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A Family of New q-Extensions of the Humbert Functions

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Abstract. By using the generating-function method as a starting point, the authors construct several (presumably new) basic (or q-) extensions of the Humbert function. Various potentially useful properties of these q-Humbert functions including (for example) explicit representations, recurrence relations and differential recurrence relations are derived by applying the defining generating functions.

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1. Introduction and Preliminaries

The theory of the quantum (q-) calculus has recently been applied in many areas of pure and applied mathematics as well as in such other disciplines as (for example) biology, physics, electrochemistry, economics, probability theory, and statistics. Many special functions of mathematical physics have been shown to admit generalizations to a base q, which are usually referred to as q-special functions. The q-special functions have important roles in many branches of mathematics and mathematical physics. Numerous investigations related to the q-special functions and q-polynomials can be found in many books and monographs as well as in journal articles in these field (see, for example, [2, 3, 5, 6, 7, 10, 11, 12, 14, 15, 17, 18, 19, 20, 21, 22, 29]).

Motivated essentially by the potential for applications of the q-analysis and the q-extensions of higher transcendental Bessel-type functions in diverse areas of mathematical,

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physical, engineering and statistical sciences (see the recent papers [1, 2, 5, 10, 11, 12, 13, 14, 15, 18, 28] and the references therein). The main purpose of this paper is to derive and investigate explicit formulas for the various families of the q-Humbert functions for $0 < q < 1$ by using the q-calculus in the theory of special functions. The work presented here is organized as follows. We define the numerous (known or new) q-Humbert functions and discuss some of their significant properties such as explicit representations, recurrence relations and generating functions in Section 2. In Section 3, we investigate pure recurrence relations and differential recurrence relations for the q-Humbert functions.

For the convenience of the reader, we first provide a summary of the mathematical notations and some basic definitions of the q-calculus for $|q| < 1$, which we need to use in this work.

For real or complex numbers λ and μ ($\lambda, \mu \in \mathbb{C}$), a q-number (or, equivalently, a basic or quantun number) $[\lambda]_q$ is defined as follows:

$$[\lambda]_q = \frac{1-q^\lambda}{1-q} = [\lambda+\mu]_q - q^\lambda\,[\mu]_q \qquad (|q|<1;\ \lambda,\mu \in \mathbb{C}), \tag{1}$$

where $\mathbb{C}$ denotes the set of complex numbers. Correspondingly, for $n \in \mathbb{N}_0$, the q-factorial $[n]_q!$ is defined by

$$[0]_q! = 1 \qquad \text{and} \qquad [n]_q! = \prod_{k=1}^{n}[k]_q = \prod_{k=1}^{n}\left(\frac{1-q^k}{1-q}\right) = \frac{(q;q)_n}{(1-q)^n} \qquad (n \in \mathbb{N}), \tag{2}$$

where

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} \qquad (\mathbb{N} = \{1,2,3,\cdots\}).$$

Here, and in what follows, for $q, \lambda, \mu \in \mathbb{C}$ $(|q| < 1)$, the basic (or q-) shifted factorial $(\lambda;q)_\mu$ is defined by (see, for example, [3], [6] and [24]; see also the recent works [7, 19, 23, 26, 27] dealing with the q-analysis)

$$(\lambda;q)_\mu = \prod_{j=0}^{\infty}\left(\frac{1-\lambda q^j}{1-\lambda q^{\mu+j}}\right) \qquad (|q|<1;\ \lambda,\mu \in \mathbb{C}), \tag{3}$$

so that

$$(\lambda;q)_n := \begin{cases} 1 & (n=0) \\ \prod\limits_{j=0}^{n-1}\left(1-\lambda q^j\right) & (n \in \mathbb{N}) \end{cases} \tag{4}$$

and

$$(\lambda;q)_\infty := \prod_{j=0}^{\infty}\left(1-\lambda q^j\right) \qquad (|q|<1;\ \lambda \in \mathbb{C}). \tag{5}$$

In the usual notation, the q-exponential function $\mathrm{e}_q(z)$ of the *first kind* and the q-exponential function $\mathrm{E}_q(z)$ of the *second kind* are defined by

$$\mathrm{e}_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{[(1-q)z]^n}{(q;q)_n} = \frac{1}{\big((1-q)z;q\big)_\infty} \tag{6}$$

and

$$\mathrm{E}_q(z) := \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{[(1-q)z]^n}{(q;q)_n}, \tag{7}$$

respectively.

