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Solving a quartic equation and certain equations with degree n

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Abstract. In this paper we describe a new method to solve the general quartic equation. This approch is simple and different from ferrari's method, in addition we can solve some algebraic equations with degree n by using the technics proposed in this method.

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Key Words and Phrases: Quartic equation, cubic equation, Ferrari's method

1. Introduction

From the results of Galois (1832) [1] and Abel (1826) [1], we know that a general equation with degree n can not be solved in radicals. Now, we can solve every quartic by using the Ferrari's method [3] which published in 1545. In this paper we propose a new method of solving a general quartic which is different from the Ferrari's method, and we can determine certain equations of degree $n \geq 5$ solvable by the proposed method.

2. Preliminary results

Lemma 1. *Let P be a polynomial function with degree $n \geq 1$ in $\mathbb{C}[x]$ such that*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0.$$

If α is a root of P, and λ is a complexe number, then $\lambda\alpha$ is a root of $\lambda * P$ defined as

$$(\lambda * P)(x) = a_n x^n + a_{n-1}\lambda x^{n-1} + a_{n-2}\lambda^2 x^{n-2} ... + a_1 \lambda^{n-1} x + a_0 \lambda^n.$$

Lemma 2. *Every equation of degree $n \geq 1$ as*

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0,$$

can be written as

$$y^n + b_{n-2}y^{n-2} + ... + b_1 y + b_0 = 0.$$

Proof:
We can use this change of variable : $y = x + \dfrac{a_{n-1}}{n a_n}$.

Notation : Now we consider this notation : $\sqrt{\gamma}$ is a root of $z^2 = \gamma$.

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3. Ferrari's method

By lemma 2, all quartic equation can be reduced to a simple form with only tree parameters. That is,

$$(E) \qquad X^4 + TX^2 = ZX + W,$$

where $T, Z \, and \, W$ are real numbers.
This equation is equivalent to :

$$(E) \qquad X^4 + TX^2 - ZX - W = 0.$$

Let $y \in \mathbb{R}^*$, we have :

$$\begin{aligned} X^4 + TX^2 - ZX - W &= (X^2 + y)^2 - 2yX^2 + TX^2 - y^2 - ZX - W \\ &= (X^2 + y)^2 - 2yX^2 + TX^2 - y^2 - ZX - W \\ &= (X^2 + y)^2 + (T - 2y)X^2 - ZX - y^2 - W. \end{aligned}$$

We can choose that value of $y = y_0$ when this term $(2y - T)X^2 + ZX + y^2 + W$ is a square, for that his discriminant is zero.
We get :

$$Z^2 - 4(y_0^2 + W)(2y_0 - T) = 0.$$

Then we deduce that y_0 is a root of this cubic equation :

$$8y^3 - 4Ty^2 + 8Wy - Z^2 = 0.$$

Remark 1. *To get the value of y_0, we can apply one of methods for solving the cubic equation (for exmaple : Cardan's method [3] and [1]).*

4. Kulkarni's method

This method was published in 2009 [4] (rectified in 2011 [2]) is similar to Ferrari's method. It's based on the possibility of writing every quartic as the difference of two squares, for that we solve a system with 4 unknowns, and we obtained a new cubic equation. We can see the proof as follows
Consider this general equation

$$x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0.$$

Consider another quartic equation shown below.

$$(x^2 + b_1x + b_0)^2 - p(x + c)^2 = 0.$$

We attempt to represent the first quartic in the form of the Latter. For that we obtained the system

$$\begin{cases} 2b_1 = a_3 \\ b_1^2 + 2b_0 - p = a_2 \\ 2(b_0b_1 - pc_0) = a_1 \\ b_0^2 - pc_0^2 = a_0. \end{cases}$$

Let $F_1 = a_2 - (\dfrac{a_3}{2})^2$, and $p_1 = p + F_1$.
After calculus, we obtained this cubic equation with the unknown p_1 :

$$p_1^3 - a_2p_1^2 + (a_3a_1 - 4a_0)p_1 + 4a_0F_1 - a_1^2 = 0.$$

We find the values of p, c, b_1, b_0, and finally we get this decomposition :

$$x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = [x^2 + (b_1 - \sqrt{p})x + b_0 - \sqrt{p}c][x^2 + (b_1 + \sqrt{p})x + b_0 + \sqrt{p}c].$$

