A Note on the Non-existence of Limit Cycles of the FitzHugh-Nagumo System

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Abstract. The condition for the non-existence of limit cycles of the FitzHugh-Nagumo system which is well-known as the simplified nerve system of Hodgkin-Huxley model is improved by constructing some plane curve.

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1. Introduction

The following two dimensional autonomous system is called the FitzHugh-Nagumo system ([1], [4] and [5] etc.):

$$\begin{cases} \dot{w} = v - \frac{1}{3}w^3 + w + I \\ \dot{v} = \rho(a - w - bv), \end{cases} \tag{1}$$

where the dot (˙) denotes differentiation and a, ρ, b are real constants such that

[C1] $a \in \mathbb{R}$, $\rho > 0$, $0 < b < 1$.

The variable w corresponds to the potential difference through the axon membrane and v represents the potassium activation (sodium inactivation). The quantity I is the current through the membrane.

The study for System (1) is enormous and many results have been published. The system for special values of I is investigated by using numerical methods and phase space analysis. Our purpose in this paper is to improve the sufficient condition for the non-existence of the limit cycles of System (1) by constructing some plane curve.

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System (1) has a unique equilibrium point (x_I, y_I) for each $I \in \mathbb{R}$. Instead of the parameter I we introduce a new parameter η. By some transformation $\eta = x_I$, $x = w - \eta$ and $y = v - a/b + \eta/b + \rho b(w - \eta)$, System (1) is transformed to the following polynomial system:

$$\begin{cases} \dot{x} = y - \left\{ \dfrac{1}{3}x^3 + \eta x^2 + (\eta^2 - \eta_0^2)x \right\} \\ \dot{y} = -\dfrac{\rho b}{3}\left\{ x^3 + 3\eta x^2 + 3\left(\eta^2 + \dfrac{1}{b} - 1 \right)x \right\}, \end{cases} \quad (2)$$

where $\eta_0^2 = 1 - \rho b$. The above system is called the FitzHugh nerve system and has a unique equilibrium point $E(0,0)$. This is a system of the Liénard type. In the previous paper ([2] or [5]) the following result have been given.

Proposition 1. *If the condition*

[C2] $\rho b \geq 1$
    or

[C3] $\eta^2 \geq \eta_0^2$ *and* $\eta^4 - 4\eta^2\eta_0^2 + \eta_0^4 + 2\left(\dfrac{1}{b} - 1 \right)\eta^2 - 4\left(\dfrac{1}{b} - 1 \right)\eta_0^2 + 4\left(\dfrac{1}{b} - 1 \right)^2 \geq 0$
    or

[C4] $2\left(\eta_0^2 + \frac{1}{b} - 1 \right)^3 < \eta^2\left\{ \eta^2 + 3\left(\frac{1}{b} - 1 \right) \right\}^2$

is satisfied, then System (2) *has no limit cycles.*

As is shown in Figure 1, the diagonal area denotes the condition in Proposition 1 (also see [2]). The dot · is the point($\rho = 0.01^2$, $b = 10/11$ and $\eta^2 = 1.01^2$) for a system given in the example of §3.
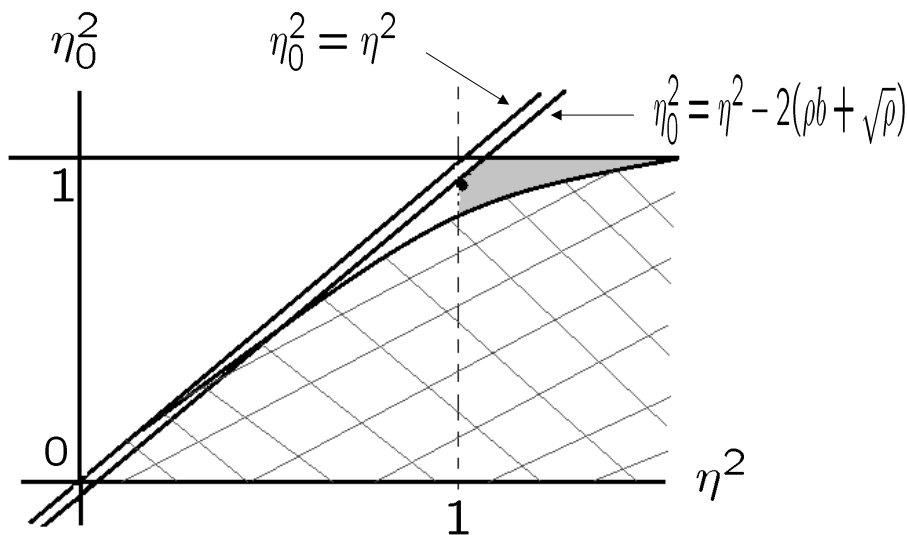

