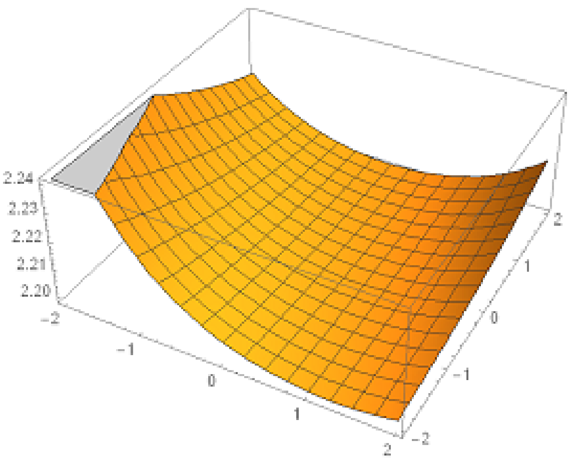

Figure 7: The plot of the solution(52)when $\lambda = 0.4, \mu = -0.2, A_1 = 0.1, A_2 = -0.4, V = 0.3, b = -0.1.$

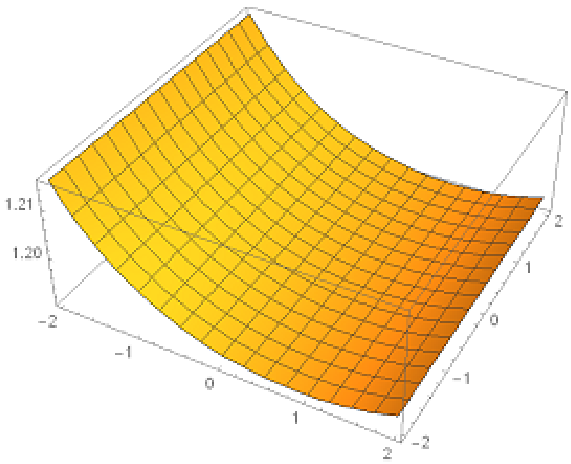

Figure 8: The plot of the solution(55)when $\lambda = -0.3, \mu = 0.2, A_1 = -0.1, A_2 = -0.4, V = -0.3, b = 0.1.$

3.2. Traveling wave solutions of the Boussinesq equation

Here, we discuss the traveling wave solution of the nonlinear the Boussinesq equation (2). We use the transformation

$$u\,(x,t) = u\,(\xi)\ ,\ \xi = x - V\,t\,, \tag{56}$$

where V is a constant to be determined later. Converting Eq.(2) into an ODE for $u(\xi)$ by using Eq.(56),we have

$$\left(V^2 - 1\right)\,u'' - \left(u^2\right)'' + u'''' = 0 \tag{57}$$

Integrating Eq.(57) twice with respect to ξ and taking the constant of integration as zero, we obtain

$$\left(V^2-1\right)\, u - u^2 + u'' = 0\,. \tag{58}$$

3.2.1. Modified $\left(\frac{g'}{g^n}\right)$- expansion method for the Boussinesq equation

Here, we use the direct method $\left(\frac{g'}{g^n}\right)$-expansion method when $n=2$ to find the traveling wave solutions of Eq.(58) and by balancing the order of $u''\, and\, u^2$, we get $m=2$. Let us assume the solution of Eq.(58) has the following form:

$$u\,(\xi) = a_0 + a_1\left(\frac{g'}{g^2}\right) + a_2\left(\frac{g'}{g^2}\right)^2 , \tag{59}$$

Noting $\left(\frac{g'}{g^2}\right)' = a + c\left(\frac{g'}{g^2}\right)^2$, we have the concrete form of u', $u''\, and\, u^2$, then substitute them into Eq.(58), collect all terms with same order of $\left(\frac{g'}{g^2}\right)$, and set the coefficients of all powers of $\left(\frac{w}{g}\right)^m$ to zeros. We will get a system of algebraic equations for a_0, a_1, a_2, $C\, and\, V$. and after some algebraic calculations we have

$$V = \pm\sqrt{1+4\,c\,a}\,,\; a_0 = -2\,c\,a\,,\; a_1 = 0\,, a_2 = -6\,c^2 \tag{60}$$

Substituting Eq.(60) and the general solution of Eq.(14) into Eq.(59), then we find the general solution of Eq(2) has the following families:

Family 1. If. $c\,a > 0$,

$$g\,(\xi) = \frac{2\,c}{\ln\left[\frac{c}{a}\,\left(A_1\, sin\,(\sqrt{a\,c}\,\xi) - A_2\, cos\,(\sqrt{ac}\,\xi)\right)^2\right]}\,, \tag{61}$$

and

$$\frac{g'}{g^2} = \sqrt{\frac{a}{c}}\left[\frac{A_1\, cos\,(\sqrt{a\,c}\,\xi) + A_2\, sin\,(\sqrt{a\,c}\,\xi)}{A_1\, sin\,(\sqrt{a\,c}\,\xi) - A_2\, cos\,(\sqrt{ac}\,\xi)}\right]. \tag{62}$$

,then the solution of Eq.(58) takes the form:

$$u\,(\xi) = -2\,c\,a - 6\,c^2\frac{a}{c}\left(\frac{A_1\, cos\,(\sqrt{a\,c}\,\xi) + A_2\, sin\,(\sqrt{a\,c}\,\xi)}{A_1\, sin\,(\sqrt{a\,c}\,\xi) - A_2\, cos\,(\sqrt{ac}\,\xi)}\right)^2 . \tag{63}$$

where $\xi = x \pm \sqrt{1+4\,c\,a}\,t$. Consequently, the traveling wave solution for the nonlinear the Boussinesq equation (2) take the form:

$$u_1\,(x,t) = -2\,c\,a - 6\,c\,a\left(\frac{A_1\, cos\left(\sqrt{a\,c}\,\left(x\pm\sqrt{1+4\,c\,a}\right)\right) + A_2\, sin\left(\sqrt{a\,c}\,\left(x\pm\sqrt{1+4\,c\,a}\right)\right)}{A_1\, sin\left(\sqrt{a\,c}\,\left(x\pm\sqrt{1+4\,c\,a}\right)\right) - A_2\, cos\left(\sqrt{ac}\,\left(x\pm\sqrt{1+4\,c\,a}\right)\right)}\right)^2 . \tag{64}$$

Family 2. If. $c\,a < 0$,

$$g(\xi) = -\frac{2\,c}{2\,c\,\sqrt{|a\,c|}\,\xi - \ln\left[\frac{c}{4\,a}\left(A_1\,e^{2\,\sqrt{|a\,c|}\,\xi} - A_2\right)^2\right]} \tag{65}$$

and

$$\frac{g'}{g^2} = \frac{1}{2c}\left[2c\,\sqrt{|a\,c|} - \frac{4\sqrt{|a\,c|}\,A_1\,e^{2\,\sqrt{|a\,c|}\,\xi}}{A_1\,e^{2\,\sqrt{|a\,c|}\,\xi} - A_2}\right] \tag{66}$$

,then the solution of Eq.(58) takes the form:

$$u(\xi) = -2\,c\,a - 6\,c^2\frac{1}{4\,c^2}\left(\left[2c\,\sqrt{|a\,c|} - \frac{4\sqrt{|a\,c|}\,A_1\,e^{2\,\sqrt{|a\,c|}\,\xi}}{A_1\,e^{2\,\sqrt{|a\,c|}\,\xi} - A_2}\right]\right)^2, \tag{67}$$

where $\xi = x \pm \sqrt{1 + 4\,c\,a}\,t$.

Consequently, the traveling wave solution for the nonlinear the Boussinesq equation (2) take the form:

$$u_2(x,t) = -2\,c\,a - \frac{3}{2}a\left(2c\sqrt{|a\,c|} - \frac{4\sqrt{|ac|}\,A_1 e^{2\sqrt{|ac|}\left(x\pm\sqrt{1+4act}\right)}}{A_1 e^{2\sqrt{|ac|}\left(x\pm\sqrt{1+4act}\right)} - A_2}\right)^2 \tag{68}$$

