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Derivatives of $x^n(x-1)(x-a)$ with Rational Roots

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Abstract. Let $n \geq 3$ denote an integer and $a \neq 0, 1$ denote a rational number. For the family of polynomials $f(x) = x^n(x-1)(x-a)$ with fixed value of n, we show that there exist infinitely many values of a such that the first two derivatives of $f(x)$ have rational roots. We find two examples of n and a for which the first three derivatives of $f(x)$ have rational roots.

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1. Introduction

A polynomial $f(x)$ with rational coefficients is called rational derived if $f(x)$ and all of its derivatives have rational roots. The following example of a rational derived quartic was given by Carroll [4].

$$\begin{aligned} f(x) &= (x+167)^2(x-141)(x-193),\\ f'(x) &= 4(x+167)(x+2)(x-169),\\ f''(x) &= 12(x-97)(x+97),\\ f'''(x) &= 24x. \end{aligned}$$

The most well known, open problem associated with rational derived polynomials is determining whether or not there exists a rational derived quartic polynomial with distinct roots. The reader should consult Buchholz and Kelly [2], Buchholz and MacDougall [3], and Stroeker [9] for background on this problem. We may also consider polynomials with rational roots, whose first l derivatives also have rational roots. The notation $\overline{D}(n, l, \mathbb{Q})$, introduced in [3], denotes the set of polynomials of degree n such that they and their first l derivatives have rational roots. Quartic polynomials with four roots in an integral domain $\mathscr{D}$ and whose first

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derivatives have three roots in $\mathscr{D}$ were treated by Groves in [6] and in [7]. In this paper we study polynomials of the type

$$f(x) = f_{n,a}(x) = x^n(x-1)(x-a)$$

where the integer n satisfies $n \geq 3$ and $a \neq 0,1$ is a rational number. Our main result confirms that for each integer $n \geq 3$ there exist infinitely many rational numbers $a \neq 0,1$ such that $f_{n,a}(x) \in \overline{D}(n+2,2,\mathbb{Q})$. We state this in Theorem 1 and give a proof in Section 2. In Section 3 we consider the sets $\overline{D}(n+2,l,\mathbb{Q})$ where $l \geq 3$, finding two pairs (n,a) such that $f_{n,a}(x) \in \overline{D}(n+2,3,\mathbb{Q})$, then finish by asking two questions.

We now state our main theorem. The case where $n=2$ of our theorem is covered in [2] and [3].

Theorem 1. *For each integer $n \geq 3$, there exist infinitely many rational numbers a such that $f_{n,a}(x) \in \overline{D}(n+2,2,\mathbb{Q})$. For fixed n, these values of a have the form*

$$a = \frac{(2n^2+kn+4n+k)(kn+k-2)}{2(n+1)^3k}, \tag{1}$$

where k satisfies

$$c_4k^4 + c_3k^3 + c_2k^2 + c_1k + c_0 = Y^2 \tag{2}$$

for some rational number Y, with

$$\begin{aligned} c_0 =&16n^4(n+1)^2(n+2)^2, \\ c_1 =&-32n^2(n+1)^3(n+2)^2, \\ c_2 =&8n(n+1)^4(n+2)(3n^2+4n-2), \\ c_3 =&8n(n+1)^5(n+2), \\ c_4 =&n^2(n+1)^6. \end{aligned}$$

If we set

$$a^* = \frac{(n^3+2n^2-2)(n^3+2n^2+2n+2)}{2n(n+1)^2(n^3+2n^2-2n-2)}, \tag{3}$$

then for each $n \geq 3$, $f_{n,a^*} \in \overline{D}(n+2,2,\mathbb{Q})$.

2. Proof Of Theorem

We will derive, for each $n \geq 3$, a parametrizing elliptic curve that specifies the values of a described in our theorem. A confirmation that these curves have positive rank will be given as well. Finally, by finding a point $(x(n), y(n))$ on this family of curves, we will obtain the value of a^* given in our theorem.