Figure 1: Demonstrating that System (2) has no limit cycles.

Our main results are the following

Theorem 1. *Assume that the condition*

[C5] $\eta_0^2 \leq \eta^2 - 2(\rho b + \sqrt{\rho})$

is satisfied in addition to the condition [C2] *or* [C3] *or* [C4]. *Then System* (2) *has no limit cycles.*

Theorem 2. *Assume that the condition* [C2] *or* [C3] *or* [C4] *or* [C5] *is satisfied. Then the unique equilibrium point E of System* (2) *is globally asymptotically stable.*

The shaded area in Figure 1 is the improved part by our result.

The above theorems are easily proved in the next section and the phase portrait as an example illustrating our results will be given in §3.

2. Proofs

To prove our main results, we shall introduce a useful tool for the following generalized Liénard system:

$$\begin{cases} \dot{x} = h(y) - F(x) \\ \dot{y} = -g(x), \end{cases} \tag{3}$$

where F, g and $h \in C^1$, $x(x - a_2)F(x) > 0$ for $x \in (a_2, +\infty)$ and $a_2 < 0$, $g(x)/x > 0$, $h(y)/y > 0$ and $dh/dy > 0$.

Let a^* is the positive number such that $G(a^*) = G(a_2)$, where $G(x) = \int_0^x g(\xi)d\xi$. We have from [3] the following

Lemma 1. *If there exists a C^1-function $\varphi(x)$ satisfying the condition*

$$\varphi(\alpha) = 0, \quad \varphi'(x) > 0, \quad \frac{\varphi'(x)[F(x) - \varphi(x)]}{\frac{dh}{dy}[h^{-1}(\varphi(x))]} \geq g(x)$$

for $x \geq \alpha \in (0, a^*]$, then System (3) *has no limit cycles.*

We shall prove Theorem 1. We can set

$$h(y) = y,$$
$$F(x) = \frac{1}{3}x^3 + \eta x^2 + (\eta^2 - \eta_0^2)x,$$
$$g(x) = \frac{\rho b}{3}\left\{x^3 + 3\eta x^2 + 3\left(\eta^2 + \frac{1}{b} - 1\right)x\right\},$$
$$G(x) = \int_0^x g(\xi)d\xi.$$

Remark that System (2) has no limit cycles if $\eta_0^2 \leq (1/4)\eta^2$. So we assume that the pair (η^2, η_0^2) satisfies the inequality $(1/4)\eta^2 < \eta_0^2 < \eta^2$. Then we have two different real numbers

$$a_1 = \frac{-3\eta - \sqrt{3(4\eta_0^2 - \eta^2)}}{2}, \; a_2 = \frac{-3\eta + \sqrt{3(4\eta_0^2 - \eta^2)}}{2}$$

and $a_1 < a_2 < 0$ as the solutions of the equation $F(x) = 0$. Moreover, we note that there exists a unique solution a^* satisfying the equation $G(a^*) = G(a_2)$. Taking the supplement function $\varphi(x) = a(x - \alpha)$ with $a > \rho b \, (> 0)$ and $\alpha \in (0, a^*]$, we have

