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Δ-weak character amenability of certain Banach algebras

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Abstract. In this paper we introduce the notion of Δ-weak character amenable Banach algebras and investigate Δ-weak character amenability of certain Banach algebras such as projective tensor product $A\widehat{\otimes}B$, Lau product $A\times_\theta B$, where $\theta \in \Delta(B)$, abstract Segal algebras and module extension Banach algebras.

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1. Introduction

Let A be a Banach algebra and let $\varphi \in \Delta(A)$, consisting of all nonzero characters on A. The concept of φ-amenability was first introduced by Kaniuth *et al.* in [6]. Specifically, A is called φ-amenable if there exist a $m \in A^{**}$ such that

(i) $m(\varphi) = 1$;

(ii) $m(f.a) = \varphi(a)m(f)$ $(a \in A, f \in A^*)$.

Monfared in [10], introduced and studied the notion of character amenable Banach algebra. A was called character amenable if it has a bounded right approximate identity and it is φ-amenable for all $\varphi \in \Delta(A)$. Many aspects of φ-amenability have been investigated in [4, 6, 9].

Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Following [7], A is called Δ-weak φ-amenable if, there exists a $m \in A^{**}$ such that

(i) $m(\varphi) = 0$;

(ii) $m(\psi.a) = \psi(a)$ $(a \in \ker(\varphi), \psi \in \Delta(A))$.

In this paper we use above definition with a slight difference. In fact we say that A is Δ-weak φ-amenable if, there exists a $m \in A^{**}$ such that

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(i) $m(\varphi) = 0$;

(ii) $m(\psi.a) = \psi(a)$ $(a \in A, \psi \in \Delta(A) \setminus \{\varphi\})$.

The aim of the present work is to study Δ-weak character amenability of certain Banach algebras such as projective tensor product $A\widehat{\otimes}B$, Lau product $A \times_\theta B$, where $\theta \in \Delta(B)$, abstract Segal algebras and module extension Banach algebras. Indeed, we show that $A\widehat{\otimes}B$ (resp. $A \times_\theta B$) is Δ-weak character amenable if and only if both A and B are Δ-weak character amenable. For abstract Segal algebra B with respect to A, we investigate relation between Δ-weak character amenability of A and B. Finally, for a Banach algebra A and A-bimodule X we show that $A \oplus_1 X$ is Δ-weak character amenable if and only if A is Δ-weak character amenable.

2. Δ-weak character amenability of $A\widehat{\otimes}B$

We commence this section with the following definition:

Definition 1. *Let A be a Banach algebra. The net $(a_\alpha)_\alpha$ in A is called a Δ-weak approximate identity if, $|\varphi(aa_\alpha) - \varphi(a)| \longrightarrow 0$, for each $a \in A$ and $\varphi \in \Delta(A)$.*

Note that the approximate identity and Δ-weak approximate identity of a Banach algebra are different. Jones and Lahr proved that if $S = \mathbb{Q}^+$ the semigroup algebra $l^1(S)$ has a bounded Δ-weak approximate identity, but it does not have any bounded or unbounded approximate identity (see [3]).

Definition 2. *Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. We say that A is Δ-weak φ-amenable if, there exists a $m \in A^{**}$ such that*

(i) $m(\varphi) = 0$;

(ii) $m(\psi.a) = \psi(a)$ $(a \in A, \psi \in \Delta(A) \setminus \{\varphi\})$.

Definition 3. *Let A be a Banach algebra. We say that A is Δ-weak character amenable if it is Δ-weak φ-amenable for every $\varphi \in \Delta(A) \cup \{0\}$.*

Lemma 1. *Let A be a Banach algebra such that $0 < |\Delta(A)| \leq 2$. Then A is Δ-weak character amenable.*

Proof. If A has only one character, the proof is easy. Let $\Delta(A) = \{\varphi, \psi\}$, where $\varphi \neq \psi$. Hence, by Theorem 3.3.14 of [5], there exists a $a_0 \in A$ with $\varphi(a_0) = 0$ and $\psi(a_0) = 1$. Put $m = \widehat{a_0}$. Then $m(\varphi) = \widehat{a_0}(\varphi) = \varphi(a_0) = 0$ and for every $a \in A$, we have

$$m(\psi.a) = \widehat{a_0}(\psi.a) = \psi.a(a_0) = \psi(aa_0) = \psi(a).$$

So, A is Δ-weak φ-amenable. A Similar argument shows that A is Δ-weak ψ-amenable. Therefore A is Δ-weak character amenable.

