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Measuring Departure from Diagonals-Parameter Symmetry for Ordinal Square Contingency Tables

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Abstract. For ordinal square contingency tables, Tomizawa et al. [6] considered the measure to represent the degree of departure from the diagonals-parameter symmetry model. This measure attains the maximum value when one of probabilities for any two symmetric cells with respect to the main diagonal in the table is zero. The present paper proposes an improved measure which can attain the maximum value even when the probabilities are not zeros.

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1. Introduction

Consider an $R \times R$ square contingency table with the same ordered row and column classifications. Let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table ($i = 1,\dots,R;\ j = 1,\dots,R$). Goodman [2] considered the diagonals-parameter symmetry (DPS) model defined by

$$p_{ij} = \begin{cases} \Delta_{j-i}\phi_{ij} & (i < j), \\ \phi_{ij} & (i \geq j), \end{cases}$$

where $\phi_{ij} = \phi_{ji}$. Special cases with $\left\{\Delta_{j-i} = \Delta\right\}$ and $\left\{\Delta_{j-i} = 1\right\}$ are the conditional symmetry model (McCullagh [3]) and the symmetry model (Bishop et al. [1]), respectively. Let X and Y denote the row and column variables, respectively. The DPS model can be expressed as

$$p^U_{i,i+k} = p^L_{i+k,i} \quad (k = 1,\dots,R-2; i = 1,\dots,R-k),$$

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where

$$p^U_{i,i+k} = \frac{p_{i,i+k}}{\delta^U_k}\ [= \mathrm{P}(X = i, Y = i + k|\ |Y - X| = k, X < Y)],$$
$$p^L_{i+k,i} = \frac{p_{i+k,i}}{\delta^L_k}\ [= \mathrm{P}(X = i + k, Y = i|\ |Y - X| = k, X > Y)],$$
$$\delta^U_k = \sum_{i=1}^{R-k} p_{i,i+k}\ [= \mathrm{P}(|Y - X| = k, X < Y)],$$
$$\delta^L_k = \sum_{i=1}^{R-k} p_{i+k,i}\ [= \mathrm{P}(|Y - X| = k, X > Y)].$$

This indicates that for distance k from the diagonal, there is a structure of symmetry between two conditional distributions $\{p^U_{i,i+k}\}$ and $\{p^L_{i+k,i}\}$.

When the DPS model does not hold, we are interested in measuring the degree of departure from the DPS. Tomizawa et al. [6] proposed the measure (denoted by $\Phi^{(\lambda)}_{DPS}$) to represent the degree of departure from the DPS. (See Section 2 for $\Phi^{(\lambda)}_{DPS}$). The $\Phi^{(\lambda)}_{DPS}$ lies between 0 and 1. Also $\Phi^{(\lambda)}_{DPS} = 0$ when the DPS model holds, and $\Phi^{(\lambda)}_{DPS} = 1$ when one of the probabilities for any two symmetric cells with respect to the main diagonal in the table is zero.

However, for square contingency tables, all cell probabilities $\{p_{ij}\}$ are positive in many cases. Thus, the measure $\Phi^{(\lambda)}_{DPS}$ may not be suitable for such data because it cannot attain the maximum value. So, we are now interested in the measure to represent the degree of departure from the DPS which can attain the maximum value even when each of cell probabilities $\{p_{ij}\}$ is not zero.

The purpose of this article is to propose the measure to represent the degree of departure from the DPS model when all cell probabilities $\{p_{ij}\}$ are positive.

2. New Measure for Diagonals-Parameter Symmetry

Assume that $\{\delta^U_k > 0\}$, $\{\delta^L_k > 0\}$ and $\{p_{i,i+k} + p_{i+k,i} > 0\}$. Let

$$p^c_{i,i+k} = \frac{p^U_{i,i+k}}{p^U_{i,i+k} + p^L_{i+k,i}},\ p^c_{i+k,i} = \frac{p^L_{i+k,i}}{p^U_{i,i+k} + p^L_{i+k,i}},$$

for $k = 1, \ldots, R-2$; $i = 1, \ldots, R-k$. Note that the DPS model is expressed as

$$\{p^c_{i,i+k} = p^c_{i+k,i} = \frac{1}{2}\}.$$

For a specified d which satisfies $0.5 < d \le 1$ and $\{1 - d \le p^c_{i,i+k} \le d\}$, we propose a new measure γ defined by, for λ (> -1) and d fixed,