Remark 1. It is easily seen by applying the definitions (6) and (7) that

$$\mathrm{e}_{\frac{1}{q}}(z) = \mathrm{E}_q(z), \quad \lim_{q\to 1}\{\mathrm{e}_q(z)\} = \mathrm{e}^z = \lim_{q\to 1}\{\mathrm{E}_q(z)\} \qquad \text{and} \qquad \mathrm{e}_q(z)\cdot \mathrm{E}_q(-z) = 1. \tag{8}$$

Let f be a function defined on a subset of the real axis $\mathbb{R}$ or the complex plane $\mathbb{C}$. We define the q-derivative operator D_q, which is also referred to as the Jackson derivative operator [13], as follows:

$$D_q\{f(x)\} = \frac{f(x)-f(qx)}{(1-q)x}, \tag{9}$$

which satisfies each of the following relations:

$$D_q\{x^n\} = [n]_q\, x^{n-1} \tag{10}$$

and

$$D_q\{\mathrm{e}_q(\lambda x)\} = \lambda \mathrm{e}_q(\lambda x) \qquad \text{and} \qquad D_q\{\mathrm{E}_q(\lambda x)\} = \lambda \mathrm{E}_q(\lambda q x) \tag{11}$$

for a given constant λ. Furthermore, the formula for the q-derivative by parts can be written as follows:

$$D_q\left\{f_1(x)f_2(x)\right\} = D_q\{f_1(x)\}f_2(x) + f_1(qx)D_q\{f_2(x)\}, \tag{12}$$

$$\begin{aligned} D_q\left\{f_1(x)f_2(x)f_3(x)\right\} = {} & D_q\{f_1(x)\}f_2f_3(x) + f_1(qx)D_q\{f_2(x)\}f_3(x) \\ & + f_1(qx)f_2(qx)D_q\{f_3(x)\}, \end{aligned} \tag{13}$$

and so on.

In the earlier works [8, 9, 16, 30], the Humbert function $J_{m,n}(x)$ is defined by means of the generating function:

$$\exp\left[\frac{x}{3}\left(u+t-\frac{1}{ut}\right)\right] = \sum_{m,n=-\infty}^{\infty} J_{m,n}(x)u^m t^n \tag{14}$$

or, more generally, as follows:

$$
\begin{aligned}
J_{m,n}(x) &= \left(\frac{x}{3}\right)^{m+n} \frac{1}{\Gamma(m+1)\Gamma(n+1)} \, {}_0F_2\bigg(\text{—}\,; m+1, n+1; -\frac{x^3}{27}\bigg) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, \Gamma(m+k+1)\Gamma(n+k+1)} \left(\frac{x}{3}\right)^{m+n+3k}.
\end{aligned} \tag{15}
$$

In the next sections, we propose to generalize and extend this class of Humbert functions. By using the above approach based upon the quantum (or q-) analysis, we define and investigate the generalized q-Humbert functions. We also present a particular case of functions belonging to the family of q-Humbert functions which are known as the q-Humbert functions of the third kind.

2. Definitions and Basic Properties of the New q-Extensions of the Humbert Functions

In this section, we apply the notion of q-generating functions to get explicit formulas for the generalized q-Humbert functions and obtain several interesting and significant properties for these functions.

Definition 1. By using the product of symmetric q-exponential functions $\mathrm{e}_q(z)$, we define the following generating function for the q-Humbert function $J^{(1)}_{m,n}(x|q)$ of the first kind:

$$
F_1(x; u, t|q) = \mathrm{e}_q\bigg(\frac{xu}{3}\bigg)\mathrm{e}_q\bigg(\frac{xt}{3}\bigg)\mathrm{e}_q\bigg(-\frac{x}{3ut}\bigg) = \sum_{m,n=-\infty}^{\infty} J^{(1)}_{m,n}(x|q) \, u^m \, t^n, \tag{16}
$$

which, in view of (6), yields

$$
\begin{aligned}
F_1(x; u, t|q) &= \mathrm{e}_q\bigg(\frac{xu}{3}\bigg)\mathrm{e}_q\bigg(\frac{xt}{3}\bigg)\mathrm{e}_q\bigg(-\frac{x}{3ut}\bigg) \\
&= \sum_{r=0}^{\infty} \frac{1}{[r]_q!}\bigg(\frac{xu}{3}\bigg)^r \sum_{j=0}^{\infty} \frac{1}{[j]_q!}\bigg(\frac{xt}{3}\bigg)^j \sum_{k=0}^{\infty} \frac{1}{[k]_q!}\bigg(-\frac{x}{3ut}\bigg)^k \\
&= \sum_{j,r,k=0}^{\infty} \frac{(-1)^k}{[k]_q![j]_q![r]_q!}\bigg(\frac{x}{3}\bigg)^{k+j+r} u^{r-k} \, t^{j-k},
\end{aligned}
$$

so that, upon setting $r-k=m$ and $j-k=n$, we find that

$$
\begin{aligned}
&\sum_{k,j,r=0}^{\infty} \frac{(-1)^k}{[k]_q![j]_q![r]_q!}\bigg(\frac{x}{3}\bigg)^{k+j+r} u^{r-k} \, t^{j-k} \\
&\quad = \sum_{m,n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q![m+k]_q![n+k]_q!}\bigg(\frac{x}{3}\bigg)^{m+n+3k} u^m \, t^n
\end{aligned}
$$

$$= \sum_{m,n=-\infty}^{\infty} J_{m,n}^{(1)}(x|q)\, u^m\, t^n.$$

Thus, explicitly, we get the following expression for the q-Humbert function $J_{m,n}^{(1)}(x|q)$ of the first kind as a power series:

$$J_{m,n}^{(1)}(x|q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q![m+k]_q![n+k]_q!} \left(\frac{x}{3}\right)^{m+n+3k}. \tag{17}$$