The proof of the following theorem is omitted, since it can be proved in the same direction of Theorem 2.2 of [7].

Theorem 1. *Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Then A is Δ-weak φ-amenable if and only if there exists a net $(a_\alpha)_\alpha \subseteq \ker(\varphi)$ such that $|\psi(aa_\alpha) - \psi(a)| \longrightarrow 0$, for each $a \in A$ and $\psi \in \Delta(A) \setminus \{\varphi\}$.*

Example 1. (i) *Let A be a Banach algebra with a bounded approximate identity. By Theorem 1, A is Δ-weak 0-amenable.*

(ii) *Let $S = \mathbb{Q}^+$. Then the semigroup algebra $l^1(S)$ has a bounded Δ-weak approximate identity* (see [3]). *So, Theorem 1, implies that $l^1(S)$ is Δ-weak 0-amenable.*

Example 2. *Let X be a Banach space and let $\varphi \in X^* \setminus \{0\}$ with $\|\varphi\| \leq 1$. Define a product on X by $ab = \varphi(a)b$ for all $a, b \in X$. With this product X is a Banach algebra which we denote it by $A_\varphi(X)$. Clearly, $\Delta(A_\varphi(X)) = \{\varphi\}$. Therefore by Lemma 1, $A_\varphi(X)$ is Δ-weak φ-amenable.*

Example 3. *Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Suppose that A is a φ-amenable and has a bounded right approximate identity. By Corollary 2.3 of* [6], $\ker(\varphi)$ *has a bounded right approximate identity. Let $(e_\alpha)_\alpha$ be a bounded right approximate identity for* $\ker(\varphi)$. *If there exists $a_0 \in A$ with $\varphi(a_0) = 1$ and $\lim_\alpha \left|\psi(a_0 e_\alpha) - \psi(a_0)\right| = 0$ for all $\psi \in \Delta(A) \setminus \{\varphi\}$, then A is Δ-weak φ-amenable. For see this let $m = w^* - \lim_\alpha(\widehat{e_\alpha})$. Now, we have*

$$m(\varphi) = \lim_\alpha \widehat{e_\alpha}(\varphi) = \lim_\alpha \varphi(e_\alpha) = 0,$$

and for every $\psi \in \Delta(A) \setminus \{\varphi\}$ and $a \in \ker(\varphi)$,

$$m(\psi.a) = \lim_\alpha \widehat{e_\alpha}(\psi.a) = \lim_\alpha \psi.a(e_\alpha) = \lim_\alpha \psi(ae_\alpha) = \psi(a).$$

Let $a \in A$. Then $a - \varphi(a)a_0 \in \ker(\varphi)$ and for every $\psi \in \Delta(A) \setminus \{\varphi\}$, we have

$$m\big(\psi.(a - \varphi(a)a_0)\big) = \psi\big(a - \varphi(a)a_0\big).$$

Therefore $m(\psi.a) = \psi(a)$. So A is Δ-weak φ-amenable.

For $f \in A^*$ and $g \in B^*$, let $f \otimes g$ denote the element of $(A \widehat{\otimes} B)^*$ satisfying $(f \otimes g)(a \otimes b) = f(a)g(b)$ for all $a \in A$ and $b \in B$. Then, with this notion,

$$\Delta(A \widehat{\otimes} B) = \{\varphi \otimes \psi : \varphi \in \Delta(A), \psi \in \Delta(B)\}.$$

Theorem 2. *Let A and B be Banach algebras and let $\varphi \in \Delta(A) \cup \{0\}$ and $\psi \in \Delta(B) \cup \{0\}$. Then $A \widehat{\otimes} B$ is Δ-weak $(\varphi \otimes \psi)$-amenable if and only if A is Δ-weak φ-amenable and B is Δ-weak ψ-amenable.*