3.2.2. Modified (g') -expansion method for the Boussinesq equation

Here, we use the (g')- expansion method to find the traveling wave solutions of Eq.(58) and by balancing the order of $u''\,and\,u^2$, we get $m = 2$. Let us assume the solution of Eq.(58) has the following form:

$$u(\xi) = a_0 + a_1\left(g'\right) + a_2\left(g'\right)^2 , \tag{69}$$

Similarly, noting $(g')' = a + b\,g' + c\,(g')^2$, one substitutes the new form of $u'\,,u''\,and\,u^2$ into Eq.(58), collect all terms with same order of (g') ,and set the coefficients of all powers of $\left(\frac{w}{g}\right)^m$ to zeros. We will get a system of algebraic equations for $a_0\,,a_1\,,a_2\,,C\,and\,V$. and after some algebraic calculations we have

$$V = \pm\sqrt{4ca - b^2 + 1}\,,a_0 = -2\,c\,a - b^2\,,a_1 = -6\,b\,c\,,a_2 = -6c^2\,, \tag{70}$$

Substituting Eq.(70) and the general solution of Eq.(16) into Eq.(69), then we find the general solution of Eq(2) has the following families:

Family 3. If. $\Delta = 4\,a\,c - b^2 > 0$,

$$g(\xi) = \frac{1}{2\,c}\left[\ln\left(1 + tan^2\left(\frac{1}{2}\sqrt{\Delta}\,\xi\right)\right) - b\,\xi\right] \tag{71}$$

and

$$g'(\xi) = \frac{1}{2\,c}\left[\sqrt{\Delta}\,tan\left(\frac{1}{2}\sqrt{\Delta}\,\xi\right) - b\right] \tag{72}$$

,then the solution of Eq.(58) takes the form:

$$u(\xi) = 2\,c\,a - b^2 - 6\,b\,c\frac{1}{2\,c}\left[\sqrt{4\,a\,c - b^2}\,tan\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\xi\right) - b\right]\ldots$$
$$- 6\,c^2\frac{1}{4\,c^2}\left(\left[\sqrt{4\,a\,c - b^2}\,tan\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\xi\right) - b\right]\right)^2, \tag{73}$$

Where $\xi = x \pm \sqrt{4\,c\,a - b^2 + 1}\,t$

Consequently, the traveling wave solution for the nonlinear the Boussinesq equation (2) take the form:

$$u_3(x,t) = -2\,c\,a - b^2 - 3\,b\left[\sqrt{4\,a\,c - b^2}\,tan\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\left(x \pm \sqrt{4\,c\,a - b^2 + 1}\right)\right) - b\right]\ldots$$
$$-\frac{3}{2}\left(\left[\sqrt{4\,a\,c - b^2}\,tan\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\left(x \pm \sqrt{4\,c\,a - b^2 + 1}\right)\right) - b\right]\right)^2, \tag{74}$$

Family 4. If. $\Delta = 4\,a\,c - b^2 < 0$,

$$g(\xi) = \frac{1}{2\,c}\left[\ln\left(tanh^2\left(\frac{1}{2}\sqrt{-\Delta}\,\xi\right) - 1\right) - b\,\xi\right] \tag{75}$$

and

$$g'(\xi) = \frac{1}{2\,c}\left[\sqrt{-\Delta}\,tanh\left(\frac{1}{2}\sqrt{-\Delta}\,\xi\right) + b\right] \tag{76}$$

,then the solution of Eq.(58) takes the form:

$$u(\xi) = -2\,c\,a - b^2 - 6\,b\,c\frac{1}{2c}\left[\sqrt{-(4\,a\,c - b^2)}\,tanh\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\xi\right) + b\right]$$
$$- 6c^2\frac{1}{4\,c^2}\left(\left[\sqrt{-(4\,a\,c - b^2)}\,tanh\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\xi\right) + b\right]\right)^2, \tag{77}$$

Where $\xi = x \pm \sqrt{4\,c\,a - b^2 + 1}\,t$ Consequently, the traveling wave solution for the nonlinear the Boussinesq equation (2) take the form:

$$u_4(x,t) = -2\,c\,a - b^2 - 3\,b\left[\sqrt{-(4\,a\,c - b^2)}\,tanh\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\xi\right) + b\right]$$
$$-\frac{3}{2}\left(\left[\sqrt{-(4\,a\,c - b^2)}\,tanh\left(\frac{1}{2}\sqrt{4\,a\,c - b^2}\,\xi\right) + b\right]\right)^2, \tag{78}$$