Remark 2. The function $J_{m,n}^{(1)}(x|q)$ is a q-analogue of the Humbert function $J_{m,n}(x)$ defined by the generating function (14) :

$$\lim_{q\to 1} \left\{ J_{m,n}^{(1)}(qx|q) \right\} = J_{m,n}(x).$$

For $m, n \in \mathbb{N}_0$, by applying the relationships in (2), we can write the equation (17) as follows:

$$J_{m,n}^{(1)}(x|q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k (q;q)_{m+k} (q;q)_{n+k}} \left(\frac{(1-q)x}{3}\right)^{m+n+3k}, \tag{18}$$

which, in light of the following elementary identity:

$$(q;q)_{n+k} = (q;q)_k (q^{k+1};q)_n = (q;q)_n (q^{n+1};q)_k, \tag{19}$$

we get another representation of the q-Humbert function $J_{m,n}^{(1)}(x|q)$ in (18):

$$J_{m,n}^{(1)}(x|q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[(q;q)_k]^3 (q^{k+1};q)_m (q^{k+1};q)_n} \left(\frac{(1-q)x}{3}\right)^{m+n+3k} \tag{20}$$

or, equivalently, we have

$$J_{m,n}^{(1)}(x|q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k (q;q)_m (q;q)_n (q^{m+1};q)_k (q^{n+1};q)_k} \left(\frac{(1-q)x}{3}\right)^{m+n+3k}. \tag{21}$$

Lemma 1. *If m and n are integers, then the q-Humbert function $J_{m,n}^{(1)}(x|q)$ of the first kind satisfies the following relation*:

$$J_{-m,n}^{(1)}(x|q) = (-1)^m J_{m,n+m}^{(1)}(x|q). \tag{22}$$

Proof. From Definition 1 of the q-Humbert function $J_{m,n}^{(1)}(x|q)$ of the first kind, we have

$$
\begin{aligned}
J^{(1)}_{-m,n}(x|q) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q![-m+k]_q![n+k]_q!} \left(\frac{x}{3}\right)^{-m+n+3k} \\
&= \sum_{k=m}^{\infty} \frac{(-1)^k}{[k]_q![-m+k]_q![n+k]_q!} \left(\frac{x}{3}\right)^{-m+n+3k},
\end{aligned}
$$

which, upon setting $s = k - m$, yields

$$
J^{(1)}_{-m,n}(x|q) = \sum_{s=0}^{\infty} \frac{(-1)^{s+m}}{[s]_q![s+m]_q![n+m+s]_q!} \left(\frac{x}{3}\right)^{2m+n+3s} = (-1)^m J^{(1)}_{m,n+m}(x|q).
$$

In a similar manner, we can obtain the next result Theorem 1 below).

Theorem 1. *If m and n are integers, then the q-Humbert function $J^{(1)}_{m,n}(x|q)$ of the first kind satisfies the following relation*:

$$
J^{(1)}_{m,-n}(x|q) = (-1)^n J^{(1)}_{m+n,m}(x|q). \tag{23}
$$

Theorem 2. *If m and n are integers, then the q-Humbert function $J^{(1)}_{m,n}(x|q)$ of the first kind satisfies the following relation*:

$$
J^{(1)}_{-m,-n}(x|q) = (-1)^m J^{(1)}_{m,m-n}(x|q) = (-1)^n J^{(1)}_{n-m,n}(x|q). \tag{24}
$$

Proof. From Definition 1 of the q-Humbert function $J^{(1)}_{m,n}(x|q)$ of the first kind, we have

$$
\begin{aligned}
J^{(1)}_{-m,-n}(x|q) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q![-m+k]_q![-n+k]_q!} \left(\frac{x}{3}\right)^{-m-n+3k} \\
&= \sum_{k=\max\{m,n\}}^{\infty} \frac{(-1)^k}{[k]_q![-m+k]_q![-n+k]_q!} \left(\frac{x}{3}\right)^{-m-n+3k},
\end{aligned}
$$

which, upon setting $s = k - m$, yields

$$
J^{(1)}_{-m,-n}(x|q) = \sum_{s=0}^{\infty} \frac{(-1)^{s+m}}{[s]_q![m+s]_q![m-n+s]_q!} \left(\frac{x}{3}\right)^{2m-n+3s} = (-1)^m J^{(1)}_{m,m-n}(x|q).
$$