Proof. The first derivative of $f(x)$ is given by

$$f'(x) = x^{n-1}((n+2)x^2 - (n+1)(a+1)x + na).$$

In order for $f'(x)$ to have rational roots, the discriminant of the quadratic factor of $f'(x)$ must equal the square of a rational number. This discriminant, written as a quadratic in a, is given by

$$(n+1)^2a^2 - (2n^2+4n-2)a + (n+1)^2. \tag{4}$$

Using a standard approach, we parametrize the values of a for which (4) is equal to a rational square and obtain

$$a = \frac{(2n^2+kn+4n+k)(kn+k-2)}{2(n+1)^3k},$$

where k denotes a nonzero rational number. This establishes (1). Assuming that a has the value given by (1), substituting it into $f(x)$, then examining the second derivative $f''(x)$, we find another quadratic factor whose discriminant must equal the square of a rational number. This discriminant, written as a quartic in k, is

$$\begin{aligned} n^2(n+1)^6k^4 + 8n(n+1)^5(n+2)k^3 + 8n(n+1)^4(n+2)(3n^2+4n-2)k^2 \\ - 32n^2(n+1)^3(n+2)^2k + 16n^4(n+1)^2(n+2)^2. \end{aligned} \tag{5}$$

Requiring this discriminant (5) to equal Y^2 for some rational number Y establishes (2). Converting (2) to Weierstrass form yields

$$y^2 = x^3 - 27n^2(n+1)^2(n+2)^2(3n^2+1)x + 54n^3(n+1)^3(n+2)^3(3n-1)(3n+1). \tag{6}$$

For $n \geq 3$, the elliptic curve (6) has positive rank. To see this, note that for fixed $n \geq 3$, the cubic on the right hand side of (6) has three rational roots contributing three points of order two in the group Γ of rational points of (6). Therefore, from the known list of torsion subgroups [8] we conclude that the torsion subgroup of Γ is of the form

$$\mathbb{Z}_2 \times \mathbb{Z}_{2N}, \quad 1 \leq N \leq 4. \tag{7}$$

Consider the point

$$P = (-3(n+2)(3n^2+n-6)(n+1), 54(n-1)(n+1)^2(n+2)^2)$$

in Γ. P has infinite order, as can be deduced by noting (7), forming the set $\{P, 2P, 4P\}$, and checking that this set contains neither points of order 2, nor a point and its inverse. Thus the elliptic curve (6) has positive rank for $n \geq 3$, implying that there exists infinitely many rational values of a such that $f_{n,a}(x) \in \overline{D}(n+2, 2, \mathbb{Q})$. Converting the point P by means of birational transformations to a point on the quartic curve (2) yields a k-value of

$$k = \frac{4n^2(n+2)}{(n+1)(n^3+2n^2-2n-2)}. \tag{8}$$

Substituting for k from (8) into (1) yields the value of a^* given in the statement of our theorem. The fact that a^* has the property stated in our theorem is a consequence of the following calculation.

$$\begin{aligned}
f(x) =& x^n(x-1)\left(x - \frac{(n^3+2n^2-2)(n^3+2n^2+2n+2)}{2n(n+1)^2(n^3+2n^2-2n-2)}\right),\\
f'(x) =& (n+2)x^{n-1}\left(x - \frac{n^3+2n^2+2n+2}{2n(n+2)(n+1)}\right)\left(x - \frac{n(n^3+2n^2-2)}{(n+1)(n^3+2n^2-2n-2)}\right),\\
f''(x) =& (n+1)(n+2)x^{n-2}\left(x - \frac{n^3+2n^2-2}{2(n+2)(n+1)^2}\right)\left(x - \frac{(n-1)(n^3+2n^2+2n+2)}{(n+1)(n^3+2n^2-2n-2)}\right).
\end{aligned}$$