Proof. Suppose that $A \widehat{\otimes} B$ is Δ-weak $(\varphi \otimes \psi)$-amenable. So, there exists $m \in (A \widehat{\otimes} B)^{**}$ such that

$$m(\varphi \otimes \psi) = 0, \;\; m(\varphi' \otimes \psi'.a \otimes b) = \varphi' \otimes \psi'(a \otimes b),$$

for all $a \otimes b \in A \widehat{\otimes} B, \varphi' \otimes \psi' \in \Delta(A \widehat{\otimes} B)$. Choose $b_0 \in B$ such that $\psi(b_0) = 1$, and define $m_\varphi \in A^{**}$ by $m_\varphi(f) = m(f \otimes \psi)(f \in A^*)$. Then $m_\varphi(\varphi) = m(\varphi \otimes \psi) = 0$ and for every $a \in A$ and $\varphi' \in \Delta(A)$, we have

$$m_\varphi(\varphi'.a) = m(\varphi'.a \otimes \psi) = m((\varphi'.a \otimes \psi.b_0)$$

$$= m(\varphi' \otimes \psi.a \otimes b_0) = \varphi' \otimes \psi(a \otimes b_0)$$
$$= \varphi'(a).$$

Thus A is Δ-weak φ-amenable. By a similar argument one can prove that B is Δ-weak ψ-amenable.

Conversely, assume that A is Δ-weak φ-amenable and B is Δ-weak ψ-amenable. By Theorem 1, there are bounded nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\ker(\varphi)$ and $\ker(\psi)$, respectively, such that $\big|\varphi'(aa_\alpha) - \varphi'(a)\big| \longrightarrow 0$ and $\big|\psi'(bb_\beta) - \psi'(b)\big| \longrightarrow 0$ for all $a \in A, b \in B, \varphi' \in \Delta(A), \psi' \in \Delta(B)$, with $\varphi' \neq \varphi$ and $\psi' \neq \psi$. Consider the bounded net $((a_\alpha \otimes b_\beta))_{(\alpha,\beta)}$ in $A\widehat{\otimes}B$. Let $\|a_\alpha\| \leq M_1, \|b_\beta\| \leq M_2$ and let $F = \sum_{i=1}^N c_i \otimes d_i \in A\widehat{\otimes}B$. For every $\varphi' \in \Delta(A) \setminus \{0\}$ and $\psi' \in \Delta(B) \setminus \{0\}$, we have

$$\begin{aligned}
&|\varphi' \otimes \psi'(F.a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(F)|\\
&\quad = \Big|\sum_{i=1}^N \Big[\big(\varphi'(c_i a_\alpha) - \varphi'(c_i)\big)\psi'(d_i b_\beta) + \varphi'(c_i)\big(\psi'(d_i b_\beta) - \psi'(d_i)\big)\Big]\Big|\\
&\quad \leq \sum_{i=1}^N M_2\|d_i\|\|\psi'\|\big|\varphi'(c_i a_\alpha) - \varphi'(c_i)\big| + \sum_{i=1}^N \|\varphi'\|\|c_i\|\big|\psi'(d_i b_\beta) - \psi'(d_i)\big|\\
&\quad \longrightarrow 0.
\end{aligned}$$

Now let $G \in A\widehat{\otimes}B$, so there exist sequences $(c_i)_i \subseteq A$ and $(d_i)_i \subseteq B$ such that $G = \sum_{i=1}^\infty c_i \otimes d_i$ with $\sum_{i=1}^\infty \|c_i\|\|d_i\| < \infty$. Let $\varepsilon > 0$ be given, we choose $N \in \mathbb{N}$ such that $\sum_{i=N+1}^\infty \|c_i\|\|d_i\| < \varepsilon/4M_1M_2\|\varphi'\|\|\psi'\|$. Put $F = \sum_{i=1}^N c_i \otimes d_i$. Since $|\varphi' \otimes \psi'(F.a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(F)| \longrightarrow 0$, it follows that there exists (α_0, β_0) such that $|\varphi' \otimes \psi'(F.a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(F)| < \varepsilon/2$ for all $(\alpha, \beta) \geq (\alpha_0, \beta_0)$. Now for such a (α, β), we have

$$\begin{aligned}
&\big|\varphi' \otimes \psi'(G.a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(G)\big|\\
&\quad = \Big|\varphi' \otimes \psi'(F.a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(F)\\
&\qquad + \sum_{i=1+N}^\infty \big(\varphi'(c_i a_\alpha)\psi'(d_i b_\beta) - \varphi'(c_i)\psi'(d_i)\big)\Big|\\
&\quad \leq \varepsilon/2 + 2M_1M_2\|\varphi'\|\|\psi'\| \sum_{i+N}^\infty \|c_i\|\|d_i\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Hence $\big|\varphi' \otimes \psi'(G.a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(G)\big| \longrightarrow 0$. Also, clearly $\big|\varphi' \otimes \psi'(G.a_\alpha \otimes b_\beta) - \varphi' \otimes \psi'(G)\big| \longrightarrow 0$ for $\varphi' = 0$ and $\psi' = 0$ and it is easy to see that $((a_\alpha \otimes b_\beta))_{(\alpha,\beta)} \subset \ker(\varphi \otimes \psi)$. Therefore $A\widehat{\otimes}B$ is Δ-weak $(\varphi \otimes \psi)$-amenable, again by Theorem 1.