On the other hand, by setting $s = k - n$, we obtain

$$
J^{(1)}_{-m,-n}(x|q) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n}}{[s]_q![n-m+s]_q![n+s]_q!} \left(\frac{x}{3}\right)^{2n-m+3s} = (-1)^n J^{(1)}_{n-m,n}(x|q).
$$

Definition 2. The q-Humbert function $J^{(2)}_{m,n}(x|q)$ of the second kind is defined by the generating function $F_2(x;u,t|q)$ as follows:

$$F_2(x;u,t|q) = \mathrm{E}_q\left(\frac{xu}{3}\right)\mathrm{E}_q\left(\frac{xt}{3}\right)\mathrm{E}_q\left(-\frac{qx}{3ut}\right) = \sum_{m,n=-\infty}^{\infty} q^{\binom{m}{2}+\binom{n}{2}}\ J^{(2)}_{m,n}(x|q)\ u^m\ t^n, \quad (25)$$

which, in light of (7), yields

$$\begin{aligned} F_2(x;u,t|q) &= \mathrm{E}_q\left(\frac{xu}{3}\right)\mathrm{E}_q\left(\frac{xt}{3}\right)\mathrm{E}_q\left(-\frac{qx}{3ut}\right) \\ &= \sum_{r=0}^{\infty}\frac{q^{\binom{r}{2}}}{[r]_q!}\left(\frac{xu}{3}\right)^r \sum_{j=0}^{\infty}\frac{q^{\binom{j}{2}}}{[j]_q!}\left(\frac{xt}{3}\right)^j \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}}{[k]_q!}\left(-\frac{qx}{3ut}\right)^k \\ &= \sum_{k,j,r=0}^{\infty}\frac{(-1)^k q^{\binom{r}{2}+\binom{j}{2}+\binom{k+1}{2}}}{[k]_q![j]_q![r]_q!}\left(\frac{x}{3}\right)^{k+j+r} u^{r-k}\ t^{j-k}, \end{aligned}$$

so that, upon setting $r-k=m$ and $j-k=n$, we get

$$\begin{aligned} F_2(x;u,t|q) &= \mathrm{E}_q\left(\frac{xu}{3}\right)\mathrm{E}_q\left(\frac{xt}{3}\right)\mathrm{E}_q\left(-\frac{qx}{3ut}\right) \\ &= \sum_{k,j,r=0}^{\infty}\frac{(-1)^k q^{\binom{r}{2}+\binom{j}{2}+\binom{k+1}{2}}}{[k]_q![j]_q![r]_q!}\left(\frac{x}{3}\right)^{k+j+r} u^{r-k}t^{j-k} \\ &= \sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k+1}{2}}}{[k]_q![m+k]_q![n+k]_q!}\left(\frac{x}{3}\right)^{m+n+3k} u^m\ t^n \\ &= \sum_{m,n=-\infty}^{\infty} q^{\binom{m}{2}+\binom{n}{2}} J^{(2)}_{m,n}(x|q)\ u^m\ t^n. \end{aligned}$$

Thus, explicitly, we are led to the following expression for the q-Humbert function $J^{(2)}_{m,n}(x|q)$ of the second kind as power series:

$$J^{(2)}_{m,n}(x|q) = \sum_{k=0}^{\infty}\frac{(-1)^k}{[k]_q![m+k]_q![n+k]_q!}\ q^{\frac{1}{2}k[3k+2(m+n)-1]}\left(\frac{x}{3}\right)^{m+n+3k}. \quad (26)$$

Theorem 3. *The following relationship holds true between the q-Humbert functions $J^{(1)}_{m,n}(x|q)$ and $J^{(2)}_{m,n}(x|q)$ of the first and the second kind:*

$$J^{(1)}_{m,n}\left(q^{\frac{1}{3}}x\,\middle|\,\frac{1}{q}\right) = q^{\frac{1}{3}(m+n)+\binom{m}{2}+\binom{n}{2}}\ J^{(2)}_{m,n}(x|q). \quad (27)$$

