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On identities in right alternative superalgebras

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Abstract. Some fundamental identities characterizing right alternative superalgebras are found. These identities are the $\mathbb{Z}_2$-graded versions of well-known identities in right alternative algebras.

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1. Introduction

An algebra that satisfies both of the identities
$(xy)y = x(yy)$ (the right alternative identity),
$(xx)y = x(xy)$ (the left alternative identity)
is called an alternative algebra. As algebraic objects, alternative algebras were studied ([18]) in connection with some problems in projective planes (see also [9], where it is shown that alternative algebras satisfy some specific identities that are later called Moufang identities). For basics in the general theory of alternative algebras, one may refer to [3], [11] and [4], where some examples of alternative algebras could be found.

As a generalization of alternative algebras, right alternative algebras were first studied in [1]. An algebra is said to be right alternative if it satisfies only the right alternative identity. In [1] an example of a five-dimensional right alternative algebra that is not left alternative is constructed. Right alternative algebras have been further studied in, e.g., [7], [14], [15], [16] (see also [4] and references therein). In [14] some properties of right alternative algebras were used to solve a problem in projective planes.

In [2] and [8] a $\mathbb{Z}_2$-graded generalization of the Lie theory is considered with the introduction of "$\mathbb{Z}_2$-graded Lie algebras" (now called Lie superalgebras; see [6] and [12] for a survey on the subject). An extension of such a $\mathbb{Z}_2$-gradation to other types of algebras is first performed in [5]. In this scheme, alternative superalgebras were introduced in [17] and in [13] it was shown how one can obtain the graded identities defining a type of

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superalgebras from the known identities satisfied by its Grassmann envelope of the same type.

As in [7] and [14], where fundamental identities characterizing right alternative algebras were found, the purpose of this paper is to consider the explicit superization of some of these identities in case of right alternative superalgebras. We find that these superidentities could be useful as a working tool while dealing with some issues on right alternative (or even alternative) superalgebras. We stress that we do not resort to the Grassmann envelope of the corresponding right alternative superalgebra for finding these superidentities, but we rather start from the Teichmüller identity, the right superalternativity and the $\mathbb{Z}_2$-graded version of the function $g(w, x, y, z)$ defined in [7] for right alternative algebras.

In section 2 we recall some useful notions and prove (as for usual algebras) some general identities that hold in any superalgebra with consequences in right alternative or alternative superalgebras. In section 3 we first define the $\mathbb{Z}_2$-graded version of the function $g(w, x, y, z)$ and prove that it is identically zero. Next, we prove some fundamental identities in right alternative superalgebras and observe that they are the $\mathbb{Z}_2$-graded generalization of corresponding well-known identities in right alternative algebras. In section 4 we point out some implications of these identities. In particular, we get some $\mathbb{Z}_2$-graded Moufang-type identities for right alternative superalgebras.

Thoughout this paper we will work over a ground field of characteristic not 2.

2. Preliminaries

Definition 1. *A binary algebra $(A, \cdot)$ is called a (binary) superalgebra (i.e. a $\mathbb{Z}_2$-graded binary algebra) whenever the vector space A can be expressed as a direct sum of subspaces $A = A_0 \oplus A_1$, where $A_i \cdot A_j \subseteq A_{i+j}$, $i, j \in \mathbb{Z}_2$. The subspaces A_0 and A_1 are called respectively the even and odd parts of the superalgebra A; so are also called the elements from A_0 and A_1 respectively.*

All elements in A are assumed to be *homogeneous*, i.e. either even or odd. For a given element $x \in A_i$ $(i = 0, 1)$, by $\overline{x} = i$ we denote its parity. In order to reduce the number of braces, we use juxtaposition whenever applicable and so, e.g., $xy \cdot z$ means $(x \cdot y) \cdot z$. Moreover, for simplicity and where there is no danger of confusion, we write xy in place of $x \cdot y$.

In a superalgebra $(A, \cdot)$, the *supercommutator* and the *super Jordan product* of any two elements $x, y \in A$ are defined respectively as $[x, y] := xy - (-1)^{\overline{x}\,\overline{y}} yx$ and $x \circ y := xy + (-1)^{\overline{x}\,\overline{y}} yx$. For any $x, y, z \in A$, the *associator* (x, y, z) is defined as $(x, y, z) := xy \cdot z - x \cdot yz$.

Definition 2. *([2], [5]). A superalgebra $(A, \cdot)$ is called a Lie superalgebra if*

$xy = -(-1)^{\overline{x}\,\overline{y}} yx$ *(super anticommutativity),*

$xy \cdot z + (-1)^{\overline{x}(\overline{y}+\overline{z})} yz \cdot x + (-1)^{\overline{z}(\overline{x}+\overline{y})} zx \cdot y = 0$ *(super Jacobi identity)*

for all $x, y, z \in A$. A superalgebra $(A, \cdot)$ is said to be Lie-admissible, if $(A, [,])$ is a Lie superalgebra.

Definition 3. *([17]). A superalgebra A is said to be right alternative if*

$(x,y,z) = -(-1)^{\overline{y}\,\overline{z}}(x,z,y)$ *(right superalternativity)* (1)

for all $x,y,z \in A$. *If, moreover, the left superalternativity* $(x,y,z) = -(-1)^{\overline{x}\,\overline{y}}(y,x,y)$ *holds in* A*, then* A *is said to be alternative.*

If A has zero odd part, then (1) reads as $(x,y,z) = -(x,z,y)$ which is the linearized form of the right alternativity $xy \cdot y = x \cdot yy$.

