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Showing How to Imply Proving The Riemann Hypothesis

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Abstract. Using the integral convergence attaching analytic properties prove the Riemann hypothesis.

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1. Introduction

We show how the function properties prove the integral

$$\int_1^\infty \frac{\psi(x)-x}{x^{\frac{3}{2}+\varepsilon}}dx$$

converges absolutely and uniformly for $\forall\varepsilon>0$, which concludes the integral

$$\int_1^\infty \frac{\psi(x)-x}{x^{s+1}}dx$$

is analytic for $Re(s)>\frac{1}{2}$, thus this concludes the proof of the form

$$\psi(x)=x+O(x^{\frac{1}{2}+\varepsilon})$$

for $\forall\varepsilon>0$.

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2. Preliminaries

2.1. Basic Properties of The Zeta Function

For $Re(s) > 1$ we know that the zeta function defined by the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

and the Euler product

$$\prod_p (1 - p^{-s})^{-1},$$

namely

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}.$$

The series converges absolutely and uniformly for $Re(s) \geq 1+\delta$, with any $\delta > 0$, and the product converges absolutely and uniformly for $Re(s) \geq 1+\delta$, with any $\delta > 0$. The representation of the zeta function as such a product shows that $\zeta(s) \neq 0$ for $Re(s) > 1$.

In fact,we often obtain the following identities,valid for $Re(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} s \int_n^{\infty} \frac{dx}{x^{s+1}} = s \int_1^{\infty} \left(\sum_{n \leq x} 1 \right) \frac{dx}{x^{s+1}} = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx,$$

and

$$s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx.$$

The symbol $[x]$ denotes the greatest integer $\leq x$, it is called the integral part of x, the number $\{x\} = x - [x]$ is called the fractional part of x, it satisfies the inequalities $0 \leq \{x\} < 1$, with $\{x\} = 0$ if and only if x is an integer.

We know the integral

$$\int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx$$

converges absolutely, and uniformly for $Re(s) \geq \delta$, with any $\delta > 0$.

Also, for $Re(s) > 1$ we know the identity (ii):

$$\phi(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_p \frac{\log p}{p^s - 1} = s \int_1^{\infty} \frac{\psi(x)}{x^{s+1}} dx = \frac{s}{s-1} + s \int_1^{\infty} \frac{\psi(x) - x}{x^{s+1}} dx,$$

where the sum

$$\sum_p \frac{\log p}{p^s - 1}$$

is extended over all primes.

We define

$$\psi(x) = \sum_{p^m \le x} \log p = \sum_{n \le x} \Lambda(n).$$

The $\Lambda(n)$ is $\log p$ when $\exists m \ge 1$ such that $n = p^m$, or otherwise it is 0. Where the sum is taken over those integers of the form p^m that are less than or equal to x. Here p is a prime number and m is a positive integer.We also see that $\zeta(1+it) \ne 0$, $\zeta(it) \ne 0$, and when the function $\zeta'/\zeta(s)$ has no poles on the region $1 > Re(s) > \frac{1}{2}$, then which implies that the function $\zeta(s)$ has no zeros on the region $1 > Re(s) > \frac{1}{2}$.

We intimately know the result

$$-\zeta'/\zeta(s) = \sum_p \frac{\log p}{p^s - 1}$$

for $Re(s) > 1$, and the identity (ii):

$$\phi(s) = -\zeta'/\zeta(s) = \frac{s}{s-1} + s\int_1^\infty \frac{\psi(x) - x}{x^{s+1}} dx \text{ for } Re(s) > 1.$$

We define

$$\Phi(s) = \sum_p \frac{logp}{p^s}$$

for $Re(s) > 1$. Here p is a prime number. The sum defining $\Phi(s)$ converges uniformly and absolutely for $Re(s) \ge 1 + \delta$, by the same argument as for the sum defining the zeta function. We merely use the fact that given $\varepsilon > 0$,

$$\log n \le n^\varepsilon \text{ for all } n \ge n_0(\varepsilon).$$

Since

$$e^x = 1 + x + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \ldots,$$

it follows that for positive x and for all n, we have

$$x^{-n} e^x > \frac{x}{(n+1)!}.$$

For any fixed n the right-hand side tends to infinity as $x \to \infty$, it follows that e^x grows faster than any fixed power of x. We can write $x^n = o(e^x)$ to mean

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0$$

for all n. We see that $logx = o(x^\delta)$, $\delta > 0$. In other words $\log x$ grows slower than any fixed positive power n of x.