Proof. If we set

$$x \mapsto q^{\frac{1}{3}}x, \quad t \mapsto q^{-\frac{1}{3}}t \qquad \text{and} \qquad u \mapsto q^{-\frac{1}{3}}u$$

in the generating function (16), we obtain

$$\begin{aligned} F_1\left(q^{\frac{1}{3}}x; q^{-\frac{1}{3}}u, q^{-\frac{1}{3}}t \middle| \frac{1}{q}\right) &= \mathrm{e}_{\frac{1}{q}}\left(\frac{xu}{3}\right)\mathrm{e}_{\frac{1}{q}}\left(\frac{xt}{3}\right)\mathrm{e}_{\frac{1}{q}}\left(-\frac{qx}{3ut}\right) \\ &= \sum_{m,n=-\infty}^{\infty} J^{(1)}_{m,n}\left(q^{\frac{1}{3}}x \middle| \frac{1}{q}\right) q^{-\frac{1}{3}(m+n)}\, u^m\, t^n, \end{aligned}$$

which, in view of the first relationship in (8), becomes

$$\begin{aligned} F_1\left(q^{\frac{1}{3}}x; q^{-\frac{1}{3}}u, q^{-\frac{1}{3}}t \middle| \frac{1}{q}\right) &= \mathrm{E}_q\left(\frac{xu}{3}\right)\mathrm{E}_q\left(\frac{xt}{3}\right)\mathrm{E}_q\left(-\frac{qx}{3ut}\right) \\ &= \sum_{m,n=-\infty}^{\infty} J^{(1)}_{m,n}\left(q^{\frac{1}{3}}x \middle| \frac{1}{q}\right) q^{-\frac{1}{3}(m+n)}\, u^m\, t^n, \end{aligned} \tag{28}$$

where the q-exponential function $E_q(z)$ of the second kind is defined by (7).

Now, if we compare this last equation (28) with the defining generating function (25) for the q-Humbert functions $J^{(2)}_{m,n}(x|q)$ of the second kind, we are led to the assertion (27) of Theorem 3.

Remark 3. By appropriately permuting q-exponential functions $\mathrm{e}_q(z)$ and $\mathrm{E}_q(z)$ of the first and the second kind in Definition 1 and Definition 2, we can analogously define and investigate many variations of the q-Humbert functions $J^{(1)}_{m,n}(x|q)$ and $J^{(2)}_{m,n}(x|q)$ of the first and the second kind. However, since the q-exponential functions $\mathrm{e}_q(z)$ and $\mathrm{E}_q(z)$ of the first and the second kind are related by

$$\mathrm{e}_{\frac{1}{q}}(z) = \mathrm{E}_q(z) \qquad \text{and} \qquad \mathrm{E}_{\frac{1}{q}}(z) = \mathrm{e}_q(z), \tag{29}$$

we choose to present here just one such variation given by Definition 3 below.

Definition 3. The q-Humbert function $J^{(3)}_{m,n}(x|q)$ of the *third kind* is given by the generating function $F_3(x; u, t|q)$ as follows:

$$F_3(x; u, t|q) = \mathrm{e}_q\left(\frac{xu}{3}\right)\mathrm{e}_q\left(\frac{xt}{3}\right)\mathrm{E}_q\left(-\frac{qx}{3ut}\right) = \sum_{m,n=-\infty}^{\infty} J^{(3)}_{m,n}(x|q)\, u^m\, t^n, \tag{30}$$

which would readily yield the following explicit representation for q-Humbert function $J^{(3)}_{m,n}(x|q)$ of the third kind as a power series:

$$J^{(3)}_{m,n}(x|q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q![m+k]_q![n+k]_q!}\, q^{\binom{k+1}{k}} \left(\frac{x}{3}\right)^{m+n+3k}. \tag{31}$$

The following result is a straightforward consequence of Definition 1 and Definition 3 in conjunction with the third relationship in (8).

Theorem 4. (Multiplication Theorems) *The generating functions for the q-Humbert functions* $J^{(1)}_{m,n}(x|q)$ *and* $J^{(3)}_{m,n}(x|q)$ *of the first and the third kind are related as follows*:

$$F_1(x;u,t|q) = \mathrm{e}_q\left(\frac{qx}{3ut}\right)\mathrm{e}_q\left(-\frac{x}{3ut}\right)\cdot F_3(x;u,t|q) \tag{32}$$

and

$$F_3(x;u,t|q) = \mathrm{E}_q\left(-\frac{qx}{3ut}\right)\mathrm{E}_q\left(\frac{x}{3ut}\right)\cdot F_1(x;u,t|q). \tag{33}$$

3. Differential and Pure Recurrences Relations

In this section, we briefly indicate how several interesting differential and pure recurrences relations for the q-Humbert functions, which we have introduced here, can be established by applying their generating functions in different ways. We illustrate the techniques used in the case of the q-Humber function $J^{(1)}_{m,n}(x|q)$ of the first kind. Similar arguments will lead to analogous results for the q-Humbert functions $J^{(2)}_{m,n}(x|q)$ and $J^{(3)}_{m,n}(x|q)$ of the second and the third kind.