3.2.3. Modified $\left(\frac{w}{g}\right)$- expansion method for the Boussinesq equation

Here, we use the direct method $\left(\frac{w}{g}\right)$-expansion method to find the traveling wave solutions of Eq.(58) and by balancing the order of $u''\, and\, u^2$, we get $m = 2$. Let us assume the solution of Eq.(58) has the following form:

$$u(\xi) = a_0 + a_1 \left(\frac{w}{g}\right) + a_2 \left(\frac{w}{g}\right)^2 , \tag{79}$$

we have the concrete form of $u'\,, u''\, and\, u^2$,then substitute them into Eq.(58), collect all terms with same order of $\left(\frac{w}{g}\right)$, and set the coefficients of all powers of $\left(\frac{w}{g}\right)^m$ to zeros. We will get a system of algebraic equations for a_0, a_1, a_2, $C\, and\, V$ and after some algebraic calculations we have

$$V = \pm\sqrt{1 - 4\mu\,\lambda}\,, a_0 = 2\mu\lambda\,, a_1 = 0\,, a_2 = -6\mu^2 \tag{80}$$

Substituting Eq.(80) and the general solution of Eq.(18) into Eq.(79), then we find the general solution of Eq(2) has the following families:

Family 5. If. $\lambda > 0$ *and* $\mu > 0$, in this family the solution of the coupled ordinary differential equation (18) have the following:

$$\left(\frac{w}{g}\right) = \frac{\sqrt{\lambda}}{\sqrt{\mu}} \left(\frac{A_1\sqrt{\mu}\, cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{\lambda}\, sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right)}{A_1\sqrt{\mu}\, sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{\lambda}\, cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right)}\right) \tag{81}$$

,then the solution of Eq.(58) takes the form:

$$u(\xi) = 2\mu\lambda - 6\mu^2\,\frac{\lambda}{\mu} \left(\frac{A_1\sqrt{\mu}\, cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{\lambda}\, sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right)}{A_1\sqrt{\mu}\, sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{\lambda}\, cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\xi\right)}\right)^2 , \tag{82}$$

where $\xi = x \pm \sqrt{1 - 4\,\mu\,\lambda}\,t$.

Consequently, the solitary wave solution of Eq.(2) takes the following from:

$$u_5(x,t) = 2\mu\lambda - 6\mu\lambda \left(\frac{A_1\sqrt{\mu}\, cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\right)\right) + A_2\sqrt{\lambda}\, sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\right)\right)}{A_1\sqrt{\mu}\, sinh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\right)\right) + A_2\sqrt{\lambda}\, cosh\left(\sqrt{\lambda}\,\sqrt{\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\right)\right)}\right)^2 , \tag{83}$$

Family 6. If. $\lambda < 0$ *and* $\mu < 0$, in this family the solution of the coupled system ordinary (18) have the following:

$$\left(\frac{w}{g}\right) = \frac{\sqrt{-\lambda}}{\sqrt{-\mu}} \left(\frac{A_1\sqrt{-\mu}\, cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right) - A_2\sqrt{-\lambda}\, sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right)}{A_1\sqrt{-\mu}\, sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right) + A_2\sqrt{-\lambda}\, cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right)}\right) , \tag{84}$$

,then the solution of Eq.(58) takes the form:

$$u(\xi) = 2\mu\,\lambda - 6\mu^2 \frac{\lambda}{\mu}\left(\frac{A_1\sqrt{-\mu}\,cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right) - A_2\sqrt{-\lambda}\,sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right)}{A_1\sqrt{-\mu}\,sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right) + A_2\sqrt{-\lambda}\,cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\xi\right)}\right)^2, \tag{85}$$

where $\xi = x \pm \sqrt{1-4\,\mu\,\lambda}\,t$. Consequently, the solitary wave solution of Eq.(2) takes the following from:

$$u_6(x,t) = 2\mu\,\lambda \ldots$$

$$-6\mu\,\lambda\left(\frac{A_1\sqrt{-\mu}\,cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\,t\right)\right) - A_2\sqrt{-\lambda}\,sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\,t\right)\right)}{A_1\sqrt{-\mu}\,sinh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\,t\right)\right) + A_2\sqrt{-\lambda}\,cosh\left(\sqrt{-\lambda}\,\sqrt{-\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\,t\right)\right)}\right)^2, \tag{86}$$

