EUROPEAN JOURNAL OF MATHEMATICAL SCIENCES
Vol. 2, No. 2, 2013, 215-228
ISSN 2147-5512 – www.ejmathsci.com

Continuous Dependence on the Impulsive Effects of Dying Solutions of Systems Differential Equations with Variable Structure and Impulses

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Abstract. Basic object of research in this paper are systems differential equations with variable structure and impulses. Switching moments, in which a change of the structure and impulsive effects on the solutions are determined by means of the switching hyperplanes, belonging to the phase space system. Changing the structure and impulsive effects on the solutions is performed at the switching moments, which are determined by the switching hyperplanes of phase space system.
The switching moments coincide with the moments, when the trajectory of corresponding initial problem meets the switching hyperplanes.
The main aim of this studies is finding the sufficient conditions for continuous dependence of the solutions of systems differential equations, specified above.
We will clarify that:

- The solutions are dying due to the impulsive effects;
- Continuous dependence is on the perturbations in initial conditions and impulsive effects;
- Continuous dependence is on an arbitrary closed interval, which is contained in maximum interval of existence of the solutions.

2010 Mathematics Subject Classifications: 34A12, 34A37

Key Words and Phrases: Variable structure, Impulses, Variable impulsive moments, Switching functions

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1. Introduction

The applications of differential equations with variable structure (without impulsive effects) are primarily in the control theory: [6, 11, 14, 17, 18, 21, 26, 29].

Impulsive equations (with fixed structure) are used most commonly for describing and study development of dynamic processes, subjected to the discrete in time external influences: [1–5, 12–16, 19, 20, 22, 24, 25, 27, 28, 30].

Differential equations with variable structure and impulses are introduced in [23]. Some qualitative characteristics of their solutions are studied in [7–9, 14].

The following initial problem is the main object of investigation

$$\frac{dx}{dt} = f_i(t,x), \text{ if } \langle a_i, x(t)\rangle \neq \alpha_i, \quad \text{i.e.} \quad t_{i-1} < t < t_i, \tag{1}$$

$$\langle a_i, x(t_i)\rangle = \alpha_i, \quad i = 1,2,\ldots, \tag{2}$$

$$x(t_i+0) = x(t_i) + I_i(x(t_i)), \tag{3}$$

$$x(t_0) = x_0, \tag{4}$$

where

- The functions $f_i : \mathbb{R}^+ \times D \longrightarrow \mathbb{R}^n$ and domain $D \subset \mathbb{R}^n$;
- The vectors $a_i = \left(a_i^1, a_i^2, \ldots, a_i^n\right) \in \mathbb{R}^n,\ a_i \neq 0$ and the constants $\alpha_i \in \mathbb{R}$;
- $\langle .,. \rangle$ is the Euclidean scalar product in $\mathbb{R}^n$;
- The switching functions $\varphi_i = \langle a_i, x\rangle - \alpha_i, \quad i = 1,2,\ldots$;
- The switching sets $\Phi_i = \left\{x \in D; \langle a_i, x\rangle = \alpha_i\right\}, \quad i = 1,2,\ldots$, are parts of the hyperplanes, situated in phase space D of the system;
- The functions $I_i : D \longrightarrow \mathbb{R}^n$ and $\left(Id + I_i\right) : D \longrightarrow D$;
- The initial point $\left(t_0, x_0\right) \in \mathbb{R}^+ \times D$ and $\langle a_1, x_0\rangle \neq \alpha_1$.

The solution of initial problem is a piecewise continuous function with first type points of discontinuity: $t_1, t_2, \ldots$. It is continuous from the left at all points in its domain. The points $t_1, t_2, \ldots$ are named moments of switching. The functions $I_i,\ i = 1,2,\ldots$, are called impulsive. As seen from (1) and (2), the functions $\varphi_i(x) = \langle a_i, x\rangle - \alpha_i$ are linear, and their corresponding sets:

$$\Phi_i = \left\{x \in D; \langle a_i, x\rangle = a_i^1 x^1 + a_i^2 x^2 + \ldots + a_i^n x^n = \alpha_i\right\}, \quad i = 1,2,\ldots$$

are parts of the hyperplanes, situated in the phase space D. The functions and their corresponding sets $\Phi_i, i = 1,2,\ldots$, are named switching functions and switching sets.

The following notations will be used:

- $f = \{f_1, f_2, \ldots\}, \quad \varphi = \{\varphi_1, \varphi_2, \ldots\}, \quad I = \{I_1, I_2, \ldots\}$;

- $x\left(t;t_0,x_0,f,\varphi,I\right)$ is the solution of problem (1), (2), (3), (4);
- $x_i\left(t;t_0,x_0\right)$ is the solution of problem with fixed structure and without impulses

$$\frac{dx}{dt}=f_i\left(t,x\right),\ x\left(t_0\right)=x_0,\quad i=1,2,\ldots; \tag{5}$$

- The curve $\gamma\left(t_0,x_0,f,\varphi,I\right)=\left\{x\left(t;t_0,x_0,f,\varphi,I\right),\ t\in J\left(t_0,x_0,f,\varphi,I\right)\right\}$ is the trajectory of problem considered, where $J(t_0,x_0,f,\varphi,I)$ is the maximum interval of existence of the solution;
- The curve $\gamma_i\left(t_0,x_0\right)=\left\{x\left(t;t_0,x_0\right),\ t\in J\left(t_0,x_0,f_i\right)\right\}$ is the trajectory of problem (5), where $J\left(t_0,x_0,f_i\right)$ is the maximum interval of existence of the solution, $i=1,2\ldots$;
- $\parallel . \parallel$ and $\langle .,. \rangle$ are the Euclidean norm and scalar product, respectively in $\mathbb{R}^n$.

