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On a Class of Positive Linear Operators

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Abstract. A new class of positive linear operators have been introduced which contains a number of well known positive linear operators such as Gamma-Operators of Muller, Post-Widder and Modified Post-Widder Operators as particular cases.Some basic approximation properties of this class of operators have been studied in this paper.

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1. Introduction

A number of classes and sequences of positive linear operators (henceforth written as operator) both, of the summation and those defined by integrals have been introduced and studied by a number of authors during the past few decades. Some of well known operators of latter type are the Gamma-Operators of Muller [6], Post-Widder Operators [10], Modified Post-Widder Operators [5], Gauss-Weierstrass Integrals [7] , Convolution type operators [8], Baskakov Operators [1], and the operators studied by De Vore [2], Leviaton [4], Kunwar [3], Sikkema and Rathore [9].

In this paper we will study a class of operators which contains a number of well known operators as special cases. This class of operators was introduced in Kunwar [3]. Now we will give a brief description of the notations and definitions followed by the definition of the operators.

Throughout the paper IR^+ denotes the interval
$(0,\infty), < a, b >$ open interval containing $[a,b] \subseteq IR^+, \chi_{\delta,x}(\chi^c_{\delta,x})$ the characteristic function of the interval $(x-\delta, x+\delta)$ $\{IR^+ - (x-\delta, x+\delta)\}$. The spaces $M(IR^+)$, $M_b(IR^+)$, $Loc(IR^+)$, $L^1(IR^+)$ respectively denote the sets of complex valued measurable, bounded and measurable, locally integrable and Lebesgue integrable functions on IR^+.

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Now we define our operator L_n [3] and give some elementary properties of the same.

$$L_n(f;x) = D(m,n,\alpha)x^{mn+\alpha-1}\int_0^\infty u^{-mn-\alpha}e^{-n(\frac{x}{u})^m}f(u)du$$

where

$$D(m,n,\alpha) = \frac{|m|\,n^{n+\frac{\alpha-1}{m}}}{\Gamma(n+\frac{\alpha-1}{m})}, m \in IR-\{0\}, n > 0, \alpha \in IR.$$

Several well known operators are special cases of L_n : Choosing $m = 1$ and $\alpha = 2$,the operator reduces to the Gamma-Operators of Muller [6] denoted and defined by

(i) $G_n(f;x) = \frac{x^{n+1}}{n!}\int_0^\infty t^n e^{-tx} f(\frac{n}{t})dt$

Choosing $m = -1$, $\alpha = 1$ and $m = -1$, $\alpha = 0$ and by proper substitution the operators L_n reduces respectively to the Post-Widder operators S_n^1 May [5] defined by,

(ii) $S_n^1(f;t) = \frac{1}{(n-1)!}\int_0^\infty e^{-n\frac{u}{t}}u^{n-1}f(u)du$

and the operators $L_{k,t}$ [Widder 10] defined by

(iii) $L_{k,t}(f;x) = \frac{1}{k!}(\frac{k}{t})^{k-1}\int_0^\infty e^{-\frac{ku}{t}}u^k f(u)du$

We will make use of a bounding function introduced by Rathore [7] for establishing the basic convergence theorem for our operators.

Definition 1. *Let $\Omega(>1)$ be a continuous function defined on IR^+. We call Ω, a bounding function if for each compact $K \subseteq IR^+$, there exist positive numbers n_k and M_k such that*

$$L_{n_k}(\Omega;x) < M_k, x \in K$$

For our operators the bounding function is

$$\Omega(u) = u^{-a} + e^{bu^m} + u^c, \text{ where } a,b,c \geq 0.$$

For this bounding function Ω we define

$$D_\Omega = \{f \in Loc(IR^+) \text{ such that } \limsup_{u\longrightarrow 0}\frac{f(u)}{\Omega(u)} \text{ and } \limsup_{u\longrightarrow\infty}\frac{f(u)}{\Omega(u)} \text{ exist }\}$$

2. Basic Approximation

Lemma 1. *If* $0 < \delta < a < b < \infty$ *and* $f \in D_\Omega$, *then*

$$\lim_{n \longrightarrow \infty} n^k L_n(f\chi^c_{\delta,x}; x) = 0 \tag{1}$$

uniformly in $x \in [a,b]$ *for any* $k \in IR^+$.