Family 7. If. $\lambda > 0$ *and* $\mu < 0$, in this family the solution of the coupled system ordinary (18) have the following:

$$\left(\frac{w}{g}\right) = \sqrt{\frac{\lambda}{-\mu}}\left(\frac{A_1\sqrt{-\mu}\,cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right) + A_2\sqrt{\lambda}\,sin\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right)}{-A_1\sqrt{-\mu}\,sin\left(\sqrt{\lambda}\,-\mu\,\xi\right) + A_2\sqrt{\lambda}\,cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right)}\right), \tag{87}$$

,then the solution of Eq.(58) takes the form:

$$u(\xi) = 2\mu\,\lambda - 6\mu^2 \frac{\lambda}{-\mu}\left(\frac{A_1\sqrt{-\mu}\,cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right) + A_2\sqrt{\lambda}\,sin\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right)}{-A_1\sqrt{-\mu}\,sin\left(\sqrt{\lambda}\,-\mu\,\xi\right) + A_2\sqrt{\lambda}\,cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\xi\right)}\right)^2, \tag{88}$$

where $\xi = x \pm \sqrt{1-4\mu\lambda}\,t$. Consequently, the solitary wave solution of Eq.(2) takes the following from:

$$u_7(x,t) = 2\lambda\mu + 6\mu\lambda\left(\frac{A_1\sqrt{-\mu}\,cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\left(x \pm \sqrt{1-4\mu\lambda}\,t\right)\right) + A_2\sqrt{\lambda}\,sin\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\left(x \pm \sqrt{1-4\mu\lambda}\,t\right)\right)}{-A_1\sqrt{-\mu}\,sin\left(\sqrt{\lambda}\,-\mu\,\left(x \pm \sqrt{1-4\mu\lambda}\,t\right)\right) + A_2\sqrt{\lambda}\,cos\left(\sqrt{\lambda}\,\sqrt{-\mu}\,\left(x \pm \sqrt{1-4\mu\lambda}\,t\right)\right)}\right)^2, \tag{89}$$

Family 8. If. $\lambda < 0$ *and* $\mu > 0$, in this family the solution of the coupled system ordinary (18) have the following:

$$\left(\frac{w}{g}\right) = \sqrt{\frac{-\lambda}{\mu}}\left(\frac{A_1\sqrt{\mu}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right) - A_2\sqrt{-\lambda}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right)}{A_1\sqrt{\mu}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{-\lambda}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right)}\right), \tag{90}$$

,then the solution of Eq.(58) takes the form:

$$u(\xi) = 2\mu\,\lambda - 6\mu^2 \frac{-\lambda}{\mu}\left(\frac{A_1\sqrt{\mu}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right) - A_2\sqrt{-\lambda}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right)}{A_1\sqrt{\mu}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right) + A_2\sqrt{-\lambda}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\xi\right)}\right)^2, \tag{91}$$

where $\xi = x \pm \sqrt{1-4\,\mu\,\lambda}, t$. Consequently, the periodic wave solution of Eq.(2) takes the following from:

$$u_8\,(x,t) = 2\mu\,\lambda + 6\mu\,\lambda \left(\frac{A_1\sqrt{\mu}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\,t\right)\right) - A_2\sqrt{-\lambda}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\,t\right)\right)}{A_1\sqrt{\mu}\,sin\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\,t\right)\right) + A_2\sqrt{-\lambda}\,cos\left(\sqrt{-\lambda}\,\sqrt{\mu}\,\left(x \pm \sqrt{1-4\,\mu\,\lambda}\,t\right)\right)} \right)^2, \tag{92}$$

3.2.4. Numerical solutions for the exact solutions for the Boussinesq equation

Here, we study the behavior of the traveling wave solutions which are given above subsections, according to some figures. Hence , we take some special values of the parameters to show the behavior of extended rational $\left(\frac{g'}{g^2}\right)$ -expansion method, (g') -expansion method and $\left(\frac{w}{g}\right)$-expansion method for the Boussinesq equation