2.2. Basic Results

In a way, we need go through the basic definitions with their results concerning functions, and we omit some definitions and proofs which are just elementary knowledge such as:

We recall the notation: $f = O(g)$ mean that f, g are two functions of a variable x defined for all sufficiently large, and g is positive, there exists a constant $C > 0$ such that $\left|f(x)\right| \leq Cg(x)$ for all x sufficiently large. In fact,we have $\psi(x) = x + O(x^{\lambda})$ for $\frac{1}{2} \leq \lambda < 1$, and $\psi(x) = O(x)$.

We also recall the results, let ρ be all nontrivial zeros of $\zeta(s)$, we know $\psi(x) = x + O(x^{\sup Re(\rho)+\varepsilon})$ for every $\varepsilon > 0$. Intimately, we can get $\lambda \geq \sup Re(s) + \varepsilon$ by the result that $\zeta(s)$ has no zero on the region $Re(s) > \lambda$. Using a basic fact from properties of the zeta function that $\zeta(s)$ certainly has an infinite number of the nontrivial zeros in the region $Re(s) \geq \frac{1}{2}$ and the symmetry of the zeros such that we have $\lambda \geq \frac{1}{2}$.

We recall the definitions:Let U be a subset of the complex plane, we say that U is open set if for every point z in U, there is a disc $D(z, r)$ of radius centered at z such that this disc $D(z, r)$ is contained in U. And we know a set what is called closed,what is called bounded,what is called continuous,compact.We also know a series of sequence and a series of functions what is said to converge uniformly, converge absolutely. We have the usual tests for compact sets, and some for convergence, etc, those definitions with their results.

2.3. Some Theorems

We make the precise theorems omitted its proof as follows [cf. 1].

Theorem 1. *A set of complex numbers is compact if and only if it is closed and bounded.*

Theorem 2. *Let S be a compact set of complex numbers,and let f be a continuous function on S. Then the image of f is compact.*

Theorem 3. *Let S be a compact set of complex numbers, and let f be a continuous function on S. Then f is uniformly continuous, i.e. given ε there exists δ such that whenever $z, w \in S$ and $|z - w| < \delta$, then $\left|f(z) - f(w)\right| < \varepsilon$.*

Theorem 4. *Let γ be a path in an open set U and let g be a continuous function on γ (i.e. on the image $\gamma([a, b])$ if γ is defined on $[a, b]$). If z is not on γ, define*

$$f(z) = \int_{\gamma} \frac{g(\zeta)}{\zeta - z} d\zeta.$$

Then f is analytic on the complement of γ in U, and its derivatives are given by [cf. 1, pp.130-131]

$$f^{(n)}(z) = n! \int_{\gamma} \frac{g(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Let f be a continuous function on the real numbers ≥ 0 and assume that there is constants A, B such that $\left|f(t)\right| \leq Ae^{Bt}$ for all t sufficiently large. Moreover ,and assume for simplicity that f is bounded, piecewise continuous, whence $B \leq 0$. What we prove will hold under much less restrictive conditions: instead of piecewise continuous,it would suffice to assume that the integral

$$\int_a^b \left|f(t)\right| dt$$

exists for every pair of numbers $a, b \geq 0$. We shall associate to f the Laplace transform g defined by

$$g(z) = \int_0^\infty f(t)e^{-zt} dt$$

for $Re(z) > 0$, we can then apply Lemma 1 (the differentiation lemma) of subsection 4.2, whose proof applies to a function satisfying our conditions (piecewise continuous and bounded), and then we easily conclude that g is analytic for $Re(z) > 0$.