Further, the following conditions will be used:

H1. The functions $f\in C[\mathbb{R}^+\times D,\mathbb{R}^n],\ i=1,2,\ldots$.

H2. The functions $I_i\in C[\Phi_i,\mathbb{R}^n]$ and $\left(Id+I_i\right):\Phi_i\longrightarrow D,\ i=1,2,\ldots$.

H3. For any point $\left(t_0,x_0\right)\in\mathbb{R}^+\times D$ and for each $i=1,2,\ldots$ the solution of initial problem (5) exists and is unique for $t\geq t_0$.

H4. The equalities:

$$\parallel a_i\parallel=1,\quad i=1,2,\ldots$$

are fulfilled;

H5. The inequalities:

$$(\langle a_i,(Id+I_{i-1})(x)\rangle-\alpha_i).\langle a_i,f_i(t,x)\rangle<0,\ (t,x)\in\mathbb{R}^+\times D,\quad i=1,2,\ldots$$

are valid, where $I_0(x)=0$ for $x\in D$.

H6. There exist constants $C_{\langle a_i,f_i\rangle}>0$ such that

$$(\forall(t,x)\in\mathbb{R}^+\times D)\Longrightarrow|\langle a_i,f_i(t,x)\rangle|\geq C_{\langle a_i,f_i\rangle},\quad i=1,2,\ldots.$$

H7. There exist constants $C_{a_i}>0$ such that

$$(\forall x\in\Phi_i)\Longrightarrow|\langle a_{i+1},(Id+I_i)(x)\rangle-\alpha_{i+1}|\leq C_{a_{i+1}},\quad i=1,2,\ldots.$$

H8. The series $\sum_{i=1}^{\infty}\frac{C_{a_{i+1}}}{C_{\langle a_{i+1},f_{i+1}\rangle}}$ is convergent.

H9. There exist constants $C_{f_i}>0$ such that

$$(\forall(t,x)\in\mathbb{R}^+\times D)\Longrightarrow\parallel f_i(t,x)\parallel\leq C_{f_i},\quad i=1,2,\ldots.$$

H10. There exist constants $C_{LipI_i}>0$ such that

$$(\forall x',x''\in D)\Longrightarrow|I_i(x')-I_i(x'')|\leq C_{LipI_i}\parallel x'-x''\parallel,\quad i=1,2,\ldots.$$

2. The Death of Solutions

Definition 1. *We say that the solution of system* (1), (2), (3) *dies due to the impulsive effects, if:*

1. *It is fulfilled*

$$(\forall t_0 \geq 0)\,(\forall x_0 \in D)\,(\forall i = 1, 2, \ldots) \Longrightarrow J\left(t_0, x_0, f_i\right) = [t_0, \infty);$$

2. *We have*

$$\begin{gathered}(\forall t_0 \geq 0)\,(\forall x_0 \in D)\\ (\exists I_1 = I_1(t_0, x_0), I_2 = I_2(t_0, x_0), \ldots; \quad I_1, I_2, \ldots : D \longrightarrow \mathbb{R}^n)\\ (\exists t^0 = t^0(t_0, x_0, I_1, I_2, \ldots) = \ const \ \in \mathbb{R}, \quad t^0 > t_0):\\ J\left(t_0, x_0, f, \varphi, I\right) = [t_0, t^0).\end{gathered}$$

In other words, it is satisfied:

1. *For every choice of an initial point from the extended phase space of system under consideration and for any fixed right side, belonging to the set of right sides for the basic system* (1), *the solution of initial problem with fixed structure and without impulses is continuable up to infinity;*

2. *For every choice of an initial point from the extended phase space, the switching functions exist, such that the solution of corresponding problem with variable structure and impulses* (1), (2), (3), (4) *has a limited maximum interval of existence.*

Theorem 1. *Let the conditions H1–H6 be satisfied.*
Then the trajectory of problem (1), (2), (3), (4) *meets each of the hyperplanes* Φ_i, $i = 1, 2, \ldots$.