5. The proposed method

We use the lemma 2 for rewriting every quartic in the form :

$$(E) \qquad X^4 + TX^2 = ZX + W,$$

where $T, Z\, et\, W$ are complexe numbers.
Next, we can determine two reals a and b as a fuction to T,Z and W, such that the X is a root of :

$$X^2 = aX + b.$$

For that, consider the equation

$$x^2 = x + \varepsilon \qquad (1).$$

Then :

$$x^4 = x^2 + 2\varepsilon x + \varepsilon^2 \qquad (2).$$

By using the equality (1) $x^2 = x + \varepsilon$, we replace x^2 by $x + \varepsilon$ in the equation (2), we find :

$$x^4 = x(2\varepsilon + 1) + \varepsilon^2 + \varepsilon.$$

Then :

$$\begin{cases} x^2 = x + \varepsilon \\ x^4 = x(2\varepsilon + 1) + \varepsilon^2 + \varepsilon \end{cases} .$$

Let $\alpha \in \mathbb{R}$, we have:

$$\begin{cases} \alpha x^2 = \alpha x + \alpha\varepsilon \\ x^4 = x(2\varepsilon + 1) + \varepsilon^2 + \varepsilon. \end{cases}$$

We add the two equations, and we get :

$$x^4 + \alpha x^2 = x(2\varepsilon + 1 + \alpha) + \varepsilon^2 + \varepsilon + \alpha\varepsilon.$$

Let $\lambda \in \mathbb{R}$ and $X = \lambda x$, by using the lemma 1, we deduce that the X is a root of the equation :

$$X^4 + \alpha\lambda^2 X^2 = X(2\varepsilon\lambda^3 + \lambda^3 + \alpha\lambda^3) + \varepsilon^2\lambda^4 + \varepsilon\lambda^4 + \alpha\varepsilon\lambda^4.$$

To solve the quartic (E), we can represent it in the form of the above quartic, for that we can solve this system :

$$(S) : \begin{cases} T = \alpha\lambda^2 & (1') \\ Z = 2\varepsilon\lambda^3 + \lambda^3 + \alpha\lambda^3 & (2') \\ W = \varepsilon^2\lambda^4 + \varepsilon\lambda^4 + \alpha\varepsilon\lambda^4 & (3') \end{cases}$$

a root of E is given by $X = \lambda x$, where x is a root of the equation $x^2 = x + \varepsilon$.
From $(2')$ $Z = 2\varepsilon\lambda^3 + \lambda^3 + \alpha\lambda^2\lambda$, and $T = \alpha\lambda^2$ from $(1')$, we get :

$$Z = 2\varepsilon\lambda^3 + \lambda^3 + T\lambda.$$

Then :

$$\varepsilon\lambda^2 = \frac{Z - T\lambda - \lambda^3}{2\lambda}.$$

Next, we replace $\varepsilon\lambda^2$ by its value, and $\alpha\lambda^2$ by T in equation $(3')$, we get :

$$W = (\frac{Z - T\lambda - \lambda^3}{2\lambda})^2 + \lambda^2(\frac{Z - T\lambda - \lambda^3}{2\lambda}) + T(\frac{Z - T\lambda - \lambda^3}{2\lambda}),$$

$$4\lambda^2 W = (Z - T\lambda - \lambda^3)^2 + 2\lambda^3(Z - T\lambda - \lambda^3) + 2T\lambda(Z - T\lambda - \lambda^3).$$

Then :

$$4\lambda^2 W = (Z - T\lambda - \lambda^3)(Z - T\lambda - \lambda^3 + 2\lambda^3 + 2T\lambda),$$
$$4\lambda^2 W = (Z - T\lambda - \lambda^3)(Z + \lambda^3 + T\lambda.$$

Finally, we deduce:

$$4\lambda^2 W = Z^2 - (\lambda^3 + T\lambda)^2.$$

And

$$\lambda^6 + 2T\lambda^4 + \lambda^2(T^2 + 4W) - Z^2 = 0.$$

Conclusion 1.

- *λ^2 is a root of $y^3 + 2Ty^2 + y(T^2 + 4W) - Z^2 = 0$.*
- *$\alpha = T\lambda^{-2}$ and $\varepsilon = \dfrac{Z - T\lambda - \lambda^3}{2\lambda^3}$.*
- *$\mathcal{S}_\varepsilon$ is the set of roots for the quadratic $x^2 = x + \varepsilon$.*
- *Every element of the set $\mathcal{S}_E = \lambda\mathcal{S}_\varepsilon = \{\lambda x\ ,\ x \in \mathcal{S}_\varepsilon\}$ is a root of (E).*

6. Comparison between Ferrari's method and the new method

6.1. A didactic perspective

In the field of teaching and learning, the choice of a beautiful wording of the resolution of a problem Mathematics is essential to give students the generals rules of calculating which they can learn by heart. In this framework, the new method gives us a simple representation of the solving of a quartic equation over the Ferrari's method, in fact :