Corollary 1. *Let A and B be Banach algebras. Then $A\widehat{\otimes}B$ is Δ-weak character amenable if and only if both A and B are Δ-weak character amenable.*

3. Δ-weak character amenability of $A \times_\theta B$

Let A and B be Banach algebras with $\Delta(B) \neq \emptyset$. Let $\theta \in \Delta(B)$. Then the direct product $A \times B$ equipped with the algebra multiplication

$$(a_1, b_1).(a_2, b_2) = (a_1 a_2 + \theta(b_2)a_1 + \theta(b_1)a_2, b_1 b_2) \quad (a_1, a_2 \in A, b_1, b_2 \in B),$$

and the l^1-norm is a Banach algebra which is called the θ-Lau product of A and B and is denoted by $A \times_\theta B$. This type of product was introduced by Lau [8] for certain class of Banach algebras and was extended by Monfared [9] for the general case.

We note that the dual space $(A \times_\theta B)^*$ can be identified with $A^* \times B^*$, via

$$\langle (f, g), (a, b)\rangle = \langle a, f\rangle + \langle b, g\rangle \ \ (a \in A, f \in A^*, b \in B, g \in B^*).$$

Moreover, $(A \times_\theta B)^*$ is a $(A \times_\theta B)$-bimodule with the module operations given by

$$(f, g).(a, b) = \big(f.a + \theta(b)f, f(a)\theta + g.b\big) \ , \tag{1}$$

and

$$(a, b).(f, g) = \big(a.f + \theta(b)f, f(a)\theta + b.g\big) \ , \tag{2}$$

for all $a \in A, b \in B$ and $f \in A^*, g \in B^*$.

Proposition 1. *Let A be a unital Banach algebra and B be a Banach algebra and $\theta \in \Delta(B)$. Then $A \times_\theta B$ has a Δ-weak approximate identity if and only if B has a Δ-weak approximate identity.*

Proof. Let $((a_\alpha, b_\alpha))_\alpha$ be a Δ-weak approximate identity for $A \times_\theta B$. For every $\psi \in \Delta(B)$ and $b \in B$ we have,

$$\big|\psi(bb_\alpha) - \psi(b)\big| = \big|(0, \psi)\big((0, b)(a_\alpha, b_\alpha)\big) - (0, \psi)(0, b)\big| \longrightarrow 0.$$

Then $(b_\alpha)_\alpha$ is a Δ-weak approximate identity for B.

Conversely, let e_A be the identity of A and $(b_\beta)_\beta$ be a Δ-weak approximate identity for B. We claim that $(e_A - \theta(b_\beta)e_A, b_\beta))_\beta$ is a Δ-weak approximate identity for $A \times_\theta B$. In fact for every $a \in A, b \in B$ and $\varphi \in \Delta(A)$, we have

$$\begin{aligned}&\big|(\varphi, \theta)\big((a, b)(e_A - \theta(b_\beta)e_A, b_\beta)\big) - (\varphi, \theta)(a, b)\big| \\ &\quad = \big|(\varphi, \theta)\big(a + \theta(b)e_A - \theta(bb_\beta), bb_\beta)\big) - (\varphi, \theta)(a, b)\big| \\ &\quad = 0.\end{aligned}$$

Also for every $a \in A, b \in B$ and $\psi \in \Delta(B)$, we have

$$\big|(0, \psi)\big((a, b)(e_A - \theta(b_\beta)e_A, b_\beta)\big) - (0, \psi)(a, b)\big| = \big|\psi(bb_\beta) - \psi(b)\big| \longrightarrow 0.$$

Therefore $((e_A - \theta(b_\beta)e_A, b_\beta))_\beta$ is a Δ-weak approximate identity for $A \times_\theta B$.