$$L(x) = \frac{\varphi'(x)[F(x) - \varphi(x)]}{\frac{dh}{dy}[h^{-1}(\varphi(x))]} - g(x)$$

$$= \frac{1}{3}(a - \rho b)x^3 + (a - \rho b)\eta x^2$$

$$+ \left\{ a(\eta^2 - \eta_0^2 - a) - \rho b \left(\eta^2 + \frac{1}{b} - 1 \right) \right\} x + a^2 \alpha.$$

Assume that the condition [C5] is satisfied for System (2). Then we see that there exists two positive numbers $a = p_i \, (i = 1, 2)$ satisfying the equation $P(a) = -L'(0) = 0$ and $p_i > \rho b$, where $p_1 < p_2$ and $P(a)$ is the function of two degree for the variable a. In fact, we have $\eta^2 - \eta_0^2 > 2\rho b$, $P(\rho b) = \rho > 0$ and $D = (\eta^2 - \eta_0^2)^2 - 4\rho b(\eta^2 + 1/b - 1) \geq 0$, where D is the discriminant of the equation $P(a) = 0$. Taking $a = p_2$, it follows that $L(x) \geq 0$ for $x \geq 0$. Thus, we can easily check the conditions $\varphi'(x) > 0$ and $L(x) > 0$ for $x \geq \alpha \in (0, a^*]$. Hence, we conclude from Lemma 1 that System (2) has no limit cycles under the condition [C5].

The proof of Theorem 2 is proved by the same discussion as in [2] from the non-existence of limit cycles and homoclinic orbits for System (2). So we omit the details.

We shall confirm that these results are an improvement of the previous ones as seen in Figure 1.

3. A Numerical Example

We shall present the phase portrait of the following system as an example illustrating the application of Theorem 1.

Example 1. *Consider System (2) with* $\rho = 0.01^2$, $b = 10/11$ *and* $\eta^2 = 1.01^2$. *Then the pair* (η^2, η_0^2) *belongs to the shaded part in Figure 1, which is satisfied the condition* [C5], *but not the condition* [C3] *or* [C4] *in Proposition 1. Then we have* $a_2 = -0.020124663$ *and* $a^* = 0.0198841$. *Solving the equation* $L'(0) = 0$, *we can take* $a = 0.010397$ *and* $\alpha = a^*$. *Thus, we shall see from the existence of the supplement function* $\varphi(x) = a(x - \alpha)$ *that the system has no limit cycles as is shown in Figure 2 below.*
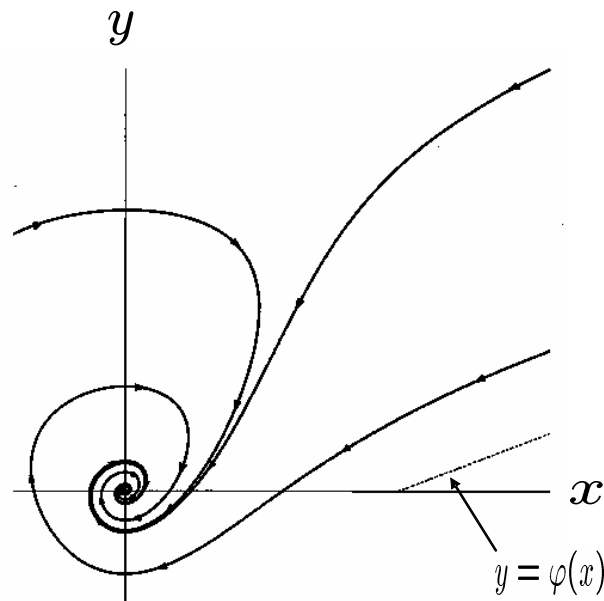
Figure 2: Demonstrating that System (2) has no limit cycles.

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