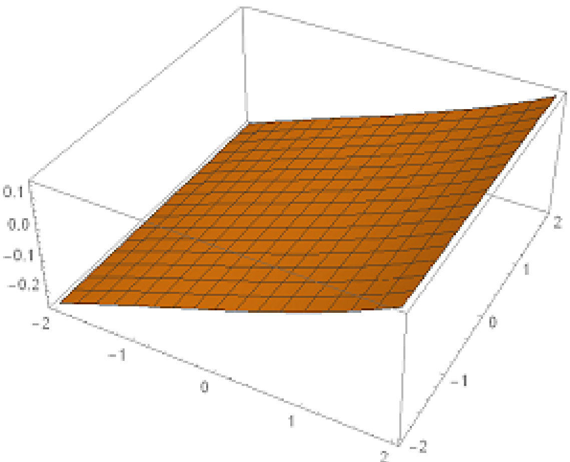

Figure 9: The plot of the solution(64)when $a = 0.3, c = 0.2, A_1 = 0.1, A_2 = 0.7$.

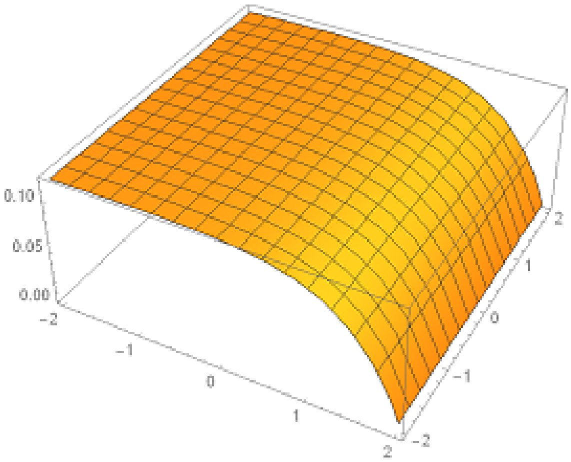

Figure 10: The plot of the solution(68)when $a = 0.2, c = -0.2, A_1 = 0.1, A_2 = 0.7$.

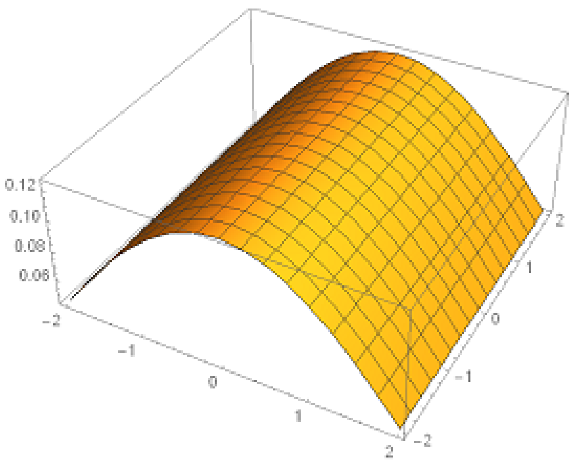

Figure 11: The plot of the solution(74)when $a = 0.3, c = 0.2, A_1 = 0.1, A_2 = 0.7.$

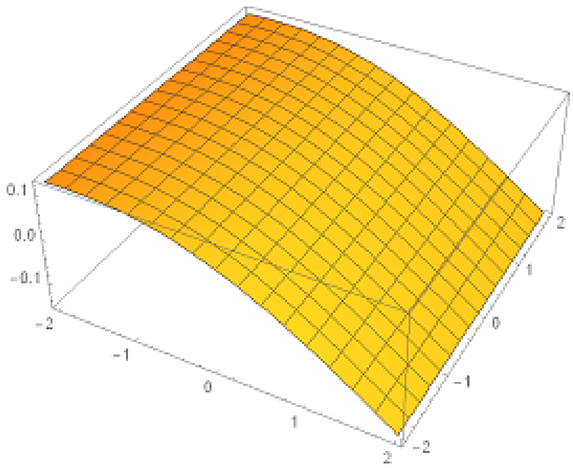

Figure 12: The plot of the solution(78)when $a = 0.2, c = -0.2, A_1 = 0.1, A_2 = 0.7.$