Theorem 5. *The q-Humbert function* $J^{(1)}_{m,n}(x|q)$ *of the first kind satisfies the following differential recurrences relations*:

$$J^{(1)}_{m-1,n}(x|q)+q^{\frac{2m-n+1}{3}}J^{(1)}_{m,n-1}(q^{\frac{1}{3}}x|q)-q^{\frac{m+n+2}{3}}J^{(1)}_{m+1,n+1}(q^{\frac{2}{3}}x|q)=3D_q\left\{J^{(1)}_{m,n}(x|q)\right\}, \tag{34}$$

$$q^{\frac{2n-m+1}{3}}J^{(1)}_{m-1,n}(q^{\frac{1}{3}}x|q)+J^{(1)}_{m,n-1}(x|q)-q^{\frac{m+n+2}{3}}J^{(1)}_{m+1,n+1}(q^{\frac{2}{3}}x|q)=3D_q\left\{J^{(1)}_{m,n}(x|q)\right\}, \tag{35}$$

$$J^{(1)}_{m-1,n}(x|q)+q^{\frac{m-2n+2}{3}}J^{(1)}_{m,n-1}(q^{\frac{2}{3}}x|q)-q^{\frac{2m-n+1}{3}}J^{(1)}_{m+1,n+1}(q^{\frac{1}{3}}x|q)=3D_q\left\{J^{(1)}_{m,n}(x|q)\right\}, \tag{36}$$

$$q^{\frac{n-2m+2}{3}}J^{(1)}_{m-1,n}(q^{\frac{2}{3}}x|q)+J^{(1)}_{m,n-1}(x|q)-q^{\frac{2n-m+1}{3}}J^{(1)}_{m+1,n+1}(q^{\frac{1}{3}}x|q)=3D_q\left\{J^{(1)}_{m,n}(x|q)\right\}, \tag{37}$$

$$q^{\frac{1-m-n}{3}}J^{(1)}_{m-1,n}(q^{\frac{1}{3}}x|q)+q^{\frac{m-2n+2}{3}}J^{(1)}_{m,n-1}(q^{\frac{2}{3}}x|q)-J^{(1)}_{m+1,n+1}(x|q)=3D_q\left\{J^{(1)}_{m,n}(x|q)\right\} \tag{38}$$

and

$$q^{\frac{n-2m+2}{3}}J^{(1)}_{m-1,n}(q^{\frac{2}{3}}x|q)+q^{\frac{1-m-n}{3}}J^{(1)}_{m,n-1}(q^{\frac{1}{3}}x|q)-J^{(1)}_{m+1,n+1}(x|q)=3D_q\left\{J^{(1)}_{m,n}(x|q)\right\}. \tag{39}$$

Proof. By applying the q-derivative operator D_q on both sides of the equation (16) and using (13), we have

$$\frac{1}{3}\bigg[ue_q\Big(\frac{xu}{3}\Big)e_q\Big(\frac{xt}{3}\Big)e_q\Big(-\frac{x}{3ut}\Big)+te_q\Big(\frac{qxu}{3}\Big)e_q\Big(\frac{xt}{3}\Big)e_q\Big(-\frac{x}{3ut}\Big) \\ -\frac{1}{ut}\,e_q\Big(\frac{qxu}{3}\Big)e_q\Big(\frac{qxt}{3}\Big)e_q\Big(-\frac{x}{3ut}\Big)\bigg] \\ =\sum_{m,n=-\infty}^{\infty}D_q\left\{J^{(1)}_{m,n}(x|q)\right\}\,u^m\,t^n. \tag{40}$$

Now, if we set

$$x\mapsto q^{\frac{1}{3}}x,\quad u\mapsto q^{\frac{2}{3}}u \qquad \text{and} \qquad t\mapsto q^{-\frac{1}{3}}t$$

in the generating relation (16), then we get the following result:

$$q^{-\frac{1}{3}}te_q\Big(\frac{q^{\frac{1}{3}}xq^{\frac{2}{3}}u}{3}\Big)e_q\Big(\frac{q^{\frac{1}{3}}xq^{-\frac{1}{3}}t}{3}\Big)e_q\Big(-\frac{q^{\frac{1}{3}}x}{3q^{\frac{2}{3}}uq^{-\frac{1}{3}}t}\Big) \\ =q^{-\frac{1}{3}}te_q\Big(\frac{qxu}{3}\Big)e_q\Big(\frac{xt}{3}\Big)e_q\Big(-\frac{x}{3ut}\Big) \\ =\sum_{m,n=-\infty}^{\infty}J^{(1)}_{m,n}(q^{\frac{1}{3}}x|q)\Big(q^{\frac{2}{3}}u\Big)^m\Big(q^{-\frac{1}{3}}t\Big)^{n+1} \\ =\sum_{m,n=-\infty}^{\infty}q^{\frac{2m-n}{3}}J^{(1)}_{m,n-1}\big(q^{\frac{1}{3}}x|q\big)\,u^m\,t^n. \tag{41}$$