3. The Three Basic Theorems

Theorem 5. *The function Φ is meromorphic for $Re(s) > \frac{1}{2}$. Furthermore, for $Re(s) \geq 1$, we have $\zeta(s) \neq 0$ and*

$$\Phi(s) - \frac{1}{s-1}$$

has no poles for $Re(s) \geq 1$.

Proof. For $Re(s) > 1$ the Euler product shows that $\zeta(s) \neq 0$. We know the result

$$-\zeta'/\zeta(s) = \sum_p \frac{\log p}{p^s - 1}$$

for $Re(s) > 1$, and the identity (ii):

$$\phi(s) = -\zeta'/\zeta(s) = \frac{s}{s-1} + s\int_1^\infty \frac{\psi(x) - x}{x^{s+1}} dx.$$

Using the geometric series we get the expansion

$$\frac{1}{p^s - 1} = \frac{1}{p^s} \times \frac{1}{1 - \frac{1}{p^s}} = \frac{1}{p^s}\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) = \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots \tag{1}$$

So the identity (ii) it satisfies

$$-\zeta'/\zeta(s) = \Phi(s) + \sum_p h_p(s), \text{ where } \left|h_p(s)\right| \leq B\frac{\log p}{\left|p^{2s}\right|} \text{ for some constant } B. \tag{2}$$

But the series

$$\sum \frac{\log n}{n^{2s}}$$

converges absolutely and uniformly for $Re(s) \geq \frac{1}{2} + \delta$, with any $\delta > 0$, and we know the intimate result that the function $\zeta(s) - \frac{1}{s-1}$ extends to an analytic function on the region $Re(s) > 0$, this result and (ii) imply that Φ is meromorphic for $Re(s) > \frac{1}{2}$, and has a pole at $s = 1$ and at the zeros of ζ, but no other poles in this region. We omit the proof that ζ has no zero on the line $Re(s) = 1$, which is just the intimate result.

We shall now prove special cases of the following theorems concerning differentiation under the integral sign which are sufficient for our applications. We first prove a general theorem that the uniform limit of analytic functions is analytic. This will allow us to define analytic functions by uniformly convergent series.

Theorem 6. *Let $\{f_n\}$ be a sequence of analytic functions on an open set U. Assume that for each compact subset $K$of U the sequence converges uniformly on K, and let the limit function be f, i.e.* $\lim f_n = f$. *Then f is analytic.*

Proof. $z_0 \in U$, and let $\bar{D_R}$ be a closed disc of radius R centered at in z_0 and contained in U. Then the sequence $\{f_n\}$ converges uniformly on $\bar{D_R}$. Let C_R be the circle which is the boundary of $\bar{D_R}$. Let $\bar{D_{\frac{R}{2}}}$ be the closed disc of radius $\frac{R}{2}$ centered at z_0. Then for $z \in \bar{D_{\frac{R}{2}}}$ we have

$$f_n(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f_n(\zeta)}{\zeta - z} d\zeta,$$

and $|\zeta - z| \geq \frac{R}{2}$. Since $\{f_n\}$ converges uniformly to $f(z)$, for $\left|z - z_0\right| \leq \frac{R}{2}$, we get

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

By Theorem 4 it follows that f is analytic on a neighborhood of z_0. Since this is true for every z_0 in U, we have proved what we wanted.

Theorem 7. *Let $\{f_n\}$ be a sequence of analytic functions on an open set U converging uniformly on every compact subset K of U to a function f. Then the sequence of derivatives $\{f_n{}'\}$ converges uniformly on every compact subset K, and* $\lim f_n{}' = f'$.