Proof. Since $f \in D_\Omega$, $\exists$ positive constants A, B and M such that $A < min\{1,a\}$ and $B > max\{1,b\}$ and $\left|f(u)\right| < M\Omega(u)$ for all $u \in (0, \frac{b}{B}) \cup (\frac{a}{A}, \infty)$.Let $J(A,B) = (0,A) \cup (B,\infty)$ then

$$\left| \int\limits_{J(A,B)} u^{mn+\alpha-2} e^{-nu^m} f(\frac{x}{u}) \chi^c_{\delta,x}(\frac{x}{u}) du \right| \leqslant M \int\limits_{J(A,B)} u^{mn+\alpha-2} e^{-nu^m} \Omega(\frac{x}{u}) du \tag{2}$$

For $\Omega(u)$, there exists $n_1, M_1 > 0$ such that $L_{n_1}(\Omega; x) < M_1$, for all $x \in [a,b]$.
For any $\varepsilon > 0$ we have

$$u^m e^{-u^m} < \frac{1}{e} - 2\varepsilon \text{ for almost all } u \in J(A,B).$$

Hence if $n > n_0 > n_1$,we have

$$\int\limits_{J(A,B)} u^{mn+\alpha-2} e^{-nu^m} \Omega(\frac{x}{u}) du \leqslant (\frac{1}{e} - 2\varepsilon)^{n-n_0} \frac{1}{D(m,n,\alpha)} L_{n_0}(\Omega; x)$$

$$\leqslant M_1 \frac{1}{D(m,n,\alpha)} (\frac{1}{e})^{n-n_0} (\frac{1}{e} - 2\varepsilon)^{n-n_0} = [\frac{M(n_0,n_1)}{M}](\frac{1}{e} - 2\varepsilon)^n \tag{3}$$

By choosing a positive δ_1 such that $\frac{b}{b+\delta} < 1 - \delta_1 < 1 + \delta_1 < \frac{b}{b-\delta}$ and using the property of the function $u^m e^{-u^m}$ for sufficiently small ε we have $u^m e^{-u^m} < \frac{1}{e} - 2\varepsilon$ for almost all $u \in IR^+ - (1 - \delta_1, 1 + \delta_1)$. Hence

$$\left| \int_A^B u^{mn+\alpha-2} e^{-nu^m} f(\frac{x}{u}) \chi^c_{\delta,x}(\frac{x}{u}) du \right| \leq (\frac{1}{e} - 2\varepsilon)^{n-n_0} \int_A^B u^{mn_0+\alpha-2} e^{-n_0 u^m} \left| f(\frac{x}{u}) \right| du$$

$$\leq (\frac{1}{e} - 2\varepsilon)^{n-n_0} D(m,n_0,\alpha) \frac{A^\alpha + B^\alpha}{e^{n_0}} \int_{\frac{a}{B}}^{\frac{b}{A}} \left| f(u) \right| du = (\frac{1}{e} - 2\varepsilon)^n M(n_0) \tag{4}$$

Since $u^m e^{-u^m}$ is continuous at u=1, there exists a $\delta_2 > 0$ s.t. $u^m e^{-u^m} > \frac{1}{e} - \varepsilon$, for all $u \in (1 - \delta_2, 1 + \delta_2)$. Therefore

$$\frac{1}{D(m,n,\alpha)} > \int_{1-\delta_2}^{1+\delta_2} u^{mn+\alpha-2} e^{-nu^m} du > \delta_2 (\frac{1}{e} - \varepsilon)^n \tag{5}$$

Thus (2), (3), (4) and (5) imply that

$$\left|L_n(f\chi^c_{\delta,x};x)\right| \leq \frac{M(n_0,n_1)+M(n_0)}{\delta_2}\frac{(\frac{1}{e}-2\varepsilon)^n}{(\frac{1}{e}-\varepsilon)^n}$$

since

$$\lim_{n\to\infty} n^k \frac{(\frac{1}{e}-2\varepsilon)^n}{(\frac{1}{e}-\varepsilon)^n} = 0 \text{ for any } k \in IR^+$$

this proves the lemma.

Next we prove the following basic approximation theorem.

Theorem 1. *If $f \in D_\Omega$ and is continuous at a point $x \in IR^+$, then there holds*

$$\lim_{n\to\infty} L_n(f;x) = f(x) \tag{6}$$

further if f is continuous on $< a, b >$, then convergence of (6) *holds uniformly in* $[a,b]$.