Next, by using the generating function (16) and setting

$$x\mapsto q^{\frac{2}{3}}x,\quad u\mapsto q^{\frac{1}{3}}u \qquad \text{and} \qquad t\mapsto q^{\frac{1}{3}}t,$$

we have

$$\frac{1}{q^{\frac{1}{3}}uq^{\frac{1}{3}}t}e_q\Big(\frac{q^{\frac{2}{3}}xq^{\frac{1}{3}}u}{3}\Big)e_q\Big(\frac{q^{\frac{2}{3}}xq^{\frac{1}{3}}t}{3}\Big)e_q\Big(-\frac{q^{\frac{2}{3}}x}{3q^{\frac{1}{3}}uq^{\frac{1}{3}}t}\Big) \\ =\frac{1}{q^{\frac{2}{3}}ut}e_q\Big(\frac{qxu}{3}\Big)e_q\Big(\frac{qxt}{3}\Big)e_q\Big(-\frac{x}{3ut}\Big) \\ =\sum_{m,n=-\infty}^{\infty}J^{(1)}_{m,n}(q^{\frac{2}{3}}x|q)\Big(q^{\frac{1}{3}}u\Big)^{m-1}\Big(q^{\frac{1}{3}}t\Big)^{n-1} \\ =\sum_{m,n=-\infty}^{\infty}J^{(1)}_{m+1,n+1}(q^{\frac{2}{3}}x|q)\Big(q^{\frac{1}{3}}u\Big)^{m}\Big(q^{\frac{1}{3}}t\Big)^{n}$$

$$= \sum_{m,n=-\infty}^{\infty} q^{\frac{m+n}{3}} J^{(1)}_{m+1,n+1}\big(q^{\frac{2}{3}}x|q\big)\, u^m\, t^n. \tag{42}$$

Finally, by using the equations (40), (41) and (42), we find that

$$\begin{aligned}\frac{1}{3}\Bigg[\sum_{m,n=-\infty}^{\infty} J^{(1)}_{m,n}(x|q)\, u^{m+1}\, t^n + \sum_{m,n=-\infty}^{\infty} q^{\frac{2m-n+1}{3}} J^{(1)}_{m,n-1}\big(q^{\frac{1}{3}}x|q\big)\, u^m\, t^n \\ - \sum_{m,n=-\infty}^{\infty} q^{\frac{m+n+2}{3}} J^{(1)}_{m+1,n+1}\big(q^{\frac{2}{3}}x|q\big)\, u^m\, t^n\Bigg] \\ = \sum_{m,n=-\infty}^{\infty} D_q \left\{J^{(1)}_{m,n}(x|q)\right\}\, u^m\, t^n,\end{aligned} \tag{43}$$

which yields the differential recurrence relation (34) asserted by Theorem 5.

The other differential recurrence relations (35) to (40) can be proven similarly.

The demonstration of each of the following results (Theorems 6 to 9 below) would run run essentially parallel to that of Theorem 5. We, therefore, choose to skip the details involved.

Theorem 6. *The q-Humbert function $J^{(1)}_{m,n}(x|q)$ satisfies the following pure recurrence relations*:

$$J^{(1)}_{m-1,n}(x|q) + q^{\frac{2m-n+1}{3}} J^{(1)}_{m,n-1}\big(q^{\frac{1}{3}}x|q\big) = q^{\frac{2n-m+1}{3}} J^{(1)}_{m-1,n}\big(q^{\frac{1}{3}}x|q\big) + J^{(1)}_{m,n-1}(x|q), \tag{44}$$

$$\begin{aligned}J^{(1)}_{m-1,n}(x|q) + q^{\frac{m-2n+2}{3}} J^{(1)}_{m,n-1}\big(q^{\frac{2}{3}}x|q\big) - q^{\frac{2m-n+1}{3}} J^{(1)}_{m+1,n+1}\big(q^{\frac{1}{3}}x|q\big) \\ = q^{\frac{n-2m+2}{3}} J^{(1)}_{m-1,n}\big(q^{\frac{2}{3}}x|q\big) + J^{(1)}_{m,n-1}(x|q) - q^{\frac{2n-m+1}{3}} J^{(1)}_{m+1,n+1}\big(q^{\frac{1}{3}}x|q\big),\end{aligned} \tag{45}$$

$$\begin{aligned}q^{\frac{1-m-n}{3}} J^{(1)}_{m-1,n}\big(q^{\frac{1}{3}}x|q\big) + q^{\frac{m-2n+2}{3}} J^{(1)}_{m,n-1}\big(q^{\frac{2}{3}}x|q\big) \\ = q^{\frac{n-2m+2}{3}} J^{(1)}_{m-1,n}\big(q^{\frac{2}{3}}x|q\big) + q^{\frac{1-m-n}{3}} J^{(1)}_{m,n-1}\big(q^{\frac{1}{3}}x|q\big),\end{aligned} \tag{46}$$