Proof. Cauchy's formula expresses the derivative $f_n{}'$ as an integral, cover the compact set with a finite number of closed discs contained in U, and sufficiently small radius. And one can argue as in the previous Theorem 6. For the sake of completeness we shortly state the similar result,and we have

$$f_n(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f_n(\zeta)}{\zeta - z} d\zeta,$$

and $|\zeta - z| \geq \frac{R}{2}$. Since $\{f_n\}$ converges uniformly to $f(z)$, i.e. $\lim f_n = f$, for $\left|z - z_0\right| \leq \frac{R}{2}$, we get

$$f(z) = \frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{\zeta - z}d\zeta.$$

By Theorem 4, we obtain a bound for the derivative of an analytic function in term of the function itself, we see that

$$f_n{}'(z) = \frac{1}{2\pi i}\int_{C_R}\frac{f_n(\zeta)}{(\zeta - z)^2}d\zeta,$$

and

$$f'(z) = \frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{(\zeta - z)^2}d\zeta,$$

and

$$\lim f_n = f.$$

Therefore

$$\lim f_n{}' = f'.$$

4. The Three Main Lemmas

4.1. Stating Some Basic Conditions

We know that the form

$$\psi(x) = x + O(x^{\frac{1}{2}+\varepsilon})$$

for $\forall \varepsilon > 0$ implies the Riemann Hypothesis.We shall now prove the main lemmas, which constitute the delicate part of the proof.

We shall also deal with integrals depending on a parameter. This means a function f of two variables, $f(t,z)$, where z in some domain U in the complex numbers, and we let $D_1 f$ and $D_2 f$ be the partial derivatives of f with respect to the first and second variable respectively.

The integral

$$\int_0^\infty f(t,z)dt = \lim_{B\to\infty}\int_0^B f(t,z)dt$$

is said to be uniformly convergent for $z \in U$ if, given ε, there exists B_0 such that $B_0 < B_1 < B_2$, then

$$\left|\int_{B_1}^{B_2} f(t,z)dt\right| < \varepsilon.$$

The integral is absolutely and uniformly convergent for $z \in U$ if the same condition holds with $f(t,z)$ replaced by the absolute value $\left|f(t,z)\right|$.

4.2. The Three Lemmas

Lemma 1 (The Differention Lemma). *Let I be an interval of real numbers, possibly infinite. Let U be an open set of complex numbers.Let $f = f(t,z)$ be a continuous function on $I \times U$. Assume:*

(i) For each compact subset K of U, the integral

$$\int_I f(t,z)dt$$

is uniformly convergent for $z \in K$.

(ii) For each t the function $z \mapsto f(t,z)$ is analytic.

Let

$$F(z) = \int_I f(t,z)dt.$$

Then F is analytic on U, $D_2 f(t,z)$ satisfies the same assumptions as f, and

$$F'(z) = \int_I D_2 f(t,z)dt.$$

Proof. Let $\{I_n\}$ be a sequence of finite closed interval, increasing to I. Let D be a disc in the z-plane whose closure is contained in U. Let γ be the circle bounding D. Then for each z in D, we have

$$f(t,z) = \frac{1}{2\pi i}\int_\gamma \frac{f(t,\zeta)}{\zeta - z}d\zeta,$$

So

$$F(z) = \frac{1}{2\pi i}\int_I \int_\gamma \frac{f(t,\zeta)}{\zeta - z}d\zeta dt.$$

If γ has radius R center z_0, consider only z such that $\left|z - z_0\right| \leq \frac{R}{2}$, then

$$\left|\frac{1}{\zeta - z}\right| \leq \frac{2}{R}.$$

For each n we can define

$$F_n(z) = \frac{1}{2\pi i}\int_{I_n} \int_\gamma \frac{f(t,\zeta)}{\zeta - z}d\zeta dt.$$

In view of the restriction on z above, we may interchange the integrals and get Cauchy's formula expresses the F_n as an integral, covering the compact set with a finite number of closed discs contained in U, and sufficiently small radius, i.e.

$$F_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\left[\int_{I_n} f(t,\zeta)dt\right]d\zeta.$$

We can apply our conditions that the integral

$$\int_I f(t,z)dt$$

is uniformly convergent and $f = f(t,z)$ is uniformly continuous on the compact subset $I \times K$ of $I \times U$, we obviously get that

$$\int_I f(t,z)dt$$

is continuous on the compact subset K of U on the γ.