Proof. By continuity of $f(u)$ at $u = x$, given $\varepsilon > 0$ arbitrary we can find a $\delta > 0$ such that

$$\left|f(u) - f(x)\right| < \frac{\varepsilon}{2}, |u-x| < \delta \tag{7}$$

where in the case of uniformity δ is independent of $x \in [a,b]$. In view of (7) for all $u \in IR^+$ there holds

$$\left|f(u)-f(x)\right| < \frac{\varepsilon}{2} + (\left|f(u)\right| + \left|f(x)\right|)\chi^c_{\delta,x}(u) \tag{8}$$

Using the linearity, poisitivity and the property that $L_n(1;x) = 1$ of L_n from the inequality (8) we have

$$\left|L_n(f;x)-f(x)\right| \leq \frac{\varepsilon}{2} + L_n((\left|f(u)\right| + \left|f(x)\right|)\chi^c_{\delta,x}(u);x)$$

since

$$(\left|f(u)\right| + \left|f(x)\right|)\chi^c_{\delta,x}(u) \in D_\Omega$$

using Lemma 1, we can find a n_0 such that

$$L_n((\left|f(u)\right| + \left|f(x)\right|)\chi^c_{\delta,x}(u);x) < \frac{\varepsilon}{2}$$

for all $n > n_0$ and ($x \in [a,b]$ in this uniformity case). Hence

$$\left|L_n(f;x)-f(x)\right| \leq \varepsilon \text{ , for } n > n_0$$

since $\varepsilon > 0$ is arbitrary, the theorem holds.

3. Voronovskaya Theorems

The existence of the third order derivative at the point $u=1$ and the non zero second order derivative at $u=1$ of the function $u^m e^{-u^m}$ ensures that the operators L_n possesses a Voronovskaya-type asymptotic formula. The main result will be followed by the following auxiliary results.

Lemma 2. *If $\delta > 0$ is sufficiently small, then the following equalities are true for the operators $L_n(f;x)$.*

(i) $\lim_{n\to\infty} mnD(m,n,\alpha)\int_{1-\delta}^{1+\delta} u^{\alpha+mn+m-3}e^{-(n+1)u^m}(1-u^m)du = \frac{2-\alpha}{e}$

(ii) $\lim_{n\to\infty} m^2nD(m,n,\alpha)\int_{1-\delta}^{1+\delta} u^{\alpha+mn+2m-1}e^{-(n+2)u^m}(1-u^m)^2du = (\frac{m}{e})^2$

(iii) $\lim_{n\to\infty} nD(m,n,\alpha)\int_{1-\delta}^{1+\delta} u^{\alpha+mn-2}e^{-nu^m}(\frac{1}{e}-u^m e^{-u^m})du = \frac{1}{2e}$

Proof. Integrating by parts, taking $u^{\alpha-2}$ as the first functon

$$mD(m,n,\alpha)\int_{1-\delta}^{1+\delta} u^{\alpha+mn+m-3}e^{-(n+1)u^m}(1-u^m)du$$

$$= D(m,n,\alpha)\left[\frac{u^{\alpha+mn+m-2}e^{-(n+1)u^m}}{n+1}\right]_{1-\delta}^{1+\delta} - \frac{(\alpha+2)}{(n+1)}\int_{1-\delta}^{1+\delta} u^{\alpha+mn+m-2}e^{-(n+1)u^m}du$$

for a given $\varepsilon > 0$ we can find a $\delta_1 (0 < \delta_1 < \delta)$ such that

$$(\frac{1}{e}-\varepsilon)D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1} u^{\alpha+mn-2}e^{-nu^m}du \le D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1} u^{\alpha+mn+m-3}e^{-(n+1)u^m}du$$

$$\le (\frac{1}{e}+\varepsilon)D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1} u^{\alpha+mn-2}e^{-nu^m}du$$

Applying Theorem 1, we have

$$\lim_{n\to\infty} D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1} u^{\alpha+mn-2}e^{-nu^m}du = 1$$

Hence if n is sufficiently large say $n > n_0$,

$$1-\varepsilon \le D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1} u^{\alpha+mn-2}e^{-nu^m}du \le 1+\varepsilon$$

therefore if $n > n_0$

$$(1-\varepsilon)(\frac{1}{e}-\varepsilon) \leq D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1} u^{\alpha+mn+m-3}e^{-(n+1)u^m}du \leq (1+\varepsilon)(\frac{1}{e}+\varepsilon)$$

Let

$$\left\|\chi^c_{\delta,1}u^m e^{-u^m}\right\|_\infty = (\frac{1}{e}-2\mu)$$

Also by the continuity of the above function, there exists a $\delta_2(0<\delta_2<\delta)$ such that

$$\inf_{|u-1|<\delta_2} u^m e^{-u^m} \geq (\frac{1}{e}-\mu)$$

Hence $D^{-1}(m,n,\alpha) \geq \delta_2(\frac{1}{e}-\mu)^n$ and therefore if n is sufficiently large

$$\left|D(m,n,\alpha)[\frac{u^{\alpha+(n+1)m-2e}e^{-(n+1)u^m}}{n+1}]^{1+\delta}_{1-\delta}\right| \leq \frac{\varepsilon}{n},\ n>n_1$$