3. Rational Roots of Higher Derivatives

It was shown by Flynn [5] that if $n = 3$, then $f_{n,a}(x) \notin \overline{D}(n+2, 3, \mathbb{Q})$ for all rational values of $a \neq 0, 1$. The idea of the proof was to form the product of the discriminants of the irreducible quadratic factors of the first three derivatives of $f_{3,a}(x)$, and require this product to be a square in $\mathbb{Q}$. This leads to a study of the rational points on a genus 2 curve, from which the result for $n = 3$ was deduced. We formed the corresponding products of discriminants for $4 \leq n \leq 1500$ and searched for rational points on the resulting genus 2 curves using the routine "RationalPoints" in Magma [1]. For these values of n, no rational numbers $a \neq 0, 1$ were found for which $f_{n,a}(x) \in \overline{D}(n+2, 3, \mathbb{Q})$. As an alternative approach, we assumed that for some $n \geq 4$, and some rational number $a \neq 0, 1$, the third derivative of $f_{n,a}(x)$ has the rational root $x = 1$. Substituting $x = 1$ into $f'''(x)$ produces an equation for a which yields

$$a = \frac{n+1}{n-1}.$$

By substituting this value of a into $f_{n,a}(x)$, we find that first two derivatives of $f_{n,a}(x)$ each have a quadratic factor, which for purposes of having rational roots, must have their discriminants equal to the square of a rational number. These discriminants are, respectively,

$$8n(n+1) \tag{9}$$

and

$$4n(3n-2). \tag{10}$$

Since $\gcd(n, n+1) = 1$, forcing (9) to equal a square in $\mathbb{Q}$ implies that

$$(n, n+1) = (2a^2, b^2) \text{ or } (a^2, 2b^2) \tag{11}$$

for positive integers a and b. Combining (10) and (11) we deduce that either

$$4(b^2-1)(3b^2-5) \tag{12}$$

or

$$4(2b^2-1)(6b^2-5) \tag{13}$$

is equal to a square in $\mathbb{Q}$. The positive integral values of b for which (12) or (13) is equal to square in $\mathbb{Q}$ can be determined by using the command "IntegralQuarticPoints" available in Magma [1]. We find that for $b = 29$, (13) yields a square from which we determine $n = 1681$. This leads to the following example of a degree 1683 polynomial $f_{n,a}(x) \in \overline{D}(1683, 3, \mathbb{Q})$.

$$\begin{aligned} f(x) =& x^{1681}(x-1)\left(x-\frac{841}{840}\right), \\ f'(x) =& \frac{x^{1680}(1190x-1189)(1188x-1189)}{840}, \\ f''(x) =& \frac{841x^{1679}(616x-615)(2295x-2296)}{420}, \\ f'''(x) =& 2827442x^{1678}(x-1)(1683x-1679). \end{aligned}$$

We also substituted $n = 1681$ into the product of discriminants mentioned at the beginning of this section. Using Magma, we searched for values of a for which this product of discriminants was equal to a rational square. We found two values of a, namely $a = \frac{841}{840}$ and $a = \frac{840}{841}$. The first value of a gave rise to the previous example. The second value of a leads to another polynomial in $\overline{D}(1683, 3, \mathbb{Q})$ that is a scaled version of the previous example.

$$\begin{aligned} f(x) =& x^{1681}(x-1)\left(x-\frac{840}{841}\right), \\ f'(x) =& \frac{x^{1680}(493x-492)(2871x-2870)}{841}, \\ f''(x) =& \frac{2x^{1679}(9251x-9225)(128673x-128576)}{841}, \\ f'''(x) =& \frac{10086x^{1678}(841x-840)(471801x-470120)}{841}. \end{aligned}$$

We finish with two questions. Are there infinitely many pairs (n, a), where $n \geq 3$ is an integer and $a \neq 0, 1$ is a rational number such that $f_{n,a}(x) \in \overline{D}(n+2, 3, \mathbb{Q})$? Does there exist an integer $n \geq 3$ and a rational number $a \neq 0, 1$ such that $f_{n,a}(x) \in \overline{D}(n+2, 4, \mathbb{Q})$?

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