Hence

$$\int_{I_n} f(t,z)dt$$

is continuous on the path γ. Then F_n is analytic by Theorem 4.

By Theorem 6 and Theorem 7 with our assumption,the integrals over I_n converge uniformly to the integral over I. Hence F is analytic,being the uniform limit of the function F_n for $\left|z - z_0\right| \leq \frac{R}{2}$. On the other hand, ${F_n}'(z)$ is obtained by Theorem 7, and converges uniformly to $F'(z)$, i.e. the uniform limit of ${F_n}'(z)$ is:

$$\lim {F_n}'(z) = F'(z) = \int_I D_2 f(t,z)dt.$$

This proves the theorem.

Lemma 2. *Let f be bounded, piecewise continuous on the real numbers ≥ 0. Let f the Laplace transform g defined by*

$$g(z) = \int_0^\infty f(t)e^{-zt}dt$$

for $Re(z) > 0$, then g is analytic in the region $Re(z) > 0$. If g extends to an analytic function for $Re(z) \geq 0$, then

$$\int_0^\infty f(t)dt$$

exists and is equal to $g(0)$.

Proof. Let B be a bound for $f(t)$, that is $\left|f(t)\right| \leq B$ for all $t \geq 0$, then we can apply the differentiation lemma to conclude that g is analytic in the region $Re(z) > 0$, we omit its proof which is just a intimate result.

For $T > 0$ define

$$g_T(z) = \int_0^T f(t)e^{-zt}dt.$$

Then g_T is an entire function,as follows at once by the differentiation lemma. We have to show that

$$\lim_{T\to\infty} g_T(0) = g(0).$$

Let $\delta > 0$ and let C be the path consisting of the line segment $Re(z) = -\delta$ and the arc of circle $|z| = R$ and $Re(z) \geq -\delta$, as shown on the Figure 1.