The trilinear function $(x,y,z) + (y,z,x) + (z,x,y)$ is shown to be very useful in the study of nonassociative algebras. Here we consider its $\mathbb{Z}_2$-graded version as

$$S(x,y,z) := (x,y,z) + (-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x) + (-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)$$

and, as for usual algebras, define "$(-1,1)$-*superalgebras*" as right alternative superalgebras satisfying $S(x,y,z) = 0$.

Consider in A the following multilinear function

$$f(w,x,y,z) := (wx,y,z) - (w,xy,z) + (w,x,yz) - w(x,y,z) - (w,x,y)z.$$

The following identities hold in any superalgebra.

Proposition 1. *Let* $(A,\cdot)$ *be a superalgebra. Then*

$$f(w,x,y,z) = 0, \tag{2}$$

$$[xy,z] - x[y,z] - (-1)^{\overline{y}\,\overline{z}}[x,z]y = (x,y,z) - (-1)^{\overline{y}\,\overline{z}}(x,z,y) + (-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y), \tag{3}$$

$$[xy,z] - [x,y]z + (-1)^{\overline{y}\,\overline{z}}[xz,y] - (-1)^{\overline{y}\,\overline{z}}[x,z]y = (-1)^{\overline{x}\,\overline{y}}(y,x,z) + (-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y), \tag{4}$$

$$[xy,z] + (-1)^{\overline{x}(\overline{y}+\overline{z})}[yz,x] + (-1)^{\overline{z}(\overline{x}+\overline{y})}[zx,y] = S(x,y,z), \tag{5}$$

$$(x \circ y) \circ z - (-1)^{\overline{y}\,\overline{z}}(x \circ z) \circ y = S(x,y,z) - (-1)^{\overline{x}\,\overline{y}}S(y,x,z) + 2(-1)^{\overline{x}\,\overline{y}}(y,x,z) - 2(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y) + [x,[y,z]] \tag{6}$$

for all w,x,y,z *in* A.

Proof. The identity (2) follows by direct expansion of associators in $f(w,x,y,z)$. Next we have

$$\begin{aligned}
&[xy,z] - x[y,z] - (-1)^{\overline{y}\,\overline{z}}[x,z]y = xy \cdot z - (-1)^{\overline{z}(\overline{x}+\overline{y})}z \cdot xy \\
&-x(yz - (-1)^{\overline{y}\,\overline{z}}zy) - (-1)^{\overline{y}\,\overline{z}}(xz - (-1)^{\overline{x}\,\overline{z}}zx)y = \{xy \cdot z - x \cdot yz\} \\
&-(-1)^{\overline{y}\,\overline{z}}\{xz \cdot y - x \cdot zy\} + (-1)^{\overline{z}(\overline{x}+\overline{y})}\{zx \cdot y - z \cdot xy\} \\
&= (x,y,z) - (-1)^{\overline{y}\,\overline{z}}(x,z,y) + (-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)
\end{aligned}$$

and so we get (3). As for (4), we compute

$$\begin{aligned}
&[xy,z] - [x,y]z + (-1)^{\overline{y}\,\overline{z}}[xz,y] - (-1)^{\overline{y}\,\overline{z}}[x,z]y = xy \cdot z \\
&-(-1)^{\overline{z}(\overline{x}+\overline{y})}z \cdot xy - (xy - (-1)^{\overline{x}\,\overline{y}}yx)z + (-1)^{\overline{y}\,\overline{z}}(xz \cdot y - (-1)^{\overline{y}(\overline{x}+\overline{z})}y \cdot xz) \\
&-(-1)^{\overline{y}\,\overline{z}}(xz - (-1)^{\overline{x}\,\overline{z}}zx)y = (-1)^{\overline{x}\,\overline{y}}(yx \cdot z - y \cdot xz) \\
&+(-1)^{\overline{z}(\overline{x}+\overline{y})}(zx \cdot y - z \cdot xy) = (-1)^{\overline{x}\,\overline{y}}(y,x,z) + (-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y),
\end{aligned}$$

which gives (4).

Expanding the associators in the right-hand side of (5) and next rearranging terms, we get its left-hand side.