$$\gamma = \frac{1}{K}\Phi^{(\lambda)}_{DPS},$$

where

$$K = 1 - \frac{\lambda 2^{\lambda}}{2^{\lambda} - 1} H,$$
$$H = \frac{1}{\lambda}\left(1 - d^{\lambda+1} - (1-d)^{\lambda+1}\right),$$
$$\Phi^{(\lambda)}_{DPS} = \frac{\sum_{k=1}^{R-2}(\delta^U_k + \delta^L_k)\Phi_k}{\sum_{t=1}^{R-2}(\delta^U_t + \delta^L_t)}$$

with

$$\Phi_k = \frac{1}{2}\sum_{i=1}^{R-k}\left(p^U_{i,i+k} + p^L_{i+k,i}\right)\left(1 - \frac{\lambda 2^{\lambda}}{2^{\lambda}-1} H_{i,i+k}\right),$$
$$H_{i,i+k} = \frac{1}{\lambda}\left(1 - (p^c_{i,i+k})^{\lambda+1} - (p^c_{i+k,i})^{\lambda+1}\right),$$

and the value at $\lambda = 0$ is taken to be continuous limit as $\lambda \to 0$. Note that when $\lambda = 0$,

$$K = 1 - \frac{1}{\log 2} H,$$

and

$$H = -d\log d - (1-d)\log(1-d).$$

Note that $\{H_{i,i+k}\}$ are the Patil and Taillie's [4] diversity index including the Shannon entropy when $\lambda = 0$.

First, consider the case of $d = 1$. Then the measure γ is identical to $\Phi^{(\lambda)}_{DPS}$. The minimum value of $H_{i,i+k}$ is 0, and the maximum value of $H_{i,i+k}$ is $(2^{\lambda}-1)/\lambda 2^{\lambda}$ (when $\lambda \neq 0$) or $\log 2$ (when $\lambda = 0$) for $k = 1,\dots,R-2; i = 1,\dots,R-k$. Also, $\Phi^{(\lambda)}_{DPS}$ attains the maximum value 1 when $\{H_{i,i+k} = 0\}$.

Next, consider the case of $d \neq 1$. Then the minimum value of $H_{i,i+k}$ is $(1 - d^{\lambda+1} - (1-d)^{\lambda+1})/\lambda$ (when $\lambda \neq 0$) or $-d\log d - (1-d)\log(1-d)$ (when $\lambda = 0$), and the maximum value of $H_{i,i+k}$ is $(2^{\lambda}-1)/\lambda 2^{\lambda}$ (when $\lambda \neq 0$), or $\log 2$ (when $\lambda = 0$) for $k = 1,\dots,R-2; i = 1,\dots,R-k$.

Thus, when the probabilities $\{p_{ij}\}$ are not positive (i.e., $\{H_{i,i+k} \neq 0\}$), $\Phi^{(\lambda)}_{DPS}$ cannot attain the maximum value 1; however, the proposed measure γ can attain the maximum value 1.

Thus we see that

i) $0 \leq \gamma \leq 1$,

ii) $\gamma = 0$ if and only if the DPS model holds, i.e., $\{p^c_{i,i+k} = p^c_{i+k,i} = \frac{1}{2}\}$, and

iii) $\gamma = 1$ if and only if the degree of departure from the DPS is the largest in the sense that $\{p^c_{i,i+k} = d \text{ or } 1-d\}$.

3. Approximate Variances for Estimated Measure

Let n_{ij} denote the observed frequency in the ith row and jth column of the table ($i = 1, \ldots, R$; $j = 1, \ldots, R$). Assume that a multinomial distribution applies to the $R \times R$ table. The sample version of γ, i.e., $\hat{\gamma}$, is given by γ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$ and $n = \sum\sum n_{ij}$. Using the delta method [1, Sec.14.6] , $\sqrt{n}\left(\hat{\gamma} - \gamma\right)$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance

$$\sigma^2 = \frac{1}{(K\Gamma)^2} \sum_{k=1}^{R-2} \sum_{i=1}^{R-k} \left[p_{i,i+k} \left(\omega_{i,i+k} \right)^2 + p_{i+k,i} \left(\omega_{i+k,i} \right)^2 \right],$$