Proof. We will show that the trajectory of problem meets the hyperplane Φ_1. From H5 it follows that at least one of the following two cases is fulfilled:
Case 1. $(\langle a_1, x\rangle - \alpha_1) < 0,\ x \in D$ and $\langle a_1, f_1(t, x)\rangle > 0,\ (t, x) \in \mathbb{R}^+ \times D$;
Case 2. $(\langle a_1, x\rangle - \alpha_1) > 0,\ x \in D$ and $\langle a_1, f_1(t, x)\rangle < 0,\ (t, x) \in \mathbb{R}^+ \times D$.
We will discuss Case 2. The first case is considered similarly. Function

$$\psi_1(t) = \langle a_1, x_1(t; t_0, x_0)\rangle - \alpha_1$$

is introduced, where $x_1(t; t_0, x_0)$ is the solution of problem (5) for $i = 1$. Function ψ_1 is defined for $J\left(t_0, x_0, f_i\right) = [t_0, \infty)$. We have

$$\psi_1(t_0) = \langle a_1, x_1(t_0; t_0, x_0)\rangle - \alpha_1 = \langle a_1, x_0\rangle - \alpha_1 > 0.$$

According to condition H6, it is satisfied

$$\frac{d}{dt}\psi_1(t) = \langle a_1, \frac{d}{dt}x_1(t; t_0, x_0)\rangle$$

$$\begin{aligned}
&=\langle a_1, f_1(t, x_1(t; t_0, x_0))\rangle \\
&=-\,|\langle a_1, f_1(t; x_1(t; t_0, x_0))\rangle| \\
&\leq -\,C_{\langle a_1, f_1\rangle} \\
&=-\ \text{const}\ < 0.
\end{aligned}$$

By the facts

$$\psi_1(t_0) > 0 \quad \text{and} \quad \frac{d}{dt}\psi_1(t) \leq -\ \text{const}\ < 0, \quad t > t_0,$$

it follows that there exists a point $t_1 > t_0$, such that

$$\langle a_1, x_1(t_1; t_0, x_0)\rangle - \alpha_1 = \psi_1(t_1) = 0.$$

It means that at the moment t_1, the trajectory $\gamma_1(t_0, x_0)$ meets hyperplane Φ_1. Given

$$\gamma(t_0, x_0, f, \varphi, I) \equiv \gamma_1(t_0, x_0) \text{ for } t_0 \leq t \leq t_1,$$

we conclude that the trajectory of problem (1), (2), (3), (4) meets also the hyperplane Φ_1 at moment t_1.

Assume that, the trajectory of problem investigated consistently meets the hyper-planes $\Phi_1, \Phi_2, \ldots, \Phi_i$, respectively in the moments $t_1, t_2, \ldots, t_i$, at which $t_1 < t_2 <, \ldots, < t_i$ is fulfilled. We will show that the trajectory $\gamma_{i+1}(t_0, x(t_i + 0; t_0, x_0))$ meets hyperplane Φ_{i+1}, whence it follows that the same is true for the studied trajectory $\gamma(t_0, x_0, f, \varphi, I)$. Again, take into account condition H5, without loss of generality, we assume that the following inequalities are valid:

$$\langle a_{i+1}(Id + I_i)(x)\rangle - \alpha_{i+1} > 0,\ x \in D \text{ and } \langle a_{i+1}, f_{i+1}(t, x)\rangle < 0,\ (t, x) \in \mathbb{R}^+ \times D. \tag{6}$$

We coincide the function ψ_{i+1}, defined by

$$\psi_{i+1}(t) = \langle a_{i+1}, x_{i+1}(t; t_i, x(t_i + 0; t_0, x_0))\rangle - \alpha_{i+1}, \quad t \geq t_i. \tag{7}$$

We have

$$\begin{aligned}
\psi_{i+1}(t) &=\langle a_{i+1}, x_{i+1}(t_i; t_i, x(t_i + 0; t_0, x_0, f, \varphi, I))\rangle - \alpha_{i+1} \\
&=\langle a_{i+1}, x(t_i + 0; t_0, x_0, f, \varphi, I)\rangle - \alpha_{i+1} \\
&=\langle a_{i+1}, x(t_i; t_0, x_0, f, \varphi, I) + I_i(x(t_i; t_0, x_0, f, \varphi, I))\rangle - \alpha_{i+1} \\
&=\langle a_{i+1}, (Id + I)(x(t_i; t_0, x_0, f, \varphi, I))\rangle - \alpha_{i+1} > 0.
\end{aligned}$$

When $t > t_i$, it is fulfilled

$$\begin{aligned}
\frac{d}{dt}\psi_{i+1}(t) &=\langle a_{i+1}, f_{i+1}(t, x_{i+1}(t; t_0, x(t; t_0, x_0, f, \varphi, I)))\rangle \\
&=-\,|\langle a_{i+1}, f_{i+1}(t, x_{i+1}(t; t_0, x(t; t_0, x_0, f, \varphi, I)))\rangle| \\
&\leq -\,C_{\langle a_{i+1}, f_{i+1}\rangle}
\end{aligned}$$

$$= - \text{ const } < 0.$$

Therefore, there exists a point $t_{i+1} > t_i$ such that

$$\psi_{i+1}(t_{i+1}) = 0 \Longleftrightarrow \langle a_{i+1}, x_{i+1}(t_{i+1}; t_0, x(t_i + 0; t_0, x_0, f, \varphi, I))\rangle - \alpha_{i+1} = 0.$$

The last equality shows that the trajectory $\gamma_{i+1}(t_0, x(t_i + 0; t_0, x_0, f, \varphi, I))$ meets the hyperplane Φ_{i+1} at the moment t_{i+1}. This also applies to the trajectory $\gamma(t_0; x_0, f, \varphi, I)$.