Expanding the left-hand side of (6), we have

$$
\begin{aligned}
&(x\circ y)\circ z-(-1)^{\overline{y}\,\overline{z}}(x\circ z)\circ y = xy\cdot z+(-1)^{\overline{x}\,\overline{y}+\overline{x}\,\overline{z}+\overline{y}\,\overline{z}}z\cdot yx-(-1)^{\overline{y}\,\overline{z}}xz\cdot y\\
&\quad -(-1)^{\overline{x}(\overline{y}+\overline{z})}y\cdot zx-(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)+(-1)^{\overline{x}\,\overline{y}}(y,x,z)\\
&\quad = xy\cdot z+(-1)^{\overline{x}\,\overline{y}+\overline{x}\,\overline{z}+\overline{y}\,\overline{z}}z\cdot yx-(-1)^{\overline{x}\,\overline{y}+\overline{x}\,\overline{z}+\overline{y}\,\overline{z}}zy\cdot x+(-1)^{\overline{x}\,\overline{y}+\overline{x}\,\overline{z}+\overline{y}\,\overline{z}}zy\cdot x\\
&\quad -(-1)^{\overline{y}\,\overline{z}}xz\cdot y-(-1)^{\overline{x}(\overline{y}+\overline{z})}y\cdot zx+(-1)^{\overline{x}(\overline{y}+\overline{z})}yz\cdot x-(-1)^{\overline{x}(\overline{y}+\overline{z})}yz\cdot x\\
&\quad -(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)+(-1)^{\overline{x}\,\overline{y}}(y,x,z)\\
&\quad = (x,y,z)-(-1)^{\overline{x}\,\overline{y}+\overline{x}\,\overline{z}+\overline{y}\,\overline{z}}(z,y,x)+(-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x)\\
&\quad -(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)+(-1)^{\overline{x}\,\overline{y}}(y,x,z)\\
&\quad +x\cdot yz-(-1)^{\overline{x}(\overline{y}+\overline{z})}[y,z]\cdot x-(-1)^{\overline{y}\,\overline{z}}xz\cdot y\\
&\quad = (x,y,z)-(-1)^{\overline{x}\,\overline{y}+\overline{x}\,\overline{z}+\overline{y}\,\overline{z}}(z,y,x)+(-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x)\\
&\quad -(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)+(-1)^{\overline{x}\,\overline{y}}(y,x,z)+[x,[y,z]]-(-1)^{\overline{y}\,\overline{z}}(x,z,y)\\
&\quad = (x,y,z)+(-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x)+(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)\\
&\quad -2(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)+2(-1)^{\overline{x}\,\overline{y}}(y,x,z)\\
&\quad -(-1)^{\overline{x}\,\overline{y}}(y,x,z)-(-1)^{\overline{y}\,\overline{z}}(x,z,y)-(-1)^{\overline{x}\,\overline{y}+\overline{x}\,\overline{z}+\overline{y}\,\overline{z}}(z,y,x)+[x,[y,z]]\\
&\quad = S(x,y,z)-(-1)^{\overline{x}\,\overline{y}}S(y,x,z)+2(-1)^{\overline{x}\,\overline{y}}(y,x,z)-2(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)+[x,[y,z]]
\end{aligned}
$$

and so we get (6).

The identity (2) is usually called the *Teichmüller identity.* Observe that, up to $(-1)^{\overline{y}\,\overline{z}}$, the identity (4) is symmetric with respect to y and z.

Upon the additional requirement of right superalternativity or alternativity on $(A,\cdot)$, we have the following two corollaries.

Corollary 1. *If $(A,\cdot)$ is a right alternative superalgebra, then*

$$(x\circ y)\circ z-(-1)^{\overline{y}\,\overline{z}}(x\circ z)\circ y = 2(x,y,z)+[x,[y,z]], \tag{7}$$

$$[x,y]z-x[y,z]-(-1)^{\overline{y}\,\overline{z}}[xz,y] = 2(x,y,z)+(-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x), \tag{8}$$

$$S(x,y,z)+(-1)^{\overline{y}\,\overline{z}}S(x,z,y)=0, \tag{9}$$

$$[x\circ y,z]+(-1)^{\overline{x}(\overline{y}+\overline{z})}[y\circ z,x]+(-1)^{\overline{z}(\overline{x}+\overline{y})}[z\circ x,y]=0, \tag{10}$$

$$[[x,y],z]+(-1)^{\overline{x}(\overline{y}+\overline{z})}[[y,z],x]+(-1)^{\overline{z}(\overline{x}+\overline{y})}[[z,x],y]=2S(x,y,z) \tag{11}$$

for all x,y,z in A. In particular, $(A,\cdot)$ is Lie-admissible if and only if $S(x,y,z)=0$ i.e. $(A,\cdot)$ is $(-1,1)$.

Proof. The application of the right superalternativity (1) to the right-hand side of (6) gives (7).

Subtracting memberwise (4) from (3) and next using the right superalternativity, we get (8).

The identity (9) follows by direct expansion of $S(x,y,z)$, $S(x,z,y)$ and the use of the right superalternativity.

As for (10), we start from (9) and replace $S(x,y,z)$ and $S(x,z,y)$ with their corresponding expressions from (5). Then, rearranging terms, we get (10) with the definition of the super Jordan product in mind.

In (3) let permute x and y and next multiply by $(-1)^{\overline{x}\,\overline{y}}$ to get

$$
\begin{aligned}
&[(-1)^{\overline{x}\,\overline{y}}yx,z]-(-1)^{\overline{x}\,\overline{y}}y[x,z]-(-1)^{\overline{x}\,\overline{y}}[y,z]x=(-1)^{\overline{x}\,\overline{y}}(y,x,z)-(-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x)\\
&+(-1)^{\overline{x}\,\overline{y}+\overline{z}(\overline{x}+\overline{y})}(z,y,x).
\end{aligned}
$$

Now, subtracting memberwise the equality above from (3), we get

$$[xy,z] - x[y,z] - (-1)^{\overline{y}\,\overline{z}}[x,z]y - [(-1)^{\overline{x}\,\overline{y}}yx,z] - (-1)^{\overline{x}\,\overline{y}}y[x,z] - (-1)^{\overline{x}\,\overline{y}}[y,z]x$$
$$= (x,y,z) - (-1)^{\overline{y}\,\overline{z}}(x,z,y) + (-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y) - (-1)^{\overline{x}\,\overline{y}}(y,x,z)$$
$$+(-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x) - (-1)^{\overline{x}\,\overline{y}+\overline{z}(\overline{x}+\overline{y})}(z,y,x)$$

i.e.

$$\{[xy,z] - [(-1)^{\overline{x}\,\overline{y}}yx,z]\} + \{(-1)^{\overline{x}(\overline{y}+\overline{z})}[y,z]x - x[y,z]\}$$
$$+\{-(-1)^{\overline{y}\,\overline{z}}[x,z]y + (-1)^{\overline{x}\,\overline{y}}y[x,z]\} = \{(x,y,z) - (-1)^{\overline{y}\,\overline{z}}(x,z,y)\}$$
$$+\{-(-1)^{\overline{x}\,\overline{y}}(y,x,z) + (-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x)\} + \{(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)$$
$$-(-1)^{\overline{x}\,\overline{y}+\overline{z}(\overline{x}+\overline{y})}(z,y,x)\}$$

and so, by the definition of the supercommutator and the right superalternativity, we come to (11).