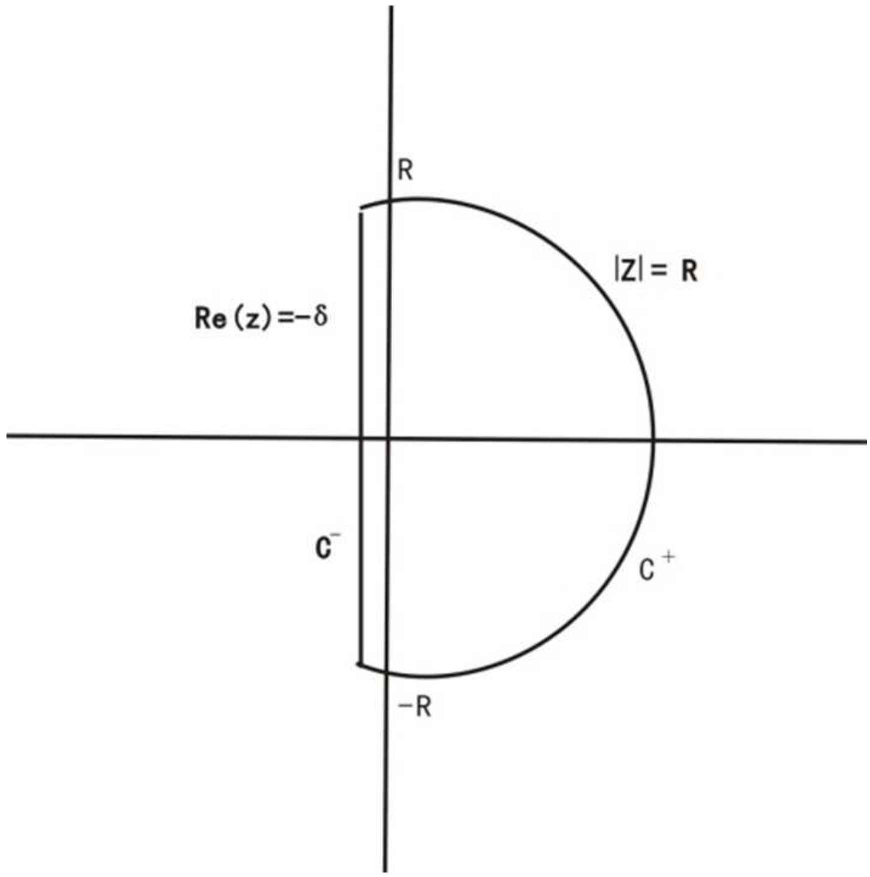

Figure 1: Line segment $Re(z) = -\delta$ and the arc of circle $|z| = R$ and $Re(z) \geq -\delta$.

By our assumption that g extends to an analytic function for $Re(z) \geq 0$, we can take δ small enough so that g is analytic on the region bounded by C, and on its boundary. Then

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C (g(z) - g_T(z))e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} = \frac{1}{2\pi i} \int_C H_T(z)dz,$$

where $H_T(z)$ abbreviates the expression under the integral sign. Let B be a bound for $f(t)$, that is $\left|f(t)\right| \leq B$ for all $t \geq 0$.

Let C^+ be the semicircle $|z| = R$, and $Re(z) \geq 0$. Then

$$\left| \frac{1}{2\pi i} \int_{C^+} H_T(z)dz \right| \leq \frac{2B}{R}. \tag{3}$$

First note that for $Re(z) > 0$ we have

$$\left|g(z) - g_T(z)\right| = \left| \int_T^\infty f(t)e^{-zt}dt \right| \leq B \int_T^\infty \left|e^{-zt}\right| dt = \frac{B}{Re(z)} e^{-Re(z)T};$$

and for $|z| = R$,

$$\left| e^{Tz}(1 + \frac{z^2}{R^2})\frac{1}{z} \right| = e^{Re(z)T} \left| \frac{R}{z} + \frac{z}{R} \right| \frac{1}{R} = e^{Re(z)T} \frac{2\,|Re(z)|}{R^2}.$$

Taking the product of the last two estimates and multiplying by the length of the semicircle gives a bound for the integral over the semicircle,and proves the claim.

Let C^- be the part of the path C with $Re(z) < 0$. We wish to estimate

$$\frac{1}{2\pi i}\int_C \left(g(z) - g_T(z)\right) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}.$$

Now we estimate separately the expression under the integral with g and g_T.

We have

$$\left| \frac{1}{2\pi i}\int_{C^-} g_T(z) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right| \leq \frac{B}{R}. \tag{4}$$

Let S^- be the semicircle with $|z| = R$ and $Re(z) < 0$. Since g_T is entire, we can replace C^- by S^- in the integral without changing the value of the integral, because the integrand has no poles to the left of the y-axis. Now we estimate the expression under the integral sign on S^-. We have

$$\left|g_T(z)\right| = \left| \int_0^T f(t) e^{-zt} dt \right| \leq B \int_0^T e^{-Re(z)t} \leq \frac{Be^{-Re(z)T}}{-Re(z)}.$$

For the other factor we use the same estimate as previously. We take the product of the two estimates,and multiply by the length of the semicircle to give the desired bound in (4).

Third, we claim that

$$\int_{C^-} g(z) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \to 0, \tag{5}$$

as $T \to \infty$.

We can write the expression under the integral sign as

$$g(z) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{1}{z} = h(z) e^{Tz}, \text{ where } h(z) \text{ is independent of } T$$

Given any compact subset K of the region defined by $Re(z) < 0$, we note that $e^{Tz} \to 0$ rapidly uniformly for $z \in K$, as $T \to \infty$.