- For Ferrari's method : The solving of this quartic

  $$(E) \qquad X^4 + TX^2 = ZX + W,$$

  leads to solving the cubic

  $$8y^3 - 4Ty^2 + 8Wy - Z^2 = 0.$$

- For the new method : We can to give the following simple representation :
  For solving the equation

  $$(E) \qquad X^4 + TX^2 = ZX + W.$$

  Consider Δ the discriminant of the quadratic with the unknown X :

  $$-yX^2 = ZX + W.$$

  To determine the real λ as defined above, for that we solve the cubic $y^3 + 2Ty^2 + y(T^2 + 4W) - Z^2 = 0$ which we can rewrite as the following **simple** form :

  $$y(y + T)^2 = \Delta.$$

6.2. Solving certain equations with degree n

We use the Ferrari's method for solving the quartic equation, However we can use the new method for solving certain equations with degree n:

6.2.1. The case of $n = 5$

We consider the equation :

$$(E_1) \qquad X^3 = aX + b.$$

We can multiply the equation in the both sides by X^2, we get $X^5 = aX^3 + bX^2$, and we replace X^3 by $aX + b$, we get :

$$X^5 = a(aX + b) + bX^2.$$

Then :

$$X^5 = bX^2 + a^2X + ab.$$

We multiply the equation E_1 by c, we get :

$$\begin{cases} X^5 = bX^2 + a^2X + ab \\ cX^3 = caX + cb. \end{cases}$$

We can do the sum of the two equations, we find :

$$(E_2) \qquad X^5 + cX^3 = bX^2 + (a^2 + ca)X + ab + cb.$$

The equation (E_2) is solvable by radicals. Moreover, every root of (E_1) is a root of (E_2).
Next, we can determine the set of the equations which we can rewrite in the form as (E_2).
Consider the equation :

$$X^5 + \alpha X^3 = \beta X^2 + \gamma X + \mu.$$

When this equation is written as (E_2), then we can find tree numbers a, b and c such that:

$$\begin{cases} \alpha = c \\ \beta = b \\ \gamma = a^2 + ca \\ \mu = ab + cb. \end{cases}$$

After the calcul we obtain :

$$\gamma = \frac{\mu^2}{\beta^2} - \alpha\frac{\mu}{\beta}. \qquad (*)$$

We deduce that every root of the equation $(E_1) : X^3 = aX + b$, is a root of (E_2), where $b = \beta$, $c = \alpha$ and $\gamma = a^2 + ca$.
Then : $a = \dfrac{\mu}{\beta} - \alpha$.
And :

$$(E_1) \Leftrightarrow X^3 = (\frac{\mu}{\beta} - \alpha)X + \beta.$$

Conclusion 2. *Every equation is a form as :*

$$(E_4) \qquad X^5 + \alpha X^3 = \beta X^2 + (\frac{\mu^2}{\beta^2} - \alpha\frac{\mu}{\beta})X + \mu,$$

is solvable by radicals. Moreover, for determine the set of roots, it is sufficient to determine the roots of :

$$X^3 = (\frac{\mu}{\beta} - \alpha)X + \beta.$$

- If $\alpha = \frac{\mu}{\beta}$, then $\gamma = 0$ and $a = 0$. The equation (E_4) becomes :

$$(E_5) \qquad X^5 + \frac{\mu}{\beta}X^3 = \beta X^2 + \mu.$$

  The above equation is solvable by radicals. Moreover, tree roots of E_5 is giving by solving the flowing equation :

$$X^3 = \beta.$$

- When $\alpha = 0$, the equation (E_4) becomes :

$$(E_6) \qquad X^5 = \beta X^2 + \frac{\mu^2}{\beta^2}X + \mu.$$

Which are solvable by radicals. For determine the roots, we can to solve the cubic:

$$X^3 = \frac{\mu}{\beta}X + \beta.$$

- Let $\mu = sin(\varphi)$, and $\beta = cos(\varphi)$, where φ are a real number.
The equation (E_5) becomes :

$$(E_7) \qquad X^5 + \tan(\varphi)X^3 = \cos(\varphi)X^2 + \sin(\varphi).$$

We can determine the set of roots by solving the following equation :

$$X^3 = cos(\varphi).$$

6.2.2. The case of $n \geq 5$

In this paragraph we see some examples of the set of the equations with degree n, which we can solve by the new method. We consider the equation with unknown X :

$$X^{n-1} = aX^{n-2} + b.$$

We multiply this equation by X, we get :

$$X^n = aX^{n-1} + bX.$$

We replace X^{n-1} by $aX^{n-2} + b$, we find :

$$X^n = a^2X^{n-2} + bX + ab.$$

Next, we multiply the first equation by c and we add the two equations, we get :