The last assertion of the corollary is obvious.

Corollary 2. *If $(A,\cdot)$ is a right alternative superalgebra, then*

$$[x\circ y,z] = (-1)^{\overline{y}\,\overline{z}}[x,z]\circ y + x\circ[y,z] + 2(x,y,z) + 2(-1)^{\overline{x}\,\overline{y}}(y,x,z) \tag{12}$$

for all x,y,z in A. Moreover, if $(A,\cdot)$ is alternative, then

$$[x\circ y,z] = (-1)^{\overline{y}\,\overline{z}}[x,z]\circ y + x\circ[y,z]. \tag{13}$$

Proof. In (3) let permute x and y and next multiply each term by $(-1)^{\overline{x}\,\overline{y}}$. Then we get

$$[(-1)^{\overline{x}\,\overline{y}}yx,z] - (-1)^{\overline{x}\,\overline{y}}y[x,z] - (-1)^{\overline{x}\,\overline{y}}(-1)^{\overline{x}\,\overline{z}}[y,z]x$$
$$= (-1)^{\overline{x}\,\overline{y}}(y,x,z) - (-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x) + (-1)^{\overline{x}\,\overline{y}+\overline{z}(\overline{x}+\overline{y})}(z,y,x). \tag{14}$$

Adding memberwise (3) and (14) and next rearranging terms, we obtain

$$[x\circ y,z] - x\circ[y,z] - (-1)^{\overline{x}\,\overline{y}}y\circ[x,z]$$
$$= \{(x,y,z) - (-1)^{\overline{y}\,\overline{z}}(x,z,y)\} + \{(-1)^{\overline{x}\,\overline{y}}\{(y,x,z) - (-1)^{\overline{x}(\overline{y}+\overline{z})}(y,z,x)\}$$
$$+\{(-1)^{\overline{x}\,\overline{y}+\overline{z}(\overline{x}+\overline{y})}(z,y,x) + (-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)\}$$
$$= 2(x,y,z) + 2(-1)^{\overline{x}\,\overline{y}}(y,x,z) \text{ (by the right superalternativity)}$$

which proves (12).

The identity (13) follows from (12) by the left superalternativity.

Remark 1. *If $(A,\cdot)$ has zero odd part, then the identities (2)-(13) reduce to their ungraded counterparts in usual algebras.*

3. Main results

Throughout this section, $(A,\cdot)$ denotes a right alternative superalgebra and we will prove some fundamental identities characterizing right alternative superalgebras.

First, we define on $(A,\cdot)$ the following multilinear function

$$g(x,w,y,z) := (-1)^{\overline{w}(\overline{y}+\overline{z})}(x,w,yz) + (-1)^{\overline{w}\,\overline{z}}(x,y,wz)$$

$$-(-1)^{\overline{w}\,\overline{z}+\overline{w}\,\overline{y}+\overline{y}\,\overline{z}}(x,w,z)y-(x,y,z)w.$$

One observes that if A has zero odd part, then the function $g(x,w,y,z)$ is precisely the one defined in [7]. As a tool in the proof of part of the results below, we show that $g(x,w,y,z)$ is identically zero.

Lemma 1. *For all w,x,y,z in A, the following identity holds:*
$$g(x,w,y,z)=0. \tag{15}$$

Proof. By (2) and right superalternativity (1), we have

$$\begin{aligned}
0 &= (-1)^{\overline{w}(\overline{y}+\overline{z})}f(x,w,y,z)-(-1)^{\overline{y}\,\overline{z}}f(x,z,y,w)+(-1)^{\overline{w}\,\overline{z}+\overline{w}\,\overline{y}+\overline{y}\,\overline{z}}f(x,w,z,y)\\
&\quad+(-1)^{\overline{w}\,\overline{z}}f(x,y,w,z)-(-1)^{\overline{y}(\overline{w}+\overline{z})}f(x,z,w,y)+f(x,y,z,w)\\
&= (-1)^{\overline{w}(\overline{y}+\overline{z})}\{(xw,y,z)-(x,wy,z)+(x,w,yz)-x(w,y,z)-(x,w,y)z\}\\
&\quad-(-1)^{\overline{y}\,\overline{z}}\{(xz,y,w)-(x,zy,w)+(x,z,yw)-x(z,y,w)-(x,z,y)w\}\\
&\quad+(-1)^{\overline{w}\,\overline{z}+\overline{w}\,\overline{y}+\overline{y}\,\overline{z}}\{(xw,z,y)-(x,wz,y)+(x,w,zy)-x(w,z,y)-(x,w,z)y\}\\
&\quad+(-1)^{\overline{w}\,\overline{z}}\{(xy,w,z)-(x,yw,z)+(x,y,wz)-x(y,w,z)-(x,y,w)z\}\\
&\quad-(-1)^{\overline{y}(\overline{w}+\overline{z})}\{(xz,w,y)-(x,zw,y)+(x,z,wy)-x(z,w,y)-(x,z,w)y\}\\
&\quad+\{(xy,z,w)-(x,yz,w)+(x,y,zw)-x(y,z,w)-(x,y,z)w\}\\
&= 2[(-1)^{\overline{w}(\overline{y}+\overline{z})}(x,w,yz)+(-1)^{\overline{w}\,\overline{z}}(x,y,wz)\\
&\quad-(-1)^{\overline{w}\,\overline{z}+\overline{w}\,\overline{y}+\overline{y}\,\overline{z}}(x,w,z)y-(x,y,z)w] \text{ (after rearranging terms)}\\
&= 2g(x,w,y,z)
\end{aligned}$$

and so we get (15).