where

$$\begin{aligned}
\Gamma &= \sum_{t=1}^{R-2} (\delta_t^U + \delta_t^L), \\
\omega_{i,i+k} &= \Phi_k - \Phi_{DPS}^{(\lambda)} + \frac{\delta_k^U + \delta_k^L}{2\delta_k^U} \left(\upsilon_{i,i+k} - \sum_{m=1}^{R-k} p_{m,m+k}^U \upsilon_{m,m+k} \right), \\
\omega_{i+k,i} &= \Phi_k - \Phi_{DPS}^{(\lambda)} + \frac{\delta_k^U + \delta_k^L}{2\delta_k^L} \left(\upsilon_{i+k,i} - \sum_{m=1}^{R-k} p_{m+k,m}^L \upsilon_{m+k,m} \right), \\
\upsilon_{st} &= \begin{cases} \dfrac{1}{2^\lambda - 1} \left[(2p_{st}^c)^\lambda + \lambda p_{ts}^c \left((2p_{st}^c)^\lambda - (2p_{ts}^c)^\lambda \right) \right] & (\lambda \neq 0), \\ \dfrac{\log(2p_{st}^c)}{\log 2} & (\lambda = 0). \end{cases}
\end{aligned}$$

Let $\hat{\sigma}^2$ denote σ^2 with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Thus the estimated approximate confidence interval for the measure γ is obtained.

4. An Example

Consider the data in Table 1, taken from Tominaga [5], pp.131-132. These data describe the cross-classification of father's and his son's occupational status categories in Japan, which were examined in 1955 and 1975.

For these data, the cell probabilities $\{p_{ij}\}$ would be positive. Therefore, we should use the measure γ with $d < 1$. For example, we choose $d = 0.999$.

We see from Table 2 that confidence intervals for γ applied to Tables 1a and 1b do not include zero for all λ. So, these would indicate that there is not a structure of the DPS in Tables 1a and 1b. In addition, for example, when $\lambda = 1$, the value of estimated measure $\hat{\gamma}$ with $d = 0.999$ is 0.146 for Table 1a and 0.336 for Table 1b. Thus, for Table 1a, the degree of departure from the DPS is estimated to be 14.6 percent of the maximum degree of departure from the DPS, and for Table 1b, it is estimated to be 33.6 percent of the maximum degree.

We see from Table 2 that the degree of departure from the DPS is greater for Table 1b than for Table 1a because for each λ, the values in the confidence intervals for γ are greater for Table 1b than for Table 1a.

Table 1: Occupational status for Japanese father-son pairs in 1955 and 1975 (Tominaga, [5, pp.131-132]).

(a) In 1955

	Son's status				
Father's status	(1)	(2)	(3)	(4)	Total
(1)	80	72	37	19	208
(2)	44	155	61	31	291
(3)	26	73	218	45	362
(4)	69	156	166	614	1005
Total	219	456	482	709	1866

(b) In 1975

	Son's status				
Father's status	(1)	(2)	(3)	(4)	Total
(1)	127	101	54	12	294
(2)	86	207	125	13	431
(3)	78	124	310	24	536
(4)	109	206	437	325	1077
Total	400	638	926	374	2338

Table 2: When $d = 0.999$, the estimated measure $\hat{\gamma}$, estimated approximate standard errors for $\hat{\gamma}$, and approximate 95% confidence intervals for γ, applied to Tables 1a and 1b.

(a) For Table 1a

λ	$\hat{\gamma}$	Standard error	Confidence interval
-0.4	0.081	0.016	(0.049, 0.113)
0.0	0.110	0.022	(0.068, 0.153)
0.6	0.138	0.026	(0.086, 0.189)
1.0	0.146	0.028	(0.092, 0.201)
1.4	0.149	0.028	(0.094, 0.204)

(b) For Table 1b

λ	$\hat{\gamma}$	Standard error	Confidence interval
-0.4	0.201	0.021	(0.160, 0.241)
0.0	0.265	0.025	(0.216, 0.314)
0.6	0.320	0.028	(0.265, 0.375)
1.0	0.336	0.029	(0.280, 0.393)
1.4	0.342	0.029	(0.285, 0.399)

5. Concluding Remark

We have proposed the measure γ which is the improvement of measure $\Phi^{(\lambda)}_{DPS}$, to represent the degree of departure from the DPS. The measure γ with $d < 1$, rather than $\Phi^{(\lambda)}_{DPS}$ (i.e., γ with $d = 1$) are appropriate for analyzing square table data with $\{p_{ij}\}$ being positive because then $\Phi^{(\lambda)}_{DPS}$ cannot attain the maximum value, however γ can attain maximum value 1.

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