Theorem 3. *Let A be a unital Banach algebra and B be a Banach algebra and $\theta \in \Delta(B)$. Then $A \times_\theta B$ is Δ-weak character amenable if and only if both A and B are Δ-weak character amenable.*

Proof. Suppose that $A \times_\theta B$ is Δ-weak character amenable. Let $\varphi \in \Delta(A) \cup \{0\}$. Then there exists $m \in (A \times_\theta B)^{**}$ such that $m(\varphi, \theta) = 0$ and $m(h.(a, b)) = h(a, b)$ for all $(a, b) \in A \times_\theta B$ and $h \in \Delta(A \times_\theta B)$, where $h \neq (\varphi, \theta)$. Let e_A be the identity of A and define $m_\varphi \in A^{**}$ by $m_\varphi(f) = m(f, f(e_A)\theta)(f \in A^*)$. For every $a \in A$ and $\varphi' \in \Delta(A)$, we have

$$\begin{aligned}
m_\varphi(\varphi'.a) &= m\big(\varphi'.a, (\varphi'.a)(e_A)\theta\big) \\
&= m\big(\varphi'.a, \varphi'(a)\theta\big) \\
&= m\big((\varphi', \theta).(a, 0)\big) \\
&= (\varphi', \theta)(a, 0) \\
&= \varphi'(a).
\end{aligned}$$

Also $m_\varphi(\varphi) = m(\varphi, \theta) = 0$. Thus A is a Δ-weak φ-amenable. Therefore A is Δ-weak character amenable.

Let $\psi \in \Delta(B) \cup \{0\}$. From the Δ-weak character amenability of $A \times_\theta B$ it follows that there exists a $m \in (A \times_\theta B)^{**}$ such that $m(0, \psi) = 0$ and $m(h.(a, b)) = h(a, b)$ for all $(a, b) \in A \times_\theta B$ and $h \in \Delta(A \times_\theta B)$, where $h \neq (0, \psi)$. Define $m_\psi \in B^{**}$ by $m_\psi(g) = m(0, g)$. So $m_\psi(\psi) = m(0, \psi) = 0$ and

$$m_\psi(\psi'.b) = m(0, \psi'.b) = m\big((0, \psi').(0, b)\big) = (0, \psi')(0, b') = \psi'(b),$$

for all $b \in B$ and $\psi' \in \Delta(B)$. Therefore B is Δ-weak character amenable.

Conversely, let A and B be Δ-weak character amenable. We show that for every $h \in \Delta(A \times_\theta B)$, $A \times_\theta B$ is Δ-weak h-amenable. To see this we first assume that $h = (0, \psi)$, where $\psi \in \Delta(B)$. Since B is Δ-weak character amenable by Theorem 1 there exists a net $(b_\beta)_\beta \subseteq \ker \psi$ such that $|\psi'(bb_\beta) - \psi'(b)| \longrightarrow 0$, for all $b \in B$ and $\psi' \in \Delta(B)$, where $\psi' \neq \psi$. Consider a bounded net $((e_A - \theta(b_\beta)e_A, b_\beta))_\beta \subseteq A \times_\theta B$. A similar argument as in the proof of Proposition 1, shows that

$$\big|(\varphi, \theta)\big((a, b)(e_A - \theta(b_\beta)e_A, b_\beta)\big) - (\varphi, \theta)(a, b)\big| \longrightarrow 0,$$

and

$$\big|(0, \psi)\big((a, b)(e_A - \theta(b_\beta)e_A, b_\beta)\big) - (0, \psi)(a, b)\big| \longrightarrow 0,$$

for all $\varphi \in \Delta(A), \psi \in \Delta(B)$ and $a \in A, b \in B$. Also one can easily check that $((e_A - \theta(b_\beta)e_A, b_\beta))_\beta \subseteq \ker h$. So, by Theorem 1, $A \times_\theta B$ is Δ-weak $(0, \psi)$-amenable.

Now let $h = (\varphi, \theta)$, where $\varphi \in \Delta(A)$. Since A is Δ-weak φ-amenable by Theorem 1, there exists a net $(a_\alpha)_\alpha \subseteq \ker \varphi$ such that $|\varphi'(aa_\alpha) - \varphi'(a)| \longrightarrow 0$, for all $a \in A$ and $\varphi' \in \Delta(A)$, where $\varphi' \neq \varphi$. Also since B is Δ-weak θ-amenable again by Theorem 1, there exists a net $(b_\beta)_\beta \subseteq \ker(\theta)$ such that $|\psi'(bb_\beta) - \psi'(b)| \longrightarrow 0$, for all $b \in B$ and $\psi' \in \Delta(B)$, where $\psi' \neq \theta$. Consider a bounded net $((a_\alpha, b_\beta))_{(\alpha,\beta)} \subseteq A \times_\theta B$. It is easy to see that $((a_\alpha, b_\beta))_{(\alpha,\beta)} \subseteq \ker(\varphi, \theta)$. For every $a \in A, b \in B$ and $\psi' \in \Delta(B)$, we have