We can now prove the following

Theorem 1. *In $(A,\cdot)$, the identities*

$$(wx,y,z)+(w,x,[y,z])=(-1)^{\overline{x}(\overline{y}+\overline{z})}(w,y,z)x+w(x,y,z), \tag{16}$$
$$(x,z,y\circ w)=(x,z\circ y,w)+(-1)^{\overline{w}\,\overline{y}}(x,z\circ w,y) \tag{17}$$

hold for all w,x,y,z in A.

Proof. We have

$$\begin{aligned}
0 &= f(w,x,y,z)-g(w,z,x,y) \text{ (by (2) and (15))}\\
&= (wx,y,z)-(w,xy,z)+(w,x,yz)-w(x,y,z)\\
&\quad-(w,x,y)z-(-1)^{\overline{z}(\overline{x}+\overline{y})}(w,z,xy)-(-1)^{\overline{y}\,\overline{z}}(w,x,zy)\\
&\quad+(-1)^{\overline{x}(\overline{y}+\overline{z})+\overline{y}\,\overline{z}}(w,z,y)x+(w,x,y)z\\
&= (wx,y,z)+(w,x,[y,z])-(-1)^{\overline{x}(\overline{y}+\overline{z})}(w,y,z)x-w(x,y,z)\\
&\quad\text{(by right superalternativity)}
\end{aligned}$$

which yields (16). As for (17), we proceed as follows.

$$\begin{aligned}
0 &= (-1)^{\overline{w}\,\overline{y}}f(x,z,w,y)+f(x,z,y,w) \text{ (by (2))}\\
&= \{(-1)^{\overline{w}\,\overline{y}}(xz,w,y)-(-1)^{\overline{w}\,\overline{y}}(x,zw,y)+(-1)^{\overline{w}\,\overline{y}}(x,z,wy)\\
&\quad-(-1)^{\overline{w}\,\overline{y}}x(z,w,y)-(-1)^{\overline{w}\,\overline{y}}(x,z,w)y\}\\
&\quad+\{(xz,y,w)-(x,zy,w)+(x,z,yw)-x(z,y,w)-(x,z,y)w\}\\
&= -(-1)^{\overline{w}\,\overline{y}}(x,zw,y)-(x,zy,w)+(x,z,yw)
\end{aligned}$$

$+(-1)^{\overline{w}\,\overline{y}}(x,z,wy)+(-1)^{\overline{w}(\overline{y}+\overline{z})}(x,w,z)y+(-1)^{\overline{y}\,\overline{z}}(x,y,z)w$
$+[(-1)^{\overline{w}\,\overline{y}}(xz,w,y)+(xz,y,w)-(-1)^{\overline{w}\,\overline{y}}x(z,w,y)-x(z,y,w)]$
$=(x,zw,y)-(x,zy,w)+(x,z,yw)+(-1)^{\overline{w}\,\overline{y}}(x,z,wy)$
$+(-1)^{\overline{w}(\overline{y}+\overline{z})}(x,w,z)y+(-1)^{\overline{y}\,\overline{z}}(x,y,z)w$ (since, by right superalternativity, the expression in bracket above is zero)
$=-(-1)^{\overline{w}\,\overline{y}}(x,zw,y)-(x,zy,w)+(x,z,yw)$
$+(-1)^{\overline{w}\,\overline{y}}(x,z,wy)+(-1)^{\overline{w}(\overline{y}+\overline{z})+\overline{y}\,\overline{z}}(x,w,yz)+(-1)^{\overline{z}(\overline{w}+\overline{y})}(x,y,wz)$ (by (15))
$=-(-1)^{\overline{w}\,\overline{y}}(x,zw,y)-(x,zy,w)+(x,z,y\circ w)$
$-(-1)^{\overline{y}\,\overline{z}}(x,yz,w)-(-1)^{\overline{w}(\overline{y}+\overline{z})}(x,wz,y)$,
which leads to (17).

In order to prove the identity (23) below, we first prove that the following identity holds in $(A,\cdot)$.

Lemma 2. *The identity*

$$\begin{aligned}&(-1)^{(\bar{t}+\overline{y})(\overline{w}+\overline{z})+\bar{t}\,\overline{y}}[(x,yz,w)+(-1)^{\overline{w}\,\overline{z}}(x,yw,z)+(x,y,z)w+(-1)^{\overline{w}\,\overline{z}}(x,y,w)z]\cdot t\\ &\quad+[(x,tz,w)+(-1)^{\overline{w}\,\overline{z}}(x,tw,z)+(x,t,z)w+(-1)^{\overline{w}\,\overline{z}}(x,t,w)z]\cdot y\\ &=-(x,t\cdot(z\circ w),y)+(-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w))\end{aligned}\tag{18}$$

holds for all t,w,x,y,z in A.