The proof of a theorem follows by induction.

The theorem is proved.

Theorem 2. *Let the conditions H1–H7 be satisfied.*

Then the following estimates are valid

$$t_{i+1} - t_i \le \frac{C_{a_{i+1}}}{C_{\langle a_{i+1}, f_{i+1}\rangle}}, \quad i = 1, 2, \dots.$$

Proof. Let i be arbitrary natural number. Consider the function ψ_{i+1}, defined by equality (7). Directly, we get the next equality

$$\psi_{i+1}(t) = \begin{cases} \langle a_{i+1}, x(t_i + 0; t_0, x_0, f, \varphi, I)\rangle - \alpha_{i+1} & \\ = \langle a_{i+1}, x(t_i; t_0, x_0, f, \varphi, I) + I_i(x(t_i; t_0, x_0, f, \varphi, I))\rangle - \alpha_{i+1}, & t = t_i; \\ \langle a_{i+1}, x(t; t_0, x_0, f, \varphi, I)\rangle - \alpha_{i+1}, & t_i < t \le t_{i+1}. \end{cases}$$

Suppose again that the inequalities (6) are valid. Under condition H7, we find

$$\begin{aligned} \psi_{i+1}(t_{i+1}) - \psi_{i+1}(t_i) =& \langle a_{i+1}, x(t_{i+1}; t_0, x_0, f, \varphi, I)\rangle - \langle a_{i+1}, x(t_i + 0; t_0, x_0, f, \varphi, I)\rangle \\ =& - \langle a_{i+1}, x_i(t_i; t_0, x_0) + I_i(x(t_i; t_0, x_0))\rangle + \alpha_{i+1} \\ =& |\langle a_{i+1}, (Id + I_i)(x_i(t_i; t_0, x_0))\rangle - \alpha_{i+1}| \\ \le& C_{a_{i+1}}. \end{aligned} \tag{8}$$

On the other hand, using the conditions H6 and H4, we consistently obtain

$$\begin{aligned} \psi_{i+1}(t_{i+1}) - \psi_{i+1}(t_i) =& \frac{d}{dt}\psi_{i+1}(t^*)(t_{i+1} - t_i) \\ =& \frac{d}{dt}\big(\langle a_{i+1}, x(t^*; t_0, x_0, f, \varphi, I)\rangle - \alpha_{i+1}\big).(t_{i+1} - t_i) \\ =& \frac{d}{dt}\big(\langle a_{i+1}, x_{i+1}(t^*; t_0, x(t_i + 0; t_0, x_0, f, \varphi, I))\rangle - \alpha_{i+1}\big).(t_{i+1} - t_i) \\ =& \langle a_{i+1}, f_{i+1}(t^*, x_{i+1}(t^*; t_0, x(t_i + 0; t_0, x_0, f, \varphi, I)))\rangle.(t_{i+1} - t_i) \\ \ge& \parallel a_{i+1} \parallel .C_{\langle a_{i+1}, f_{i+1}\rangle}.(t_{i+1} - t_i) \\ =& C_{\langle a_{i+1}, f_{i+1}\rangle}.(t_{i+1} - t_i), \end{aligned} \tag{9}$$

where the point t^* satisfies inequalities $t_i < t^* < t_{i+1}$. By (8) and (9) it follows the wanted estimate.

The theorem is proved.

Theorem 3. *Let the conditions H1–H8 be satisfied.*
Then the solutions of system (1), (2), (3) *die due to the impulsive effects.*

Proof. It is valid

$$J(t_0, x_0, f, \varphi, I) = [t_0, t_1] \bigcup (t_1, t_2] \bigcup (t_2, t_3] \bigcup \ldots = [t_0, t^0),$$

where

$$\begin{aligned} t^0 &= \lim_{i\to\infty} t_i \\ &= t_1 - \lim_{i\to\infty} \big((t_2 - t_1) + (t_3 - t_2) + \ldots + (t_i - t_{i-1})\big) \\ &= t_1 + \sum_{i=1}^{\infty} (t_{i+1} - t_i) \\ &\le t_1 + \sum_{i=1}^{\infty} \frac{C_{\varphi_{i+1}}}{C_{\langle a_{i+1}, f_{i+1}\rangle}} < \infty. \end{aligned}$$

The theorem is proved.

3. Continuous Dependence on the Impulsive Effects

The perturbed initial problem is introduced:

$$\frac{dx^*}{dt} = f_i(t, x^*), \text{ if } \langle a_i, x^*(t)\rangle \neq \alpha_i, \quad \text{i.e.} \quad t^*_{i-1} < t < t^*_i, \tag{10}$$

$$\langle a_i, x^*(t)\rangle = \alpha_i, \quad i = 1, 2, \ldots, \tag{11}$$

$$x^*(t^*_i + 0) = x^*(t^*_i) + I^*_i(x^*(t^*_i)), \tag{12}$$

$$x^*(t^*_0) = x^*_0, \tag{13}$$

where

- The functions $I^*_i : D \longrightarrow \mathbb{R}^n$, $(Id + I^*_i) : D \longrightarrow D$ and $I^* = \{I^*_1, I^*_2, \ldots\}$;
- The initial point $(t^*_0, x^*_0) \in \mathbb{R}^+ \times D$ and $\langle a_1, x^*_0\rangle \neq \alpha_i$;
- The solution of problem (10), (11), (12), (13) is denoted by $x^*(t; t^*_0, x^*_0, f, \varphi, I^*)$;
- The maximum interval of existence of the solution of problem (10), (11), (12), (13) is denoted by $J^*(t^*_0, x^*_0, f, \varphi, I^*)$;

