EUROPEAN JOURNAL OF MATHEMATICAL SCIENCES
Vol. 1, No. 1, 2012, 1-16
www.ejmathsci.com

On Interval-Valued Fuzzy Prime Ideals of a Semiring

Tapan K. Dutta[1], Sukhendu Kar[2,*], Sudipta Purkait[2]

[1] *Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kolkata - 700019, West Bengal, India*
[2] *Department of Mathematics, Jadavpur University, Kolkata - 700032, West Bengal, India*

Abstract. In this paper we introduce the notion of interval-valued fuzzy prime ideal of a semiring, interval-valued fuzzy completely prime ideal of a semiring and study some important properties of these two ideals in the context of interval-valued fuzzy algebra.

2010 Mathematics Subject Classifications: 08A72

Key Words and Phrases: Semiring, Interval Number, Interval-valued Fuzzy Ideal, Interval-valued Fuzzy Prime Ideal, Interval-valued Fuzzy Completely Prime Ideal.

---

1. Introduction

In 1965, L. Zadeh first introduced the theory of fuzzy sets in his pioneer paper [14]. After the birth of fuzzy set theory it has achieved manifold applications than the ordinary set theory. From the beginning of fuzzy set theory it was clear that this theory was an extraordinary tool for representing human knowledge. However, L. Zadeh himself established that sometimes, in decision-making processes, knowledge is better represented by means of some generalizations of fuzzy sets. The so-called extensions of fuzzy set theory arise in this way. Interval-valued fuzzy sets were introduced independently by Zadeh [15], Grattan-Guiness [5], Jahn [7], Sambuc [12] in the same year 1975 as a generalization of ordinary fuzzy set. In the field of applications, the success of the use of fuzzy set theory depends on the choice of the membership function that we take. However, there are applications in which experts do not have precise knowledge of the function that should be taken. In these cases, it is appropriate to represent the membership degree of each element to the fuzzy set by means of an interval. From these considerations arises the extension of fuzzy sets called theory of Interval-valued Fuzzy Sets (IVFSs) that is, fuzzy sets such that the membership degree of each element of the fuzzy set is given by a closed subinterval of the interval $[0,1]$. Thus an interval-valued fuzzy

---

*Corresponding author.

Email addresses: duttatapankumar@yahoo.co.in (T. Dutta), karsukhendu@yahoo.co.in (S. Kar), sanuiitg@gmail.com (S. Purkait)

set (IVFS) is defined by an interval-valued membership function. It is important to note that not only vagueness (lack of sharp class boundaries), but also a feature of uncertainty (lack of information) can be addressed intuitively by interval valued fuzzy set. After the introduction of the concept of interval-valued fuzzy set some authors [2, 6, 8, 10, 13] investigated the topics related to interval-valued fuzzy set and obtained many meaningful conclusions. There are natural ways to fuzzify various algebraic structures and it has been done successfully by many mathematicians. A. Rosenfeld [11] is the father of fuzzy abstract algebra. He first studied the notion of fuzzy subgroup in 1971. After that in 1979, N. Kuroki [9] introduced the concept of fuzzy semigroup. In 1993, J. Ahsan, K. Saifullah and M. Farid Khan [1] introduced the notion of fuzzy semirings. In 1994, T.K. Dutta and B.K. Biswas [3] characterized fuzzy prime ideals of a semiring. Recently, many results of semiring theory are investigated by many researchers in fuzzy context. In this paper, we initiate the study of fuzzification of some concepts of semiring by using the concept of interval valued fuzzy set. Our main purpose of this paper is to study prime ideals and completely prime ideals in semirings with the help of interval valued fuzzy sets.

2. Preliminaries

Firstly, we recall some definitions and results of semirings and fuzzy algebra which we shall use in this paper.

Definition 1 ([4]). *A non-empty set S together with two binary operations '$+$' and '$\cdot$' is said to be a semiring if*

(i) $(S,+)$ *is an abelian semigroup;*

(ii) $(S,\cdot)$ *is a semigroup and*

(iii) $a\cdot(b+c)=a\cdot b+a\cdot c$ *and* $(b+c)\cdot a=b\cdot a+c\cdot a$ *for all* $a,b,c\in S$.