In view of Theorem 1, it is clear that there exists a n_2 such that

$$D(m,n,\alpha)\int_{(1-\delta,1+\delta)/(1-\delta_1,1+\delta_1)} u^{\alpha+mn-2}e^{-nu^m}\chi^c_{\delta_1,1}(u)du \leq \varepsilon,\ n>n_2$$

Making use of the above estimates and the fact that ε is arbitrary, we have (i).
(ii) The proof uses similar analysis and the fact that

$$\lim_{n\to\infty}\frac{D(m,n,\alpha)}{D(m,n+1,\alpha)} = e^{-1}$$

Therefore we leave the proof to the reader.
(iii) Given an arbitrary $\varepsilon>0$, there exists a $\delta_0(0<\delta_0<\frac{1}{1+\delta})$ such that

$$-\frac{1}{2}(1-\varepsilon)u^{2(m-1)}e^{1-2u^m}(1-u^m)^2 \leq u^m e^{-u^m}-e^{-1}$$
$$\leq -\frac{(1+\varepsilon)u^{2(m-1)}}{2}e^{1-2u^m}(1-u^m)^2,\ \left|u^{-1}-1\right|<\delta_0$$

Now, using the arguments given in the proof of part (i), the proof easily follows. This completes the proof of the lemma.

The main results of this section are given in

Theorem 2. *If $f\in D_\Omega$, and at a certain point $x\in IR^+, f''$ exists, then there holds*

$$L_n(f;x)-f(x) = \frac{xf'(x)[3-2\alpha+m]}{2nm^2}+\frac{x^2f''(x)}{2nm^2}+o(\frac{1}{n}) \text{ as } n\to\infty. \tag{9}$$

Further, if f'' exists and is continuous on$< a,b >$, then (9) holds uniformly on $[a,b]$.

Proof. Using L'Hospital's rule we have

$$\lim_{u\to 1}\frac{f(\frac{x}{u})-f(x)-\frac{xf'(x)[mu^{m-1}(1-u^m)+(m-3)u^m e^{1-u^m}+(3-m)]}{m^2}}{e^{-1}-u^m e^{-u^m}}+\frac{\frac{2xf'(x)+x^2f''(x)(u^m e^{1-u^m})}{\frac{m^2}{e^2}}}{e^{-1}-u^m e^{-u^m}}=0$$

Hence, given an arbitrary $\varepsilon>0$ there exists a $\delta>0$ such that if u satisfies $\left|\frac{x}{u}-x\right|<\delta$, there holds

$$f(\frac{x}{u})-f(x)-\frac{xf'(x)}{m^2}[mu^{m-1}e^{1-u^m}+(m-3)u^m e^{1-u^m}+(3-m)]+$$
$$(2xf'(x)+x^2f''(x)(u^m e^{1-u^m}))/\frac{m^2}{e^2}\le\varepsilon(e^{-1}-u^m e^{-u^m})$$

Moreover, it is easily seen that in the uniformity case the above δ can be chosen independent of $x\in[a,b]$. Multiplying the inequality by $u^{\alpha+mn-2}e^{-nu^m}nD(m,n,\alpha)$ and integrating between the limits $(1-\delta,1+\delta)$ and making use of Lemma 2, we have

$$\frac{-\varepsilon}{2e}\le\limsup nD(m,n,\alpha)\int_{1-\delta}^{1+\delta}[f(\frac{x}{u})-f(x)]u^{\alpha+mn-2}e^{-nu^m}du$$
$$+\frac{xf'(x)(\alpha-2)}{m^2}+\frac{xf'(x)(3-m)}{2m^2}-\frac{[2xf'(x)+x^2f''(x)]}{2m^2}\le\frac{\varepsilon}{2e}$$
$$\lim_{n\to\infty}nD(m,n,\alpha)\int_{(0,\infty)-(1-\delta,1+\delta)}[f(\frac{x}{u})-f(x)]u^{\alpha+mn-2}e^{-nu^m}du=0$$

which holds uniformly in $x\in[a,b]$, in the uniformity case.

Hence

$$\frac{-\varepsilon}{2e}\le\limsup[L_n(f;x)-f(x)]+\frac{xf(x)(\alpha-2)}{m^2}+\frac{xf'(x)(3-m)}{2m^2}-\frac{[2xf'(x)+x^2f''(x)]}{2m^2}\le\frac{\varepsilon}{2e}$$

In view of the fact that $\varepsilon>0$ arbitrary, the result follows.