$$X^n + cX^{n-1} = (a^2 + ca)X^{n-2} + bX + ab + bc.$$

Consider the equation

$$X^n + \alpha X^{n-1} = \beta X^{n-2} + \gamma X + \mu,$$

where $\alpha = c$, $\beta = a^2 + ca$,$\gamma = b$ and $\mu = ab + bc$.
We deduce :

$$\beta = \frac{\mu^2}{\gamma^2} - \frac{\alpha\mu}{\gamma}$$

Conclusion 3. *Wen can determine a root of every equation which is written as the following form :*

$$(E_n) \qquad X^n + \alpha X^{n-1} = (\frac{\mu^2}{\gamma^2} - \frac{\alpha\mu}{\gamma})X^{n-2} + \gamma X + \mu.$$

For that, every root of $X^{n-1} = (\frac{\mu}{\gamma} - \alpha)X^{n-2} + \gamma$ *is a root of the above equation* E_n.

6.2.3. In the context of the improvement of the method

The question we can ask now about this new method, is do we improve the new method in the sense of determining a largest set of algebraic equations which we can to solve by the new method?.
It is known that the resolution of every equation by radicals is not always possible from the results given by Galois and Abel, despite that, the classification of every equation by his solvability by radicals is essential.
In this paragraph, I would like to give a few remarks about the generalizability of some form sets of equations solvable by the new method. We begin by studying the general case, and after we move to the case of $n = 5$.

Definition 1. *If we can determine a root of a equation (E) by using of the new methode, this equation is called solvable by the new method.*

- In general : Let $\mathcal{E}$ a set of equations with degree n solvables by the new method.
  Consider $\mathcal{C} = \{(a_0, a_1, ..., a_n) \ / \ P(x) = \sum_{i=0}^{n} a_i x^i \in \mathcal{E}\}$.
  For all number $\lambda \in \mathbf{R}^*$, we put :

  $$\lambda * \mathcal{C} = \{(\lambda^n a_0, \lambda^{n-1} a_1, ..., \lambda a_{n-1}, a_n) \ / \ (a_0, a_1, ..., a_n) \in \mathcal{C}\}$$

  We have $\mathcal{C} \subset \bigcup_{\lambda \in \mathbf{R}} \lambda * \mathcal{C}$ (because $1 * \mathcal{C} = \mathcal{C}$).
  When $\mathcal{C} = \bigcup_{\lambda \in \mathbf{R}} \lambda * \mathcal{C}$, the generalisation of the form concerning the Ã©quations of $\mathcal{E}$ from the lemma 2 it is not possible.

  Example 1. *In the case of the following equation (E_n)*

  $$(E_n) \qquad X^n + \alpha X^{n-1} = (\frac{\mu^2}{\gamma^2} - \frac{\alpha\mu}{\gamma})X^{n-2} + \gamma X + \mu$$

  We can check the following equality :

  $$\mathcal{C} = \{(-\mu, -\gamma, 0, ..., 0, -(\frac{\mu^2}{\gamma^2} - \frac{\alpha\mu}{\gamma}), \alpha, 1) \ / \ (\alpha, \mu, \gamma) \in \mathbf{R}^2 \times \mathbf{R}^* \ \}$$

  On the other hand, we can verify $\lambda * \mathcal{C} \subset \mathcal{C}$, for every real number λ, then $\mathcal{C} = \bigcup_{\lambda \in \mathbf{R}} \lambda * \mathcal{C}$.
  We deduce that it is impossible to genaralise the form of the equation E_n by using the lemma2, and this remark is still checked for the equation $(E4)$.

- For the case of $n = 5$, we can find many examples of the sets of the quintics which we can solve by the new method ((E_4) for example). Unfortunately, we can't improve this form of the equations by using the lemma2.
  Now, in the sence to generalise the form of (E_4), we can put the following question :
  Can we find the real ε which we can add it to one of coeffficients of (E_4) (different from the leading coefficient), and we obtained a new form of equations solvables by the new method ?
  For example, if $\varepsilon = 1$, we add the ε to the coefficient of x into (E_4), we get this new equation :

  $$(E_4^{'}) \qquad X^5 + \alpha X^3 = \beta X^2 + (\frac{\mu^2}{\beta^2} - \alpha\frac{\mu}{\beta} + 1)X + \mu.$$

  From the lemma 1 and 2, we can verify that the problem of resolution for every quintic reduces to problem of solving the equation of the form above, then we deduce that if the set of equations of the form $(E_4)^{'}$ are solvable by the new method, then every quintic is solvable by radicals, which is absurd.

Conclusion 4. *We can not add a real non-zero in one of the coefficients of (E4) different from the leading coefficient for to get a another form of equations solvable by the new method.*

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