Let $(S,+,\cdot)$ be a semiring. If there exists an element '0_S' $\in S$ such that $a+0_S=a=0_S+a$ and $a\cdot 0_S=0_S=0_S\cdot a$ for all $a\in S$; then '0_S' is called the zero element of S. If there exists an element '1_S' $\in S$ such that $a\cdot 1_S=a=1_S\cdot a$ for all $a\in S$, then '1_S' is called the identity element of S.

A semiring may or may not have a zero and an identity element.

• Throughout this paper we consider a semiring $(S,+,\cdot)$ with zero element '0_S' and identity element '1_S'. Unless otherwise stated a semiring $(S,+,\cdot)$ will be denoted simply by S and multiplication '$\cdot$' will be denoted by juxtaposition.

Definition 2 ([4]). *A semiring* $(S,+,\cdot)$ *is said to be commutative if* $(S,\cdot)$ *is commutative.*

Definition 3 ([4]). *Let I be a nonempty subset of a semiring S. Then*

(i) I is said to be a left ideal of S if $(I,+)$ is a subsemigroup of $(S,+)$ and $sa\in I$ for all $s\in S$ and for all $a\in I$.

(ii) I is said to be a right ideal of S if $(I,+)$ is a subsemigroup of $(S,+)$ and $as \in I$ for all $s \in S$ and for all $a \in I$.

(iii) I is said to be an ideal of S if it is both a left ideal and a right ideal of S.

Definition 4 ([4])**.** *A proper ideal P of a semiring S is said to be prime if $AB \subseteq P$ for any two ideals A, B of S implies that either $A \subseteq P$ or $B \subseteq P$.*

Proposition 1 ([4])**.** *The following conditions in a semiring S are equivalent:*

(i) P is a prime ideal of S.

(ii) $\{arb : r \in S\} \subseteq P$ if and only if $a \in P$ or $b \in P$.

Definition 5 ([4])**.** *A proper ideal I of a semiring S is said to be completely prime if $ab \in I$ for a, b in S implies that $a \in I$ or $b \in I$.*

Proposition 2 ([4])**.** *A prime ideal P of a semiring S is completely prime if and only if $ab \in P$ implies that $ba \in P$ for any $a, b \in S$.*

Definition 6 ([8])**.** *An interval number on $[0,1]$, denoted by $\widetilde{a}$, is defined as the closed subinterval of $[0,1]$, where $\widetilde{a} = [a^-, a^+]$ satisfying $0 \le a^- \le a^+ \le 1$.*

• For any two interval numbers $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$ we define :

(i) $\widetilde{a} \le \widetilde{b}$ if and only if $a^- \le b^-$ and $a^+ \le b^+$.

(ii) $\widetilde{a} = \widetilde{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$.

(iii) $\widetilde{a} < \widetilde{b}$ if and only if $\widetilde{a} \neq \widetilde{b}$ and $\widetilde{a} \le \widetilde{b}$.

Note 1. *We write $\widetilde{a} \ge \widetilde{b}$ whenever $\widetilde{b} \le \widetilde{a}$ and $\widetilde{a} > \widetilde{b}$ whenever $\widetilde{b} < \widetilde{a}$. We denote the interval number $[0,0]$ by $\widetilde{0}$ and $[1,1]$ by $\widetilde{1}$.*

Definition 7 ([2])**.** *Let $\{\widetilde{a_i} : i \in \Lambda\}$ be a family of interval numbers, where $\widetilde{a_i} = [a_i^-, a_i^+]$. Then we define $\sup_{i\in\Lambda}\{\widetilde{a_i}\} = [\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+]$ and $\inf_{i\in\Lambda}\{\widetilde{a_i}\} = [\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+]$.*

• Suppose $D[0,1]$ denotes the set of all interval numbers on $[0,1]$.

Definition 8 ([14])**.** *Let S be a non-empty set. A mapping $\mu : S \longrightarrow [0,1]$ is called a fuzzy subset of S.*

Definition 9 ([8])**.** *Let S be a non-empty set. A mapping $\widetilde{\mu} : S \longrightarrow D[0,1]$ is called an interval-valued fuzzy subset of S.*

• For simplicity, throughout this paper we shall use the term '$i.v.$ fuzzy subset' for 'interval-valued fuzzy subset'.