Definition 2. *We say that the dying solution* $x(t; t_0, x_0, f, \varphi, I)$ *of initial problem* (1), (2), (3), (4) *depends continuously on:*

- *The initial condition, if $I^* = I$ and*

$$(\forall \varepsilon = \textit{ const } > 0)\,(\forall \eta = \textit{ const } > 0)\,(\forall T \in J(t_0, x_0, f, \varphi, I) \bigcap J^*(t_0^*, x_0^*, f, \varphi, I))$$
$$(\exists \delta = \delta(\varepsilon, \eta, T, t_0, x_0) > 0)\,(\exists \eta_i = \eta_i(\varepsilon, \eta, T, t_0, x_0) \geq 0, \quad i = 1, 2, \dots, \ \textstyle\sum_{i=1}^{\infty} \eta_i < \eta) :$$
$$(\forall t_0^* \in \mathbb{R}^+, \ |t_0^* - t_0| < \delta)(\forall x_0^* \in D \bigcap B_\delta(x_0), \ \langle a_1, x_0^* \rangle \neq \alpha_1)$$
$$\Longrightarrow \| x^*(t; t_0^*, x_0^*, f, \varphi, I) - x(t; t_0, x_0, f, \varphi, I) \| < \varepsilon,$$
$$t \in \left[t_0^{\max}, T\right], \ |t - t_i| > \eta_i, \quad i = 1, 2, \dots;$$

- *The impulsive effects, if $(t_0^*, x_0^*) = (t_0, x_0)$ and*

$$(\forall \varepsilon = \textit{ const } > 0)\,(\forall \eta = \textit{ const } > 0)(\forall T \in J(t_0, x_0, f, \varphi, I) \bigcap J^*(t_0, x_0, f, \varphi, I^*))$$
$$(\exists \delta = \delta(\varepsilon, \eta, T, I) > 0)\,(\exists \eta_i = \eta_i(\varepsilon, \eta, T, I) \geq 0, \quad i = 1, 2, \dots, \ \textstyle\sum_{i=1}^{\infty} \eta_i < \eta) :$$
$$(\forall I_i^* : D \longrightarrow \mathbb{R}^n, \ \| I_i^*(x) - I_i(x) \| < \delta, x \in D, \quad i = 1, 2, \dots)$$
$$\Longrightarrow \| x^*(t; t_0, x_0, f, \varphi, I^*) - x(t; t_0, x_0, f, \varphi, I) \| < \varepsilon,$$
$$t \in \left[t_0, T\right], \ |t - t_i| > \eta_i, \quad i = 1, 2, \dots.$$

Theorem 4. *Let the conditions H1, H3, H5, H6, H9 and H10 be satisfied for $i = 1$.*
Then

$$(\exists C_1, C_2, C_3 \in \mathbb{R}) :$$
$$(\forall \omega = \textit{ const } > 0)\,(\exists \delta = \delta(\omega), \ 0 < \delta \leq \omega) :$$
$$(\forall t_0^* \in \mathbb{R}^+, \ |t_0^* - t_0| < \delta)(\forall x_0^* \in D, \ \| x_0^* - x_0 \| < \delta)$$
$$(\forall I_1^* \in C[D, \mathbb{R}^n], \ \| I_1^*(x) - I_1(x) \| < \delta \textit{ for } x \in D)$$

it follows:

1. *The solution $x^*(t; t_0^*, x_0^*, f, \varphi, I^*)$ of system* (10)*,* (11)*,* (12)*,* (13) *cancels the switching function φ_1 at a point t_1^*;*
2. $\| x^*(t; t_0^*, x_0^*, f, \varphi, I^*) - x(t; t_0, x_0, f, \varphi, I) \| \leq \omega, \quad t_0^{\max} < t \leq t_1^{\min}$;
3. $|t_1^* - t_1| \leq C_1 \omega$;
4. $\| x^*(t_1^*; t_0^*, x_0^*, f, \varphi, I^*) - x(t_1; t_0, x_0, f, \varphi, I) \| \leq C_2 \omega$;
5. $\| x^*(t_1^* + 0; t_0^*, x_0^*, f, \varphi, I^*) - x(t_1 + 0; t_0, x_0, f, \varphi, I) \| \leq C_3 \omega$.