$$\big|(0, \psi')\big((a, b)(a_\alpha, b_\beta)\big) - (0, \psi')\big(a, b)\big| = \big|\psi'(bb_\beta) - \psi'(b)\big| \longrightarrow 0,$$

and for every $\varphi' \in \Delta(A)$,

$$\begin{aligned}&\big|(\varphi', \theta)\big((a, b)(a_\alpha, b_\beta)\big) - (\varphi', \theta)\big((a, b)\big)\big|\\ &\quad= \big|\varphi'(aa_\alpha) + \theta(b)\varphi'(a_\alpha) - \varphi'(a) - \theta(b)\big)\big|\\ &\quad\leq \big|\varphi'(aa_\alpha) - \varphi'(a)\big| + |\theta(b)|\big|\varphi'(a_\alpha e_A) - \varphi'(e_A)\big| \longrightarrow 0.\end{aligned}$$

So, Theorem 1, yields that $A \times_\theta B$ is Δ-weak (φ, θ)-amenable. Therefore $A \times_\theta B$ is Δ-weak character amenable.

4. Δ-weak character amenability of abstract Segal algebras

We start this section with the basic definition of abstract Segal algebra; see [2] for more details. Let $(A, \|.\|_A)$ be a Banach algebra. A Banach algebra $(B, \|.\|_B)$ is an abstract Segal algebra with respect to A if:

(i) B is a dense left ideal in A;

(ii) there exists $M > 0$ such that $\|b\|_A \leq M\|b\|_B$ for all $b \in B$;

(iii) there exists $C > 0$ such that $\|ab\|_B \leq C\|a\|_A\|b\|_B$ for all $a, b \in B$.

Several authors have studied various notions of amenability for abstract Segal algebras; see, for example, [1, 11].

To prove our next result we need to quote the following lemma from [1].

Lemma 2. *Let A be a Banach algebra and let B be an abstract Segal algebra with respect to A. Then $\Delta(B) = \{\varphi|_B : \varphi \in \Delta(A)\}$.*

Theorem 4. *Let A be a Banach algebra and let B be an abstract Segal algebra with respect to A. If B is Δ-weak character amenable, then so is A. In the case that B^2 is dense in B and B has a bounded approximate identity the converse is also valid.*

Proof. Let $\varphi \in \Delta(A)$. Since B is Δ-weak $\varphi|_B$-amenable, by Theorem 1, there exists a bounded net $(b_\alpha)_\alpha$ in $\ker(\varphi|_B)$ such that $\Big|\psi|_B(bb_\alpha) - \psi|_B(b)\Big| \longrightarrow 0$, for all $b \in B$ and $\psi \in \Delta(A)$, with $\psi \neq \varphi|_B$. Let $\psi \in \Delta(A)$ and $a \in A$. From the density of B in A it follows that there exists a net $(b_i)_i \subseteq B$ such that $\lim_i b_i = a$. So

$$\Big|\psi(ab_\alpha) - \psi(a)\Big| = \lim_i \Big|\psi|_B(b_ib_\alpha) - \psi|_B(b_i)\Big| \longrightarrow 0.$$

Then Theorem 1 implies that A is Δ-weak φ-amenable. Therefore A is Δ-weak character amenable.

Conversely, suppose that A is Δ-weak character amenable. Let $\varphi|_B \in \Delta(B)$. By Theorem1, there exists a bounded net $(a_\alpha)_\alpha$ in $\ker(\varphi)$ such that $\Big|\psi(aa_\alpha) - \psi(a)\Big| \longrightarrow 0$, for all $a \in A$ and $\psi \in \Delta(A)$, with $\psi \neq \varphi$. Let $(e_i)_i$ be a bounded approximate identity for B with bound $M > 0$. Set $b_\alpha = \lim_i(e_i a_\alpha e_i)$, for all α. From the fact that B^2 is dense in B and continuity of φ, we infer that $b_\alpha \subseteq \ker(\varphi|_B)$. Moreover, for every $\psi|_B \in \Delta(B)$ and $b \in B$, we have

$$\begin{aligned}\Big|\psi|_B(bb_\alpha) - \psi|_B(b)\Big| &= \lim_i \Big|\psi|_B(be_i a_\alpha e_i) - \psi|_B(b)\Big| \\ &= \lim_i \Big|\psi|_B(be_i^2 a_\alpha) - \psi|_B(b)\Big| \\ &= \Big|\psi|_B(ba_\alpha) - \psi|_B(b)\Big| \longrightarrow 0.\end{aligned}$$