Note 2. *If $\widetilde{\mu}$ be an $i.v.$ fuzzy subset of a set S, then $\widetilde{\mu}(x)$ is an interval number, where $x \in S$. Suppose $\widetilde{\mu}(x) = [\alpha, \beta]$ for some $x \in S$. Then we have $0 \leq \alpha \leq \beta \leq 1$. So we can always define two fuzzy subsets μ^- and μ^+ of S such that $\mu^-(x) = \alpha$ and $\mu^+(x) = \beta$. Thus we have $\widetilde{\mu}(x) = [\mu^-(x), \mu^+(x)]$ for all $x \in S$. Conversely, suppose we have two fuzzy subsets μ^- and μ^+ of S such that $0 \leq \mu^-(x) \leq \mu^+(x) \leq 1$ for all $x \in S$. Then we can define an $i.v.$ fuzzy subset $\widetilde{\mu}$ of S such that $\widetilde{\mu}(x) = [\mu^-(x), \mu^+(x)]$ for all $x \in S$.*

Definition 10. *Let $X \neq \emptyset$ be a set and $A \subseteq X$. Then the interval-valued characteristic function $\widetilde{\chi}_A$ of A is an $i.v.$ fuzzy subset of X, defined as follows:*

$$\widetilde{\chi}_A(x) = \begin{cases} \widetilde{1} & \text{when } x \in A. \\ \widetilde{0} & \text{when } x \notin A. \end{cases}$$

Definition 11. *Let $\widetilde{\mu_1}$ and $\widetilde{\mu_2}$ be two $i.v.$ fuzzy subsets of a non-empty set X. Then $\widetilde{\mu_1}$ is said to be subset of $\widetilde{\mu_2}$, denoted by $\widetilde{\mu_1} \subseteq \widetilde{\mu_2}$, if $\widetilde{\mu_1}(x) \leq \widetilde{\mu_2}(x)$ i.e. $\mu_1^-(x) \leq \mu_2^-(x)$ and $\mu_1^+(x) \leq \mu_2^+(x)$ for all $x \in X$, where $\widetilde{\mu_1}(x) = [\mu_1^-(x), \mu_1^+(x)]$ and $\widetilde{\mu_2}(x) = [\mu_2^-(x), \mu_2^+(x)]$.*

Definition 12 ([8])**.** *Let $\widetilde{\mu}$ be an $i.v.$ fuzzy subset of a non-empty set X and $[\alpha, \beta] \in D[0,1]$. Then the level subset of $\widetilde{\mu}$, denoted by $\overline{U}(\widetilde{\mu}, [\alpha, \beta])$, is defined as :*
$\overline{U}(\widetilde{\mu}, [\alpha, \beta]) = \left\{x \in X : \widetilde{\mu}(x) \geq [\alpha, \beta]\right\}$.

If we consider two interval numbers $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ such that $[\alpha_1, \beta_1] > [\alpha_2, \beta_2]$, then we have $[\alpha_1, \beta_1] \geq [\alpha_2, \beta_2]$ and $[\alpha_1, \beta_1] \neq [\alpha_2, \beta_2]$.

In this case, we find that $\overline{U}(\widetilde{\mu}, [\alpha_1, \beta_1]) \subseteq \overline{U}(\widetilde{\mu}, [\alpha_2, \beta_2])$, since for any $x \in \overline{U}(\widetilde{\mu}, [\alpha_1, \beta_1]) \Longrightarrow \widetilde{\mu}(x) \geq [\alpha_1, \beta_1] \geq [\alpha_2, \beta_2] \Longrightarrow x \in \overline{U}(\widetilde{\mu}, [\alpha_2, \beta_2])$.

So we have the following result :

Proposition 3. *If $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ be two interval numbers such that $[\alpha_1, \beta_1] > [\alpha_2, \beta_2]$ then $\overline{U}(\widetilde{\mu}, [\alpha_1, \beta_1] \subseteq \overline{U}(\widetilde{\mu}, [\alpha_2, \beta_2])$.*

Definition 13 ([8])**.** *The interval Min-norm is a function $Min^i : D[0,1] \times D[0,1] \longrightarrow D[0,1]$, defined by: $Min^i(\widetilde{a}, \widetilde{b}) = [min(a^-, b^-), min(a^+, b^+)]$ for all $\widetilde{a}, \widetilde{b} \in D[0,1]$, where $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$.*