Proof. Write (15) as

$$(x,y,wz)=-(-1)^{\overline{w}\,\overline{y}}(x,w,yz)+(-1)^{\overline{y}(\overline{w}+\overline{z})}(x,w,z)y+(-1)^{\overline{w}\,\overline{z}}(x,y,z)w.\tag{19}$$

Now in (19) replace y with t, w with y and z with $z\circ w$ to get

$$\begin{aligned}(x,t,y\cdot(z\circ w))&=-(-1)^{\bar{t}\,\overline{y}}(x,y,t\cdot(z\circ w))+(-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,z\circ w)y\\ &\quad+(-1)^{\bar{t}(\overline{w}+\overline{y}+\overline{z})}(x,y,z\circ w)t\end{aligned}$$

i.e.

$$\begin{aligned}&(-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w))-(x,t\cdot(z\circ w),y)\\ &=(x,t,z\circ w)y+(-1)^{(\bar{t}+\overline{y})(\overline{w}+\overline{z})+\bar{t}\,\overline{y}}(x,y,z\circ w)t.\end{aligned}\tag{20}$$

Next we have

$$\begin{aligned}(x,t,z\circ w)&=(x,t,zw)+(-1)^{\overline{w}\,\overline{z}}(x,t,wz)\\ &=-(-1)^{\bar{t}\,\overline{z}}(x,z,tw)+(-1)^{\overline{w}\,\overline{z}}(x,t,w)z-(-1)^{\overline{w}(\bar{t}+\overline{z})}(x,w,tz)\\ &\quad+(x,t,z)w \text{ (by (19))}\end{aligned}$$

i.e., by the right superalternativity,

$$(x,t,z\circ w)=(-1)^{\overline{w}\,\overline{z}}(x,tw,z)+(-1)^{\overline{w}\,\overline{z}}(x,t,w)z+(x,tz,w)+(x,t,z)w.\tag{21}$$

Likewise, one has

$$(x,y,z\circ w)=(-1)^{\overline{w}\,\overline{z}}(x,yw,z)+(-1)^{\overline{w}\,\overline{z}}(x,y,w)z+(x,yz,w)+(x,y,z)w.\tag{22}$$

Therefore, from (20) and replacing $(x,t,z\circ w)$ and $(x,y,z\circ w)$ with their expressions from (21) and (22) respectively, we have

$$\begin{aligned}
&(-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w))-(x,t\cdot(z\circ w),y)\\
&=(x,t,z\circ w)y+(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\,\overline{y}}(x,y,z\circ w)t\\
&=[(-1)^{\overline{w}\,\overline{z}}(x,tw,z)+(-1)^{\overline{w}\,\overline{z}}(x,t,w)z+(x,tz,w)+(x,t,z)w]\cdot y\\
&+(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\,\overline{y}}[(-1)^{\overline{w}\,\overline{z}}(x,yw,z)+(-1)^{\overline{w}\,\overline{z}}(x,y,w)z+(x,yz,w)+(x,y,z)w]\cdot t
\end{aligned}$$

which proves (18).

We are now in position to prove the following

Theorem 2. *The identity*

$$\begin{aligned}
&(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\,\overline{y}}[(x,y,z)w+(-1)^{\overline{w}\,\overline{z}}(x,y,w)z]\cdot t\\
&\;+[(x,t,z)w+(-1)^{\overline{w}\,\overline{z}}(x,t,w)z]\cdot y\\
&\;-(-1)^{\overline{t}(\overline{y}+\overline{z})+\overline{y}(\overline{w}+\overline{z})}(x,y,z)\cdot tw-(-1)^{(\overline{t}+\overline{z})(\overline{w}+\overline{y})+\overline{w}\,\overline{y}}(x,y,w)\cdot tz\\
&\;-(-1)^{\overline{w}\,\overline{y}}(x,t,z)\cdot yw-(-1)^{\overline{z}(\overline{w}+\overline{y})}(x,t,w)\cdot yz=0
\end{aligned}\tag{23}$$

holds for all t,w,x,y,z *in* A.

Proof. From (18) we have

$$\begin{aligned}
&(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\,\overline{y}}[(x,y,z)w+(-1)^{\overline{w}\,\overline{z}}(x,y,w)z]\cdot t\\
&\;+[(x,t,z)w+(-1)^{\overline{w}\,\overline{z}}(x,t,w)z]\cdot y\\
&\;=(-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w))-(x,t\cdot(z\circ w),y)\\
&\;-(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\,\overline{y}}[(-1)^{\overline{w}\,\overline{z}}(x,yw,z)+(x,yz,w)]t\\
&\;-[(-1)^{\overline{w}\,\overline{z}}(x,tw,z)+(x,tz,w)]y.
\end{aligned}\tag{24}$$

Now, starting from the left-hand side of (23), we make the following transformations.

$$\begin{aligned}
&\{(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\,\overline{y}}[(x,y,z)w+(-1)^{\overline{w}\,\overline{z}}(x,y,w)z]\cdot t\\
&\;+[(x,t,z)w+(-1)^{\overline{w}\,\overline{z}}(x,t,w)z]\cdot y\}\\
&\;-(-1)^{\overline{t}(\overline{y}+\overline{z})+\overline{y}(\overline{w}+\overline{z})}(x,y,z)\cdot tw-(-1)^{(\overline{t}+\overline{z})(\overline{w}+\overline{y})+\overline{w}\,\overline{y}}(x,y,w)\cdot tz\\
&\;-(-1)^{\overline{w}\,\overline{y}}(x,t,z)\cdot yw-(-1)^{\overline{z}(\overline{w}+\overline{y})}(x,t,w)\cdot yz\\
&\;=\{(-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w))-(x,t\cdot(z\circ w),y)\\
&\;-(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\,\overline{y}}[(-1)^{\overline{w}\,\overline{z}}(x,yw,z)+(x,yz,w)]t\\
&\;-[(-1)^{\overline{w}\,\overline{z}}(x,tw,z)+(x,tz,w)]y\}\\
&\;-(-1)^{\overline{t}(\overline{y}+\overline{z})+\overline{y}(\overline{w}+\overline{z})}(x,y,z)\cdot tw-(-1)^{(\overline{t}+\overline{z})(\overline{w}+\overline{y})+\overline{w}\,\overline{y}}(x,y,w)\cdot tz\\
&\;-(-1)^{\overline{w}\,\overline{y}}(x,t,z)\cdot yw-(-1)^{\overline{z}(\overline{w}+\overline{y})}(x,t,w)\cdot yz
\end{aligned}$$