$$\begin{aligned}q^{\frac{2m-n+1}{3}} J^{(1)}_{m,n-1}\big(q^{\frac{1}{3}}x|q\big) - q^{\frac{m+n+2}{3}} J^{(1)}_{m+1,n+1}\big(q^{\frac{2}{3}}x|q\big) \\ = q^{\frac{m-2n+2}{3}} J^{(1)}_{m,n-1}\big(q^{\frac{2}{3}}x|q\big) - q^{\frac{2m-n+1}{3}} J^{(1)}_{m+1,n+1}\big(q^{\frac{1}{3}}x|q\big)\end{aligned} \tag{47}$$

and

$$\begin{aligned}q^{\frac{2n-m+1}{3}} J^{(1)}_{m-1,n}\big(q^{\frac{1}{3}}x|q\big) - q^{\frac{m+n+2}{3}} J^{(1)}_{m+1,n+1}\big(q^{\frac{2}{3}}x|q\big) \\ = q^{\frac{n-2m+2}{3}} J^{(1)}_{m-1,n}\big(q^{\frac{2}{3}}x|q\big) - q^{\frac{2n-m+1}{3}} J^{(1)}_{m+1,n+1}\big(q^{\frac{1}{3}}x|q\big).\end{aligned} \tag{48}$$

Theorem 7. *The q-Humbert function $J^{(1)}_{m,n}(x|q)$ satisfies the following pure recurrence relations*:

$$\frac{3}{x}\ [n]_q\ J^{(1)}_{m,n}(x|q) = J^{(1)}_{m,n-1}(x|q) + q^n\ J^{(1)}_{m+1,n+1}(x|q) \tag{49}$$

and

$$\frac{3}{x}\ [m]_q J^{(1)}_{m,n}(x|q) = J^{(1)}_{m-1,n}(x|q) + q^m\ J^{(1)}_{m+1,n+1}(x|q). \tag{50}$$

Theorem 8. *The q-Humbert function $J^{(1)}_{m,n}(x|q)$ satisfies the following pure recurrence relations*:

$$q^{-\frac{m+n-1}{3}}\ J^{(1)}_{m,n-1}(q^{\frac{1}{3}}x|q) = J^{(1)}_{m,n-1}(x|q) + \frac{(1-q)x}{3}\ J^{(1)}_{m+1,n}(x|q), \tag{51}$$

$$q^{-\frac{m+n-1}{3}}\ J^{(1)}_{m-1,n}(q^{\frac{1}{3}}x|q) = J^{(1)}_{m-1,n}(x|q) + \frac{(1-q)x}{3}\ J^{(1)}_{m,n+1}(x|q) \tag{52}$$

and

$$q^{-\frac{m+n-2}{3}} J^{(1)}_{m-1,n-1}(q^{\frac{1}{3}}x|q) = J^{(1)}_{m-1,n-1}(x|q) + \frac{(1-q)x}{3}\ J^{(1)}_{m,n}(x|q). \tag{53}$$

Theorem 9. *The q-Humbert function $J^{(1)}_{m,n}(x|q)$ satisfies the following properties*:

$$\frac{x}{3}\left(J^{(1)}_{m,n}(x|q) + q^{\frac{2m-n+1}{3}} J^{(1)}_{m+2,n+1}(q^{\frac{1}{3}}x|q)\right) = [m+1]_q\ J^{(1)}_{m+1,n}(x|q), \tag{54}$$

$$\frac{x}{3}\left(J^{(1)}_{m+2,n+1}(x|q) + q^{-\frac{m+n}{3}} J^{(1)}_{m,n}(q^{\frac{1}{3}}x|q)\right) = [m+1]_q\ J^{(1)}_{m+1,n}(x|q), \tag{55}$$

$$\frac{x}{3}\left(J^{(1)}_{m,n}(x|q) + q^{\frac{2n-m+1}{3}} J^{(1)}_{m+1,n+2}(q^{\frac{1}{3}}x|q)\right) = [n+1]_q\ J^{(1)}_{m,n+1}(x|q) \tag{56}$$

and

$$\frac{x}{3}\left(J^{(1)}_{m+1,n+2}(x|q) + q^{-\frac{m+n}{3}} J^{(1)}_{m,n}(q^{\frac{1}{3}}x|q)\right) = [n+1]_q\ J^{(1)}_{m,n+1}(x|q). \tag{57}$$

4. Concluding Remarks and Observations

In our present investigation, we have introduced and studied the properties of three families of q-Humbert functions. Each of these q-Humbert functions allows us to describe many aspects of computational and quantum (or q-) analysis. We have also explored how these classes of q-Humbert functions can be described in terms of q-Humbert functions of a different class. Our investigation of these three families of q-Humbert functions is potentially useful in motivating further researches on this subject

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