Definition 14. *The interval Max-norm is a function $Max^i : D[0,1] \times D[0,1] \longrightarrow D[0,1]$, defined by: $Max^i(\widetilde{a}, \widetilde{b}) = [max(a^-, b^-), max(a^+, b^+)]$ for all $\widetilde{a}, \widetilde{b} \in D[0,1]$, where $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$.*

Definition 15. *Let $\widetilde{\mu_1}$ and $\widetilde{\mu_2}$ be two $i.v.$ fuzzy subsets of a non-empty set X and '$\cdot$' be a binary composition on X. Then the composition of these two $i.v.$ fuzzy subsets is an $i.v.$ fuzzy subset of X, defined by :*

$$(\widetilde{\mu_1} \circ \widetilde{\mu_2})(x) = \begin{cases} \sup_{x = a \cdot b} \left\{ Min^i\left(\widetilde{\mu_1}(a), \widetilde{\mu_2}(b)\right)\right\}, & \text{when } x = a \cdot b \text{ for some } a, b \in X. \\ \widetilde{0}, & \text{otherwise}. \end{cases}$$

When we consider the composition of two $i.v.$ fuzzy subsets of a semiring, the second case will not arise as we consider a semiring with identity element in this paper.

Thus the composition of two $i.v.$ fuzzy subsets $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ of a semiring S is given by $(\widetilde{\mu}_1 \circ \widetilde{\mu}_2)(x) = \sup_{x=ab}\left\{Min^i\left(\widetilde{\mu}_1(a), \widetilde{\mu}_2(b)\right)\right\}$ for all $x \in S$ such that $x = ab$ for some $a, b \in S$.

• **Throughout this paper we assume that any two interval numbers in $D[0,1]$ are comparable i.e. for any two interval numbers $\widetilde{a}$ and $\widetilde{b}$ in $D[0,1]$, we have either $\widetilde{a} \leq \widetilde{b}$ or $\widetilde{a} > \widetilde{b}$.**

2.1. Interval-Valued Fuzzy Prime Ideal of a Semiring

Definition 16. *A non-empty $i.v.$ fuzzy subset $\widetilde{\mu}$ of a semiring S (i.e. $\widetilde{\mu}(x) \neq \widetilde{0}$ for some $x \in S$) is said to be interval-valued fuzzy ideal of S if*

(i) $\widetilde{\mu}(x + y) \geq Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$

(ii) $\widetilde{\mu}(xy) \geq Max^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$ *for all* $x, y \in S$.

Example 1. *We consider the semiring $\mathbb{N}_0$ of non-negative integers with respect to usual addition and multiplication. Let $\widetilde{\mu}$ be an $i.v.$ fuzzy subset of $\mathbb{N}_0$, defined by :*

$$\widetilde{\mu}(x) = \begin{cases} \widetilde{1} & \text{if } x = 0, \\ [0.5, 0.6] & \text{if } x \text{ is non-zero even}, \\ [0.3, 0.4] & \text{if } x \text{ is odd}. \end{cases}$$

Then $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of $\mathbb{N}_0$.

Remark 1. *Let $\widetilde{\mu}$ be an $i.v.$ fuzzy ideal of a semiring S. Then $\widetilde{\mu}(0_S) \geq \widetilde{\mu}(x)$ for all $x \in S$.*

Lemma 1. *Let S be a semiring and A be a subset of S. Then A is an ideal of S if and only if $\widetilde{\chi}_A$ is an $i.v.$ fuzzy ideal of S.*