Proof. At first, we will show that the trajectory of perturbed problem meets the hyperplane Φ_1. Based on condition H5, we suppose that the next inequalities are valid:

$$(\langle a_1, x \rangle - \alpha_1) < 0, \ x \in D \text{ and } \langle a_1, f_1(t, x) \rangle > 0, \ (t, x) \in \mathbb{R}^+ \times D.$$

The case of reverse inequalities, is considered similar. From the inequalities above, it follows that $(\langle a_1, x_0 \rangle - \alpha_1) < 0$. We have

$$(\exists \delta = \text{ const } > 0) :$$
$$(\forall x_0^* \in D, \ \| x_0^* - x_0 \| < \delta) \Longrightarrow (\langle a_1, x_0^* \rangle - \alpha_1) < 0.$$

The function $\psi_1^*(t) = \langle a_1, x_1(t; t_0^*, x_0^*)\rangle - \alpha_1$ is introduced, where $t \geq t_0^*$ and $x_1(t; t_0^*, x_0^*)$ is the solution of problem

$$\frac{dx}{dt} = f_1(t, x), \quad x(t_0^*) = x_0^*.$$

For $t = t_0^*$, it is true

$$\psi_1^*(t_0^*) = \langle a_1, x_1(t_0^*; t_0^*, x_0^*)\rangle - \alpha_1 = \langle a_1, x_0^*\rangle - \alpha_1 < 0. \tag{14}$$

On the other hand, using condition H6 for $t > t_0^*$, we have

$$\frac{d}{dt}\psi_1^*(t) = \langle a_1, f_1(t, x(t; t_0^*, x_0^*))\rangle \geq C_{\langle a_1, f_1\rangle} = \text{ const } > 0. \tag{15}$$

By (14) and (15) it follows that there exists a point $t_1^* > t_0^*$, such that

$$\langle a_1, x_1(t_1^*; t_0^*, x_0^*)\rangle - \alpha_1 = \psi_1^*(t_1^*) = 0.$$

From the last equality, it follows that the solution of system (10), (11), (12), (13) cancels the switching function φ_1 at the point t_1^* .

According to the theorem of continuous dependence of solutions of systems differential equations with fixed structure and without impulses on the initial condition (see Theorem 7.1, S 7, Chapter I, [10]- further for brevity, called the theorem of continuous dependence), we obtain that

$$\begin{gathered}(\forall \omega = \text{ const } > 0)\,(\exists \delta = \delta(\omega),\ 0 < \delta \leq \omega):\\ (\forall t_0^* \in \mathbb{R}^+,\ |t_0^* - t_0| < \delta)(\forall x_0^* \in D,\ \| x_0^* - x_0 \| < \delta)\end{gathered}$$

it follows

$$\begin{aligned}&\| x_1(t; t_0^*, x_0^*) - x_1(t; t_0, x_0) \| \\ &= \| x^*(t; t_0^*, x_0^*, f, \varphi . I^*) - x(t; t_0, x_0, f, \varphi, I) \| \leq \omega, \quad t_0^{\max} < t \leq t_1^{\min}.\end{aligned} \tag{16}$$

Let $t_1 = t_1^{\min} = \min\{t_1^*, t_1\}$. Then in particular, from the inequality (16) for $t = t_1$, it follows

$$\begin{aligned}\omega &\geq \| x_1(t_1; t_0^*, x_0^*) - x_1(t_1; t_0, x_0) \| \\ &= \| x_1(t_1; t_0^*, x_0^*) - x_1(t_1; t_0, x_0) \| \,.\, \| a_1 \| \\ &\geq |\langle x_1(t_1; t_0^*, x_0^*) - x_1(t_1; t_0, x_0), a_1\rangle| \\ &= |\langle x_1(t_1; t_0^*, x_0^*), a_1\rangle - \alpha_1| \\ &= |\langle x_1(t_1; t_0^*, x_0^*), a_1\rangle - \langle x_1(t_1^*; t_0^*, x_0^*), a_1\rangle| \\ &= |\langle x_1(t_1; t_0^*, x_0^*) - x_1(t_1^*; t_0^*, x_0^*), a_1\rangle| \\ &= \left|\left\langle \int_{t_1}^{t_1^*} f_1(\tau, x_1(\tau; t_0^*, x_0^*)) d\tau, a_1 \right\rangle\right| \\ &= \left| \int_{t_1}^{t_1^*} \langle f_1(\tau, x_1(\tau; t_0^*, x_0^*)), a_1\rangle d\tau \right|\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_1}^{t_1^*} \left\langle f_1(\tau, x_1(\tau; t_0^*, x_0^*)), a_1 \right\rangle d\tau \\
&\geq C_{\langle a_1, f_1 \rangle}(t_1^* - t_1),
\end{aligned}
$$

whence, we obtain the estimate

$$
|t_1^* - t_1| = t_1^* - t_1 \leq \frac{\omega}{C_{\langle a_1, f_1 \rangle}} = C_1 \omega.
$$

Statement 4 of theorem is proved by using the following sequence of inequalities:

$$
\begin{aligned}
\| x^*(t_1^*; t_0^*, x_0^*, f, \varphi, I^*) &- x(t_1; t_0, x_0, f, \varphi, I) \| \\
&= \| x_1(t_1^*; t_0^*, x_0^*) - x_1(t_1; t_0, x_0) \| \\
&\leq \| x_1(t_1^*; t_0^*, x_0^*) - x_1(t_1; t_0^*, x_0^*) \| + \| x_1(t_1; t_0^*, x_0^*) - x_1(t_1; t_0, x_0) \| \\
&\leq \left\| \int_{t_1}^{t_1^*} f_1(\tau, x_1(\tau; t_0^*, x_0^*)) d\tau \right\| + \omega \\
&\leq C_{f_1}(t_1^* - t_1) + \omega \\
&\leq C_{f_1} C_1 \omega + \omega \\
&= C_2 \omega.
\end{aligned}
$$