The word "rapidly" means that the expression divided by any power T^N also tends to 0 uniformly for z in K, as $T \to \infty$. From this our claim (5) follows easily.

We may now prove the Lemma 2. We have

$$\int_0^\infty f(t)dt = \lim_{T\to\infty} g_T(0),$$

if this limit exists.
But given ε, pick R so large that $\frac{2B}{R} < \varepsilon$. Then by (5), pick T so large that

$$\left| \int_{C^-} g(z)e^{Tz} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} \right| < \varepsilon.$$

Then by (3), (4) and (5) we get $\left|g(0) - g_T(0)\right| < 3\varepsilon$. This proves Lemma 2.

We claim that it also concludes the integral

$$\int_1^\infty \frac{\psi(x) - x}{x^{\frac{3}{2}+\varepsilon}} dx$$

is convergent for $\forall \varepsilon > 0$.

Observe that the function ψ is piecewise continuous. In fact, it is locally constant: there is no change in ψ between prime numbers. The application of Lemma 2 is to prove:

Lemma 3. *The pair of integrals*

$$\int_1^\infty \frac{\psi(x) - x}{x^{\frac{3}{2}+\varepsilon}} dx,$$

and

$$\int_1^\infty \frac{\left|\psi(x) - x\right|}{x^{\frac{3}{2}+\varepsilon}} dx$$

converge for $\forall \varepsilon > 0$.

Proof. Let

$$f_1(t) = \frac{\psi(e^t) - e^t}{e^{\left(1-(\lambda_0-\lambda)\right)t}},$$

and

$$f_2(t) = \frac{\left|\psi(e^t) - e^t\right|}{e^{\left(1-(\lambda_0-\lambda)\right)t}},$$

where λ satisfies the conditions $\psi(x) = x + O(x^\lambda)$, $\frac{1}{2} \le \lambda < 1$, given $\lambda_0 \le \lambda - \frac{1}{2} - \varepsilon$ for $\forall \varepsilon > 0$, show that given any $\lambda_0 \le \lambda - \frac{1}{2} - \varepsilon$ satisfies $1 - (\lambda_0 - \lambda) \ge \frac{3}{2} + \varepsilon$ for $\forall \varepsilon > 0$ including the case $1 - (\lambda_0 - \lambda) = \frac{3}{2} + \varepsilon$, and where λ_0 is dependent of ε, λ can be independent of ε.

Then f_i is certainly piecewise continuous, and is bounded by the formulas $\psi(x) = O(x)$ and $\psi(x) = x + O(x^\lambda)$ for $\frac{1}{2} \le \lambda < 1$, where $i = 1, 2$. Making the substitution $x = e^t$ in the desired integral, $dx = e^t dt$, we see that

$$\int_1^\infty \frac{\psi(x) - x}{x^{\frac{3}{2}+\varepsilon}} dx = \int_0^\infty f_1(t) dt,$$

and

$$\int_1^\infty \frac{\left|\psi(x) - x\right|}{x^{\frac{3}{2}+\varepsilon}} dx = \int_0^\infty f_2(t) dt.$$

Therefore these suffice to prove that the integrals on the right converge, it suffices to prove that the Laplace transform g_i of f_i is analytic for $Re(z) \geq 0$, so we have to compute the Laplace transform g_i. We claim that in the case $i = 1$,

$$g_1(z) = \frac{\phi\left(z + 1 - (\lambda_0 - \lambda)\right)}{z + 1 - (\lambda_0 - \lambda)} - \frac{1}{z - (\lambda_0 - \lambda)}.$$

Once we have proved the formula, we can then apply Theorem 5 and the identity (ii) to conclude that $g_1(z)$ is analytic for $Re(z) \geq 0$, thus concluding the part of the proof of Lemma 3.