Proof. Let A be an ideal of S. Then $0_S \in A$. So $\widetilde{\chi}_A(0_S) = \widetilde{1}$ and hence $\widetilde{\chi}_A$ is non-empty. Now suppose that $x, y \in S$.
<u>Case 1</u>: Let $Max^i(\widetilde{\chi}_A(x), \widetilde{\chi}_A(y)) = \widetilde{0}$. Then $\widetilde{\chi}_A(x) = \widetilde{0}$ and $\widetilde{\chi}_A(y) = \widetilde{0}$. So $\widetilde{\chi}_A(xy) \geq \widetilde{0} = Max^i(\widetilde{\chi}_A(x), \widetilde{\chi}_A(y))$ and $\widetilde{\chi}_A(x + y) \geq \widetilde{0} = Min^i(\widetilde{\chi}_A(x), \widetilde{\chi}_A(y))$.
<u>Case 2</u>: Let $Max^i(\widetilde{\chi}_A(x), \widetilde{\chi}_A(y)) = \widetilde{1}$. Then $\widetilde{\chi}_A(x) = \widetilde{1}$ or $\widetilde{\chi}_A(y) = \widetilde{1}$. This implies that $x \in A$ or $y \in A$. Then $xy \in A$, since A is an ideal of S. This shows that $\widetilde{\chi}_A(xy) = \widetilde{1} = Max^i(\widetilde{\chi}_A(x), \widetilde{\chi}_A(y))$. Now

$$Max^i(\widetilde{\chi}_A(x), \widetilde{\chi}_A(y)) = \widetilde{1} \Longrightarrow Min^i(\widetilde{\chi}_A(x), \widetilde{\chi}_A(y)) = \widetilde{0} \text{ or } \widetilde{1}.$$

$$Min^i(\widetilde{\chi}_A(x), \widetilde{\chi}_A(y)) = \widetilde{0} \Longrightarrow \widetilde{\chi}_A(x + y) \geq Min^i(\widetilde{\chi}_A(x), \widetilde{\chi}_A(y)).$$

$Min^i(\widetilde{\chi}_A(x), \widetilde{\chi}_A(y)) = \widetilde{1} \Longrightarrow \widetilde{\chi}_A(x) = \widetilde{1}$ and $\widetilde{\chi}_A(y) = \widetilde{1} \Longrightarrow x \in A$ and $y \in A \Longrightarrow x + y \in A$ (since A is an ideal of S) $\Longrightarrow \widetilde{\chi}_A(x + y) = \widetilde{1} = Min^i(\widetilde{\chi}_A(x), \widetilde{\chi}_A(y))$. Consequently, $\widetilde{\chi}_A$ is an $i.v.$ fuzzy ideal of S.
Conversely, let $\widetilde{\chi}_A$ be an $i.v.$ fuzzy ideal of S. Then $\widetilde{\chi}_A$ is non-empty. So $\widetilde{\chi}_A(s) \neq \widetilde{0}$ for some $s \in S$. This implies that $\widetilde{\chi}_A(s) = \widetilde{1}$ for some $s \in S$. So $s \in A$. Hence A is non-empty. Let $x, y \in A$.

Then $\widetilde{\chi}_A(x)=\widetilde{1}=\widetilde{\chi}_A(y)$. Now since $\widetilde{\chi}_A$ is an $i.v.$ fuzzy ideal of S, we have $\widetilde{\chi}_A(x+y)\geq Min^i(\widetilde{\chi}_A(x),\widetilde{\chi}_A(y))=Min^i(\widetilde{1},\widetilde{1})=\widetilde{1}$. So $\widetilde{\chi}_A(x+y)\geq\widetilde{1}$. Also $\widetilde{\chi}_A(x+y)\leq\widetilde{1}$, since $\widetilde{\chi}_A(z)\leq\widetilde{1}$ for all $z\in S$. Thus $\widetilde{\chi}_A(x+y)=\widetilde{1}$. So we find that $x+y\in A$. Now, let $a\in A$ and $s_1\in S$. Then $\widetilde{\chi}_A(a)=\widetilde{1}$. Now since $\widetilde{\chi}_A$ is an $i.v.$ fuzzy of S, we have $\widetilde{\chi}_A(s_1a)\geq Max^i(\widetilde{\chi}_A(a),\widetilde{\chi}_A(s_1)=\widetilde{1}$. So $\widetilde{\chi}_A(s_1a)\geq\widetilde{1}$. Again $\widetilde{\chi}_A(s_1a)\leq\widetilde{1}$. Thus we find that $\widetilde{\chi}_A(s_1a)=\widetilde{1}$. Consequently, $s_1a\in A$. Similarly, we can show that $as_1\in A$. Hence A is an ideal of S.