Finally, the last statement of theorem follows from the inequalities:

$$
\begin{aligned}
&\| x^*(t_1^*+0; t_0^*, x_0^*, f, \varphi, I^*) - x(t_1 + 0; t_0, x_0, f, \varphi, I) \| \\
&\quad = \| x_1(t_1^*; t_0^*, x_0^*) + I_1^*(x_1(t_1^*; t_0^*, x_0^*)) - x_1(t_1; t_0, x_0) - I_1(x_1(t_1; t_0, x_0)) \| \\
&\quad \leq \| x_1(t_1^*; t_0^*, x_0^*) - x_1(t_1; t_0, x_0) \| + \| I_1^*(x_1(t_1^*; t_0^*, x_0^*)) - I_1(x_1(t_1; t_0, x_0)) \| \\
&\quad \leq C_2 \omega + \| I_1^*(x_1(t_1^*; t_0^*, x_0^*)) - I_1(x_1(t_1^*; t_0^*, x_0^*)) \| \\
&\qquad + \| I_1(x_1(t_1^*; t_0^*, x_0^*)) - I_1(x_1(t_1; t_0, x_0)) \| \\
&\quad \leq C_2 \omega + \delta + C_{LipI_1} \| x_1(t_1^*; t_0^*, x_0^*) - x_1(t_1; t_0, x_0) \| \\
&\quad \leq C_2 \omega + \omega + C_{LipI_1} C_2 \omega \\
&\quad = C_3 \omega.
\end{aligned}
$$

The theorem is proved.

Theorem 5. *Let the conditions H1–H10 be satisfied.*

Then the dying solution $x(t; t_0, x_0, f, \varphi, I)$ *of initial problem* (1), (2), (3), (4) *depends continuously on the initial condition and impulsive effects.*

Proof. For the convenience, we structure the theorem proof into several parts:
Part 1. As:

- $J(t_0, x_0, f, \varphi, I) = (t_0, t^0)$, $t^0 = \text{const}$ (see Theorem 3);

- $T \in J(t_0, x_0, f, \varphi, I) \Leftrightarrow t_0 \leq T < t^0$;
- $\lim\limits_{i \to \infty} t_i = t^0$,

it follows that the closed interval $[t_0, T]$ contains a finite number of switching moments. Let the inequalities $t_0 < t_1 < t_2 < \ldots < t_k \leq T < t_{k+1}$ be valid.

Part 2. Let ε and η be arbitrary positive constants. Let the constants $\eta_0, \eta_1, \eta_2, \ldots$ satisfy the inequalities:

$$\begin{aligned}
&0 < \eta_0 < \tfrac{1}{2}(t_1 - t_0);\\
&0 < \eta_i < \min\left\{\tfrac{1}{2}(t_i - t_{i-1}), \tfrac{1}{2}(t_{i+1} - t_i), \eta/k\right\}, \quad i = 1, 2, \ldots, k;\\
&\eta_i = 0, \quad i = k+1, k+2, \ldots.
\end{aligned}$$

It is obvious that $\sum_{i=1}^{\infty} \eta_i < \eta$. In addition, if $|t_i^* - t_i| \leq \eta_i$ for $i = 0, 1, \ldots$, then

$$t_0^{\min} \leq t_0^{\max} < t_1^{\min} \leq t_1^{\max} < t_2^{\min} \leq t_2^{\max} < \ldots.$$

Part 3. Applying Theorem 4, we have:

$$(\forall \delta_1,\ 0 < \delta_1 < \varepsilon)\,(\exists \delta_0 = \delta_0(\delta_1, \varepsilon, \eta_1, t_0, x_0) = \text{ const}, \quad 0 < \delta_0 < \delta_1):$$

$$(\forall t_0^* \in \mathbb{R}^+,\ |t_0^* - t_0| < \delta_0)\,(\forall x_0^* \in D,\ \| x_0^* - x_0 \| < \delta_0)$$
$$(\forall I_1^* \in C[D, \mathbb{R}^n],\ \| I_1^*(x) - I_1(x) \| < \delta \quad \text{for} \quad x \in D)$$

it follows:

- The solution $x^*(t; t_0^*, x_0^*, f, \varphi, I^*)$ cancels switching function φ_1 at point t_1^*;
- $\| x^*(t; t_0^*, x_0^*, f, \varphi, I^*) - x(t; t_0, x_0, f, \varphi, I) \| \leq \delta_1 < \varepsilon, \quad t_0^{\max} < t \leq t_1^{\min}$;
- $|t_1^* - t_1| \leq \min\{\delta_1, \eta_1\}$;
- $\| x^*(t_1^* + 0; t_0^*, x_0^*, f, \varphi, I^*) - x(t_1 + 0; t_0, x_0, f, \varphi, I) \| \leq \delta_1$.