Hence, B is Δ-weak $\varphi|_B$-amenable by Theorem1. Therefore B is Δ-weak character amenable.

5. Δ-weak character amenability of module extension Banach algebras

Let A be a Banach algebra and X be a Banach A-bimodule. The l^1-direct sum of A and X, denoted by $A \oplus_1 X$, with the product defined by

$$(a, x)(a', x') = (aa', a.x' + x.a') \qquad (a, a' \in A, x, x' \in X),$$

is a Banach algebra that is called the module extension Banach algebra of A and X.

Using the fact that the element $(0, x)$ is nilpotent in $A \oplus_1 X$ for all $x \in X$, it is easy to verify that

$$\Delta(A \oplus_1 X) = \{\tilde{\varphi} : \varphi \in \Delta(A)\},$$

where $\tilde{\varphi}(a, x) = \varphi(a)$ for all $a \in A$ and $x \in X$.

Theorem 5. *Let A be a Banach algebra and X be a Banach A-bimodule. Then $A \oplus_1 X$ is Δ-weak character amenable if and only if A is Δ-weak character amenable.*

Proof. Suppose that A is Δ-weak character amenable. Let $\tilde{\varphi} \in \Delta(A \oplus_1 X)$. By Theorem 1, there exists a bounded net $(a_\alpha)_\alpha$ in $\ker(\varphi)$ such that $\big|\psi(aa_\alpha) - \psi(a)\big| \longrightarrow 0$, for all $a \in A$ and $\psi \in \Delta(A)$, with $\psi \neq \varphi$. Choosing a bounded net $(a_\alpha, 0)_\alpha$ in $A \oplus_1 X$. Clearly, $(a_\alpha, 0)_\alpha \subseteq \ker(\tilde{\varphi})$. For every $a \in A, x \in X$ and $\tilde{\psi} \in \Delta(A \oplus_1 X)$, we have

$$\begin{aligned}\big|\tilde{\psi}\big((a, x)(a_\alpha, 0)\big) - \tilde{\psi}(a, x)\big| &= \big|\tilde{\psi}\big(aa_\alpha, x.a_\alpha\big) - \tilde{\psi}(a, x)\big| \\ &= \big|\psi(aa_\alpha) - \psi(a)\big| \longrightarrow 0.\end{aligned}$$

So, Theorem 1 implies that $A \oplus_1 X$ is Δ-weak $\tilde{\varphi}$-amenable. Therefore $A \oplus_1 X$ is Δ-weak character amenable.

For the converse, let $\varphi \in \Delta(A)$. Again by Theorem 1 there exists a bounded net $(a_\alpha, x_\alpha)_\alpha$ in $\ker(\tilde{\varphi})$ such that $\left|\tilde{\psi}\big((a,x)(a_\alpha, x_\alpha)\big) - \tilde{\psi}(a,x)\right| \longrightarrow 0$, for all $a \in A, x \in X$ and $\tilde{\psi} \in \Delta(A \oplus_1 X)$, with $\tilde{\psi} \neq \tilde{\varphi}$. So,

$$\begin{aligned}\left|\psi(aa_\alpha) - \varphi(a)\right| &= \left|\tilde{\psi}(aa_\alpha, a.x_\alpha + x.a_\alpha) - \tilde{\psi}(a,x)\right| \\ &= \left|\tilde{\psi}\big((a,x)(a_\alpha, x_\alpha)\big) - \tilde{\psi}(a,x)\right| \longrightarrow 0,\end{aligned}$$

for all $a \in A$ and $\psi \in \Delta(A)$. Moreover, $\varphi(a_\alpha) = \tilde{\varphi}(a_\alpha, x_\alpha) = 0$, for all α. Thus $(a_\alpha)_\alpha \subseteq \ker(\varphi)$. By Theorem 1, A is Δ-weak φ-amenable. Therefore A is Δ-weak character amenable.

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