Lemma 2. *A non-empty $i.v.$ fuzzy subset $\widetilde{\mu}$ of a semiring S is an $i.v.$ fuzzy ideal of S if and only if $\overline{U}(\widetilde{\mu},[\alpha,\beta])$ are ideals of S for all $[\alpha,\beta]\in Im\widetilde{\mu}$.*

Proof. First suppose that $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of S. Let $[\alpha,\beta]$ be an arbitrary element in $Im\widetilde{\mu}$. Now consider the level subset $\overline{U}(\widetilde{\mu},[\alpha,\beta])$. Since $[\alpha,\beta]\in Im\widetilde{\mu}$, we have $\widetilde{\mu}(s_0)=[\alpha,\beta]$ for some $s_0\in S$. This implies that $s_0\in\overline{U}(\widetilde{\mu},[\alpha,\beta])$. So, $\overline{U}(\widetilde{\mu},[\alpha,\beta])$ is non-empty. Now take $x,y\in\overline{U}(\widetilde{\mu},[\alpha,\beta])$. Then we have $\widetilde{\mu}(x)\geq[\alpha,\beta]$ and $\widetilde{\mu}(y)\geq[\alpha,\beta]$. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of S, we have $\widetilde{\mu}(x+y)\geq Min^i(\widetilde{\mu}(x),\widetilde{\mu}(y))\geq[\alpha,\beta]$. So we get $x+y\in\overline{U}(\widetilde{\mu},[\alpha,\beta])$. Again let $a\in\overline{U}(\widetilde{\mu},[\alpha,\beta])$ and $s_1\in S$. Then $\widetilde{\mu}(a)\geq[\alpha,\beta]$. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of S, we have $\widetilde{\mu}(s_1a)\geq Max^i(\widetilde{\mu}(s_1),\widetilde{\mu}(a))\geq[\alpha,\beta]$. This implies that $s_1a\in\overline{U}(\widetilde{\mu},[\alpha,\beta])$. Similarly, we can show that $as_1\in\overline{U}(\widetilde{\mu},[\alpha,\beta])$. Thus $\overline{U}(\widetilde{\mu},[\alpha,\beta])$ is an ideal of S. Since $[\alpha,\beta]$ is arbitrary, it follows that $\overline{U}(\widetilde{\mu},[\alpha,\beta])$ are ideals of S for all $[\alpha,\beta]\in Im\widetilde{\mu}$.
Conversely, suppose that $\overline{U}(\widetilde{\mu},[\alpha,\beta])$ are ideals of S for all $[\alpha,\beta]\in Im\widetilde{\mu}$. Let $x,y\in S$ and let $\widetilde{\mu}(x)=[\alpha_1,\beta_1]$ and $\widetilde{\mu}(y)=[\alpha_2,\beta_2]$. This shows that $x\in\overline{U}(\widetilde{\mu},[\alpha_1,\beta_1])$ and $y\in\overline{U}(\widetilde{\mu},[\alpha_2,\beta_2])$. Without loss of generality, we consider $[\alpha_1,\beta_1]>[\alpha_2,\beta_2]$. Then by Proposition 3, we have $\overline{U}(\widetilde{\mu},[\alpha_1,\beta_1])\subseteq\overline{U}(\widetilde{\mu},[\alpha_2,\beta_2])$. So we find that $x,y\in\overline{U}(\widetilde{\mu},[\alpha_2,\beta_2])$. Now since $\overline{U}(\widetilde{\mu},[\alpha,\beta])$ are ideals of S for all $[\alpha,\beta]\in Im\widetilde{\mu}$, $\overline{U}(\widetilde{\mu},[\alpha_2,\beta_2])$ is an ideal of S. Thus $x,y\in\overline{U}(\widetilde{\mu},[\alpha_2,\beta_2])$ implies that $x+y\in\overline{U}(\widetilde{\mu},[\alpha_2,\beta_2])$. Therefore

$$\widetilde{\mu}(x+y)\geq[\alpha_2,\beta_2]=Min^i([\alpha_1,\beta_1],[\alpha_2,\beta_2])=Min^i(\widetilde{\mu}(x),\widetilde{\mu}(y)).$$

Now let $s,t\in S$ be such that $\widetilde{\mu}(t)=[\alpha_3,\beta_3]$. Then $t\in\overline{U}(\widetilde{\mu},[\alpha_3,\beta_3])$. Therefore $st\in\overline{U}(\widetilde{\mu},[\alpha_3,\beta_3])$, since $\overline{U}(\widetilde{\mu},[\alpha_3,\beta_3])$ is an ideal of S. So $\widetilde{\mu}(st)\geq[\alpha_3,\beta_3]=\widetilde{\mu}(t)$. Similarly, if we take $\widetilde{\mu}(s)=[\alpha_3,\beta_3]$, we can prove that $\widetilde{\mu}(st)\geq\widetilde{\mu}(s)$. Consequently, $\widetilde{\mu}(st)\geq Max^i(\widetilde{\mu}(s),\widetilde{\mu}(t))$. Hence $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of S.