(replacing the expression in $\{\cdots\}$ above by its equivalent counterpart from (24))

$$\begin{aligned}
&=(-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w))-(x,t\cdot(z\circ w),y)\\
&\;-\{(-1)^{\overline{t}(\overline{y}+\overline{z})+\overline{y}(\overline{w}+\overline{z})}(x,y,z)\cdot tw+(-1)^{\overline{w}\,\overline{z}}(x,tw,z)y\}\\
&\;-\{(-1)^{(\overline{t}+\overline{z})(\overline{w}+\overline{y})+\overline{w}\,\overline{y}}(x,y,w)\cdot tz+(x,tz,w)y\}\\
&\;-\{(-1)^{\overline{w}\,\overline{y}}(x,t,z)\cdot yw+(-1)^{(\overline{t}+\overline{z})(\overline{w}+\overline{y})+\overline{t}\,\overline{z}+\overline{w}\,\overline{y}}(x,yw,z)t\}\\
&\;-\{(-1)^{\overline{z}(\overline{w}+\overline{y})}(x,t,w)\cdot yz+(-1)^{(\overline{t}+\overline{y})(\overline{w}+\overline{z})+\overline{t}\,\overline{y}}(x,yz,w)t\}
\end{aligned}$$

(rearranging terms)

$$=(-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w))-(x,t\cdot(z\circ w),y)$$

$-\{(-1)^{\overline{z}(\overline{w}+\overline{y})}(x,tw,yz) - (-1)^{\overline{w}\,\overline{z}}(x,tw\cdot z,y)\}$
$-\{(-1)^{\overline{w}\,\overline{y}}(x,tz,yw) - (x,tz\cdot w,y)\}$
$-\{(-1)^{(\overline{t}+\overline{z})(\overline{w}+\overline{y})+\overline{w}\,\overline{y}}(x,yw,tz) + (-1)^{\overline{z}(\overline{w}+\overline{y})+\overline{w}\,\overline{y}}(x,t,yw\cdot z)\}$
$-\{(-1)^{\overline{y}(\overline{t}+\overline{w})+\overline{z}(\overline{t}+\overline{y})}(x,yz,tw) + (-1)^{\overline{y}(\overline{z}+\overline{w})}(x,t,yz\cdot w)\}$ (applying (15) and next the right superalternativity to each of the expressions in $\{\cdots\}$ above)
$= (-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w)) - (x,t\cdot(z\circ w),y)$
$+(x,tz\cdot w + (-1)^{\overline{w}\,\overline{z}}tw\cdot z,y) - (-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,yz\cdot w + (-1)^{\overline{w}\,\overline{z}}yw\cdot z)$
$-\{(-1)^{\overline{z}(\overline{w}+\overline{y})}(x,tw,yz) + (-1)^{\overline{y}(\overline{t}+\overline{w})+\overline{z}(\overline{t}+\overline{y})}(x,yz,tw)\}$
$-\{(-1)^{\overline{w}\,\overline{y}}(x,tz,yw) + (-1)^{(\overline{t}+\overline{z})(\overline{w}+\overline{y})+\overline{w}\,\overline{y}}(x,yw,tz)\}$ (rearranging terms)
$= (-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w)) - (x,t\cdot(z\circ w),y)$
$+(x,tz\cdot w + (-1)^{\overline{w}\,\overline{z}}tw\cdot z,y) - (-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,yz\cdot w + (-1)^{\overline{w}\,\overline{z}}yw\cdot z)$ (applying the right superalternativity to each of the expressions in $\{\cdots\}$ above)
$= (-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w)) - (x,t\cdot(z\circ w),y)$
$+(x,t\cdot zw + (-1)^{\overline{w}\,\overline{z}}t\cdot wz,y) - (-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot zw + (-1)^{\overline{w}\,\overline{z}}y\cdot wz)$ (by the right superalternativity)
$= (-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w)) - (x,t\cdot(z\circ w),y)$
$+(x,t\cdot(z\circ w),y) - (-1)^{\overline{y}(\overline{w}+\overline{z})}(x,t,y\cdot(z\circ w))$
$= 0$
and thus we obtain (23).

Remark 2. *It is easily seen that the identities (15)-(18) and (23) are the $\mathbb{Z}_2$-graded generalization of identities*

$$(x,w,yz) + (x,y,wz) - (x,w,z)y - (x,y,z)w = 0, \tag{25}$$
$$(wx,y,z) + (w,x,[y,z]) = w(x,y,z) + (w,y,z)x, \tag{26}$$
$$(x,y^2,z) = (x,y,yz+zy), \tag{27}$$
$$(x,y,y\cdot z^2) = (x,yz,z)y + (x,y,z)z\cdot y, \tag{28}$$
$$(x,y,z)y\cdot z = (x,y,z)\cdot zy \tag{29}$$

respectively, all of which could be found in [7], [14], [16].