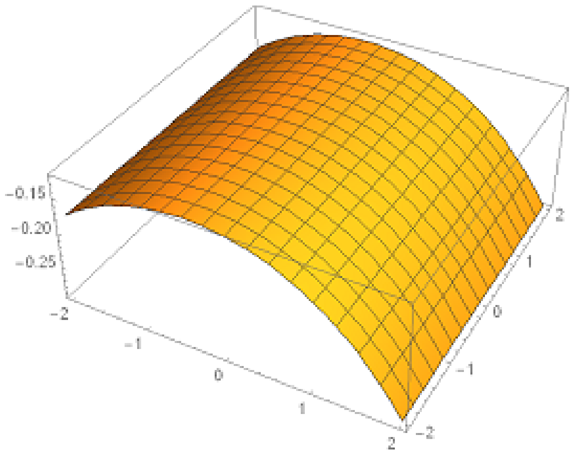

Figure 13: The plot of the solution(83) when $\lambda = 0.3, \mu = -0.2, A_1 = 0.1, A_2 = 0.7.$

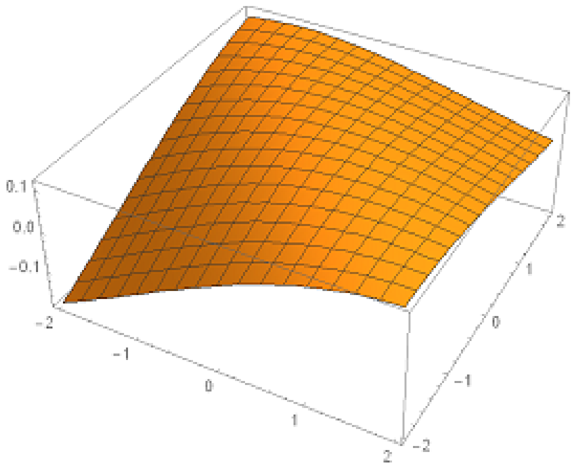

Figure 14: The plot of the solution(86) when $\lambda = -0.3, \mu = -0.2, A_1 = 0.1, A_2 = 0.7.$

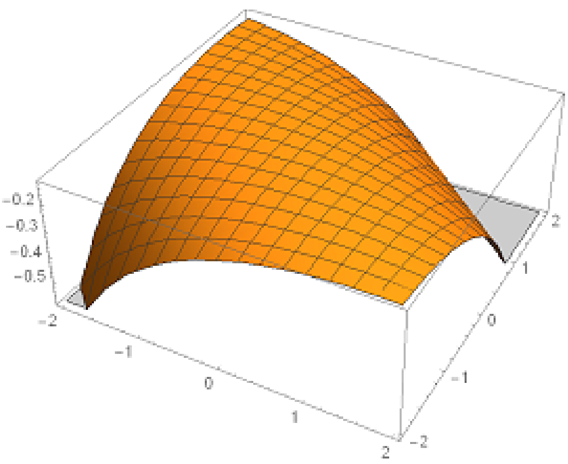

Figure 15: The plot of the solution(89) when $\lambda = 0.3, \mu = -0.2, A_1 = -0.1, A_2 = -0.7$.

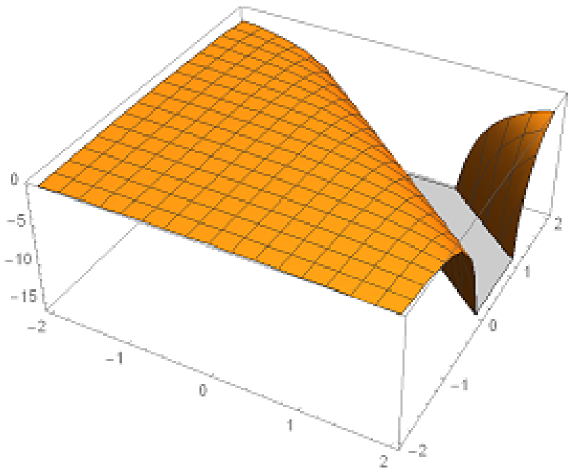

Figure 16: The plot of the solution(92) when $\lambda = -0.3, \mu = 0.3, A_1 = 0.1, A_2 = -0.1$.

4. Conclusion

In this work, we present new applications of the modified $(\frac{w}{g})$ -expansion method to construct a series of some new traveling wave solutions for some nonlinear partial differential equations, via the Zakharov Kuznetsov BBM (ZKBBM) equation and the Boussinesq equation. The performance of this method is found to be effective, powerful and reliable for solving the NPDEs. This method has the advantages of being direct and concise. Also, we believe that this method can be applied widely to many other NPDEs in the mathematical physics and this will be done in a future work.

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