Part 4. Applying Theorem 4 consistently for $i = 1, 2, \ldots, k$. We obtain:

$$(\forall \delta_{i+1},\ 0 < \delta_{i+1} < \varepsilon)\,(\exists \delta_i = \delta_i(\delta_{i+1}, \varepsilon, \eta_{i+1}, t_0, x_0) = \text{const}, \quad 0 < \delta_i < \delta_{i+1}):$$

$$(\forall t_i^* \in \mathbb{R}^+,\ |t_i^* - t_i| < \delta_i)\,(\| x^*(t_i^{\max} + 0; t_0^*, x_0^*, f, \varphi, I^*) - x(t_i^{\max} + 0; t_0, x_0, f, \varphi, I) \| \leq \delta_i)$$
$$(\forall I_{i+1}^* \in C[D, \mathbb{R}^n],\ \| I_{i+1}^*(x) - I_{i+1}(x) \| < \delta \quad for \quad x \in D)$$

it follows:

- The solution

  $$x^*(t; t_i^*, x^*(t_i^{\max} + 0; t_0^*, x_0^*, f, \varphi, I^*), f, \varphi, I^*) = x^*(t; t_0^*, x_0^*, f, \varphi, I^*)$$

  cancels the switching function φ_{i+1} at point t_{i+1}^*;

- $\| x^*(t;t_0^*,x_0^*,f,\varphi,I^*) - x(t;t_0,x_0,f,\varphi,I) \| \leq \delta_{i+1} < \varepsilon, \quad t_i^{\max} < t \leq t_{i+1}^{\min}$;
- $|t_{i+1}^* - t_{i+1}| \leq \min\{\delta_{i+1}, \eta_{i+1}\}$;
- $\| x^*(t_{i+1}^*+0;t_0^*,x_0^*,f,\varphi,I^*) - x(t_{i+1}+0;t_0,x_0,f,\varphi,I) \| \leq \delta_{i+1}$.

Part 5. The constants $\delta_0, \delta_1, \dots, \delta_{k+1}$ are defined in descending order. Firstly, we fix δ_{k+1}, $0 < \delta_{k+1} < \min\{\varepsilon, \eta_{k+1}\}$. After that, we determine consistently (in that decreasing order) the constants $\delta_k, \delta_{k-1}, \dots, \delta_0$, $(\delta_{k+1} > \delta_k >, \dots, > \delta_0)$. From the previous two parts, we receive that

$$\begin{gathered}(\forall t_0^* \in \mathbb{R}^+,\ |t_0^* - t_0| < \delta_0)\,(\forall x_0^* \in D,\ \| x_0^* - x_0 \| < \delta_0)\\ (\forall I_i^* \in C[D,\mathbb{R}^n], \| I_i^*(x) - I_i(x) \| < \delta_i < \delta_0 \quad \text{for} \quad x \in D)\\ \Longrightarrow \| x^*(t;t_0^*,x_0^*,f,\varphi,I^*) - x(t;t_0,x_0,f,\varphi,I) \| < \varepsilon,\\ t \in [t_0^{\max}, t_1^{\min}] \bigcup (t_1^{\max}, t_2^{\min}] \bigcup \dots \bigcup (t_k^{\max}, t_{k+1}^{\min}].\end{gathered}$$

Part 6. We have

$$\begin{aligned}&[t_0^{\max}, t_1^{\min}] \bigcup (t_1^{\max}, t_2^{\min}] \bigcup \dots \bigcup (t_k^{\max}, t_{k+1}^{\min}]\\ &\quad \supset [t_0^{\max}, t_{k+1}^{\min}] \setminus ((t_1^{\min}, t_1^{\max}) \bigcup (t_2^{\min}, t_2^{\max}) \bigcup \dots \bigcup (t_k^{\min}, t_k^{\max}))\\ &\quad \supset [t_0^{\max}, T] \setminus ([t_1 - \eta_1, t_1 + \eta_1] \bigcup [t_2 - \eta_2, t_2 + \eta_2] \bigcup \dots \bigcup [t_k - \eta_k, t_k + \eta_k])\\ &\quad = [t_0^{\max}, T], |t - t_i| > \eta_i, \quad i = 1, 2, \dots.\end{aligned}$$

Part 7. From the previous two parts, we find

$$\begin{gathered}(\forall t_0^* \in \mathbb{R}^+,\ |t_0^* - t_0| < \delta_0)\,(\forall x_0^* \in D,\ \| x_0^* - x_0 \| < \delta_0)\\ (\forall I_i^* \in C[D,\mathbb{R}^n],\ \| I_i^*(x) - I_i(x) \| < \delta_i < \delta_0 \quad for \quad x \in D)\\ \Longrightarrow \| x^*(t;t_0^*,x_0^*,f,\varphi,I^*) - x(t;t_0,x_0,f,\varphi,I) \| < \varepsilon,\\ t \in [t_0^{\max}, T],\ |t - t_i| > \eta_i, \quad i = 1, 2, \dots.\end{gathered}$$

The theorem is proved.

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