4. Some consequences

The main goal in this section is to prove that some $\mathbb{Z}_2$-graded Moufang-type identities hold in right alternative superalgebras. The identities in this section are more or less direct consequences of identity (15).

Theorem 3. *Let $(A,\cdot)$ be a right alternative superalgebra. Then*

$$(xy\cdot z)w + (-1)^{\overline{y}\,\overline{z}+\overline{w}\,\overline{y}+\overline{w}\,\overline{z}}(xw\cdot z)y = x(yz\cdot w) + (-1)^{\overline{y}\,\overline{z}+\overline{w}\,\overline{y}+\overline{w}\,\overline{z}}x(wz\cdot y) \tag{30}$$

for all w,x,y,z in A.

Proof. In $(A,\cdot)$, by (15) we have

$(-1)^{\overline{w}(\overline{y}+\overline{z})}(x,w,yz)+(-1)^{\overline{w}\,\overline{z}}(x,y,wz)=(-1)^{\overline{w}\,\overline{z}+\overline{w}\,\overline{y}+\overline{y}\,\overline{z}}(x,w,z)y+(x,y,z)w$

i.e., by the right superalternativity,

$$-(x,yz,w)-(-1)^{\overline{w}(\overline{y}+\overline{z})+\overline{y}\,\overline{z}}(x,wz,y)=(-1)^{\overline{w}(\overline{y}+\overline{z})+\overline{y}\,\overline{z}}(x,w,z)y+(x,y,z)w. \tag{31}$$

Next, expanding associators in (31), one gets (30).

Remark 3. *If A has zero odd part and $y=w$, from (30) one gets the right Bol identity $(xy\cdot z)y=x(yz\cdot y)$ formerly called the "right Moufang identity" (see, e.g., [9] and [4]).*

In case when $(A,\cdot)$ is alternative, then (31) yields

$$\begin{aligned}&(-1)^{\overline{x}(\overline{y}+\overline{z})}(yz,x,w)+(-1)^{\overline{x}(\overline{w}+\overline{z})+\overline{w}(\overline{y}+\overline{z})+\overline{y}\,\overline{z}}(wz,x,y)\\&=(-1)^{\overline{z}(\overline{x}+\overline{y})}(z,x,y)w+(-1)^{\overline{z}(\overline{x}+\overline{y})+\overline{w}\,\overline{y}}(z,x,w)y.\end{aligned} \tag{32}$$

If, moreover, A has zero odd part, then (32) reads as

$(yz,x,w)+(wz,x,y)=(z,x,y)w+(z,x,w)y$

which is the linearized form of the middle Moufang identity ([4])

$(yz,x,y)-(z,x,y)y=0.$

In this sense, the identity (31) is (in part) closed to the middle Moufang identity.

The following identity is proved to be valid in right alternative algebras:

$$(x,z,y\circ w)=2(x,z,w)y-2(x,y,z)w+(x,[z,y],w)+(x,[z,w],y)$$

(see [10], identity (9)). Its $\mathbb{Z}_2$-graded version is given by the following

Theorem 4. *In a right alternative superalgebra $(A,\cdot)$, the identity*

$$\begin{aligned}(x,z,y\circ w)=&\,2(-1)^{\overline{w}\,\overline{y}}(x,z,w)y-2(-1)^{\overline{y}\,\overline{z}}(x,y,z)w\\&+(x,[z,y],w)+(-1)^{\overline{w}\,\overline{y}}(x,[z,w],y)\end{aligned} \tag{33}$$

holds for all w,x,y,z in A.

Proof. We have

$$\begin{aligned}(x,z,y\circ w)&=(x,z\circ y,w)+(-1)^{\overline{w}\,\overline{y}}(x,z\circ w,y)\ \text{(see (17))}\\&=(x,zy,w)+(-1)^{\overline{y}\,\overline{z}}(x,yz,w)+(-1)^{\overline{w}\,\overline{y}}(x,zw,y)\\&\quad+(-1)^{\overline{w}(\overline{y}+\overline{z})}(x,wz,y)\\&=(x,[z,y],w)+2(-1)^{\overline{y}\,\overline{z}}(x,yz,w)\\&\quad+(-1)^{\overline{w}\,\overline{y}}(x,[z,w],y)+2(-1)^{\overline{w}(\overline{y}+\overline{z})}(x,wz,y)\\&=(x,[z,y],w)+(-1)^{\overline{w}\,\overline{y}}(x,[z,w],y)\\&\quad+2\{(-1)^{\overline{y}\,\overline{z}}(x,yz,w)+(-1)^{\overline{w}(\overline{y}+\overline{z})}(x,wz,y)\}\\&=(x,[z,y],w)+(-1)^{\overline{w}\,\overline{y}}(x,[z,w],y)\\&\quad-2(-1)^{\overline{y}\,\overline{z}}\{(-1)^{\overline{w}\,\overline{y}+\overline{w}\,\overline{z}+\overline{y}\,\overline{z}}(x,w,z)y+(x,y,z)w\}\ \text{(by (15))}\\&=(x,[z,y],w)+(-1)^{\overline{w}\,\overline{y}}(x,[z,w],y)\end{aligned}$$

$+2(-1)^{\overline{w}\,\overline{y}}(x,z,w)y - 2(-1)^{\overline{y}\,\overline{z}}(x,y,z)w$ (by the right superalternativity) and so we get (33).

The additional requirement of left superalternativity on $(A,\cdot)$ (and so $(A,\cdot)$ is alternative) yields from identities of this paper many other identities that are thusly the $\mathbb{Z}_2$-graded generalization of identities characterizing alternative algebras.

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