Corollary 1. *Choosing $m=1,\alpha=2$, we obtain the Voronovskaya formula for the Gamma-Operators of Muller.*

$$G_n(f;x)-f(x)=\frac{x^2f''(x)}{2n}+o(\frac{1}{n}),n\to\infty$$

Corollary 2. *Taking $m=-1$ and $\alpha=1$ we have the following Voronovskaya formula for the operators S_n^1.*

$$S_n^1(f;x)-f(x)=\frac{x^2f''(x)}{2n}+o(\frac{1}{n}),n\to\infty$$

Corollary 3. *With $m=-1$ and $\alpha=0$ we have the Voronovskaya formula for the operators $L_{k,t}$*

$$L_{k,t}(f;x)-f(x)=\frac{xf'(x)}{k}+\frac{x^2f''(x)}{2k}+o(\frac{1}{k}),k\to\infty$$

4. Error Estimates

In the previous section, we obtained a precise formula giving the rate of convergence of $L_n(f;x)$ to $f(x)$. The assumption on f has been the existence of its second order derivatives. If f is only assumed to be continuous, the following theorem gives an estimate of error $\left|L_n(f;x) - f(x)\right|$ in terms of the modulus of continuity of f.

Theorem 3. *For the operators $L_n(f;x)$ there holds*

$$\left|L_n(f;x) - f(x)\right| \\ \leq \omega_f(n^{-\frac{1}{2}})[1 + \min(x^2\{\frac{1}{m^2} + o(\frac{1}{n})\}, x\{\frac{1}{m^2} + o(1)\}^{\frac{1}{2}})],\ x \in IR^+, n \to \infty \quad (10)$$

where ω_f denotes the modulus of continuity of f and $o(\frac{1}{n})$ are independent of x.

Proof. Using (10), we have

$$L_n((u-x)^2; x) = x^2[\frac{1}{nm^2} + o(\frac{1}{n})], n \to \infty \quad (11)$$

By elementary properties of modulus of continuity,

$$\left|f(u) - f(x)\right| \leq \omega_f(n^{-\frac{1}{2}})[1 + n^{\frac{1}{2}}\,|u-x|] \quad (12)$$

and also

$$\left|f(u) - f(x)\right| \leq \omega_f(n^{-\frac{1}{2}})[1 + n(u-x)^2] \quad (13)$$

For all $x, u \in IR^+$, by Schwartz's inequality (11) implies

$$L_n(|u-x|; x) \leq \frac{x}{n^{\frac{1}{2}}}\{\frac{1}{m^2} + o(1)\}^{\frac{1}{2}}, n \to \infty \quad (14)$$

making use of the linearity and positivity of the operators L_n, (10) follows from $(11)-(14)$.

For functions which are continuously differentiable the error estimate (10) is rather conservative and better estimate is as follows-

Theorem 4. *If $f'(x)$exists and is uniformly continuous on IR^+ there holds,*

$$\left|L_n(f;x) - f(x)\right| \leq \frac{x\left|f'(x)\right|e^2}{m^4}[2(3-\alpha)\frac{m^2}{e} + \frac{m^3 - 3m^2}{e} + o(1)] \\ + \omega_f(n^{-\frac{1}{2}})[\frac{x}{n^{\frac{1}{2}}}(\{\frac{1}{m^2}\}^{\frac{1}{2}} + o(1)) + \frac{x^2}{2n^{\frac{1}{2}}}(\{\frac{1}{m^2}\} + o(1))] \quad (15)$$

$x \in IR^+, n \to \infty$, *where ω_f denotes the modulus of continuity of f.*

Proof. We have,

$$\left|f(u)-f(x)-(u-x)f'(x)\right| \leq \left|\int_x^u (f'(u)-f'(x))du\right| \leq \left|\int_x^u \omega_{f'}(|u-x|)du\right|$$

$$\leq \left|\int_x^u \omega_{f'}(n^{-\frac{1}{2}})(1+n^{\frac{1}{2}}|u-x|)du\right| = \omega_{f'}(n^{-\frac{1}{2}})\{|u-x|+\frac{1}{2}n^{\frac{1}{2}}(u-x)^2\} \quad (16)$$

Since by Theorem 2 we have

$$L_n((u-x);x) = \frac{x}{2nm^2}(3-2\alpha+m)+o(n^{-1})$$

The inequality (15) follows by operating (16) by L_n and making use of (11), (13) and (16).

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