Now to compute $g_i(z)$ when $i = 1$, we use the identity (ii) we obtain

$$\frac{\phi(s)}{s} - \frac{1}{s-1} = \int_1^\infty \frac{\psi(x) - x}{x^{s+1}} dx,$$

and

$$\begin{aligned} g_1(z) =& \frac{\phi\left(z + 1 - (\lambda_0 - \lambda)\right)}{z + 1 - (\lambda_0 - \lambda)} - \frac{1}{z - (\lambda_0 - \lambda)} = \int_1^\infty \frac{\psi(x) - x}{x^{z+2-(\lambda_0-\lambda)}} dx \\ =& \int_0^\infty \frac{\psi(e^t) - e^t}{e^{\left(z+2-(\lambda_0-\lambda)\right)t}} e^t dt = \int_0^\infty f_1(t) e^{-zt} dt \end{aligned}$$

where $Re\left(z + 1 - (\lambda_0 - \lambda)\right) > 1$, and so $Re(z) > \lambda_0 - \lambda$, we have $\lambda_0 - \lambda \leq -\frac{1}{2} - \varepsilon$ for $\forall \varepsilon > 0$, which by Theorem 5 and the identity (ii) show that $g_1(z)$ is analytic for $Re(z) > \lambda_0 - \lambda$, and then it is analytic for $Re(z) \geq 0$. This gives us the Laplace transform of f_1 and concludes the part of the proof of Lemma 3.

We claim that in the case $i = 2$,

$$g_2(z) = \int_0^\infty \frac{\left|\psi(e^t) - e^t\right|}{e^{\left(z+2-(\lambda_0-\lambda)\right)t}} e^t dt = \int_1^\infty \frac{\left|\psi(x) - x\right|}{x^{z+2-(\lambda_0-\lambda)}} dx = \int_0^\infty f_2(t) e^{-zt} dt,$$

where

$$f_2(t) = \frac{\left|\psi(e^t) - e^t\right|}{e^{\left(1-(\lambda_0-\lambda)\right)t}},$$

whence the function converges absolutely for $Re(z) \geq 0$ and uniformly for $Re(z) \geq \delta$ with $\delta \geq 0$ from the basic fact that the integral

$$\int_1^\infty \frac{dx}{x^n}$$

converges absolutely and uniformly for $n \geq 1 + k$, with any $k > 0$.

Given any

$$\lambda_0 \leq \lambda - \frac{1}{2} - \varepsilon$$

for $\forall \varepsilon > 0$, with

$$\psi(x) = x + O(x^\lambda), \frac{1}{2} \leq \lambda < 1.$$

When $Re(z) \geq 0$, we have $Re(z) + 2 - \lambda > 1$, $Re(z) + 2 - \lambda_0 > 1$. Then by Lemma 1 (the differentiation lemma) which show that $g_i(z)$ is analytic for $Re(z) \geq 0$ when $i = 2$. This gives us the Laplace transform of f_2 and concludes the part of the proof of Lemma3.

Hence this proves the lemma.

5. Conclusions

Using Lemma 3 concludes that the integral

$$\int_1^\infty \frac{\psi(x) - x}{x^{s+1}} dx$$

converges absolutely and uniformly for

$$Re(s) > \frac{1}{2},$$

and by Lemma 1 (the differentiation lemma) concludes the integral

$$\int_1^\infty \frac{\psi(x) - x}{x^{s+1}} dx$$

is analytic for

$$Re(s) > \frac{1}{2},$$

which immediately follows that the function $\zeta'/\zeta(s)$ has no poles on the region

$$1 > Re(s) > \frac{1}{2}$$

from the formula (ii), and which implies that the function $\zeta(s)$ has no zeros on the region

$$1 > Re(s) > \frac{1}{2}.$$

Thus concluding the proof of the form $\psi(x) = x + O(x^{\frac{1}{2}+\varepsilon})$ for $\forall \varepsilon > 0$, and it also concludes the proof of the Riemann Hypothesis.

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