Definition 17. *If $\widetilde{\mu}$ be an $i.v.$ fuzzy ideal of a semiring S, then the ideals $\overline{U}(\widetilde{\mu},[\alpha,\beta])$ of S, where $[\alpha,\beta]\in Im\widetilde{\mu}$, are called the level ideals of $\widetilde{\mu}$.*

Lemma 3. *Let S be a semiring and $\widetilde{\mu}$ be an $i.v.$ fuzzy ideal of S. Then for any $x,y\in S$; $\widetilde{\mu}(x)\geq\widetilde{\mu}(y)$ whenever $x\in<y>$, the principal ideal generated by y.*

Proof. Since S is a semiring with identity, we find that $<y>=\left\{\sum_{i=1}^{n}r_iys_i : r_i,s_i\in S \text{ and } n\in\mathbb{N}\right\}$. Now $x\in<y>$ implies that $x=\sum_{i=1}^{n}r_iys_i$ for some $r_i,s_i\in S$ and $n\in\mathbb{N}$. Then

$$\widetilde{\mu}(x)=\widetilde{\mu}\Big(\sum_{i=1}^{n}r_iys_i\Big)$$

$$
\begin{aligned}
&= \widetilde{\mu}(r_1ys_1 + r_2ys_2 + \ldots + r_nys_n)\\
&= \widetilde{\mu}\Big((r_1ys_1 + r_2ys_2 + \ldots + r_{n-1}ys_{n-1}) + r_nys_n\Big)\\
&\geq Min^i\Big(\widetilde{\mu}(r_1ys_1 + r_2ys_2 + \ldots + r_{n-1}ys_{n-1}), \widetilde{\mu}(r_nys_n)\Big) \text{ (Since } \widetilde{\mu} \text{ is an } i.v. \text{ fuzzy ideal of } S)\\
&\geq Min^i\Big(\widetilde{\mu}(r_1ys_1 + \ldots + r_{n-1}ys_{n-1}), Max^i(\widetilde{\mu}(r_ny), \widetilde{\mu}(s_n))\Big) \text{ (Since } \widetilde{\mu} \text{ is an } i.v. \text{ fuzzy ideal of } S)\\
&\geq Min^i\Big(\widetilde{\mu}(r_1ys_1 + \ldots + r_{n-1}ys_{n-1}), Max^i(Max^i(\widetilde{\mu}(r_n), \widetilde{\mu}(y)), \widetilde{\mu}(s_n))\Big)\\
&\geq Min^i\Big(\widetilde{\mu}(r_1ys_1 + \ldots + r_{n-1}ys_{n-1}), \widetilde{\mu}(y)\Big)\\
&\vdots\\
&\geq \widetilde{\mu}(y).
\end{aligned}
$$

Thus we get that $\widetilde{\mu}(x) \geq \widetilde{\mu}(y)$.

Lemma 4. *Let I be an ideal of a semiring S and $[\alpha, \beta] \leq [\gamma, \delta] \neq \widetilde{0}$ be any two interval numbers on $[0, 1]$. Then the $i.v.$ fuzzy subset $\widetilde{\mu}$ of S, defined by :*

$$
\widetilde{\mu}(x) = \begin{cases} [\gamma, \delta] & \textit{when } x \in I, \\ [\alpha, \beta] & \textit{otherwise,} \end{cases}
$$

is an $i.v.$ fuzzy ideal of S.

Proof. Since I is an ideal of S, we have $0_S \in I$. Then $\widetilde{\mu}(0_S) = [\gamma, \delta] \neq \widetilde{0}$. So $\widetilde{\mu}$ is non-empty. Now let $x, y \in S$.
Case 1: Let $Max^i(\widetilde{\mu}(x), \widetilde{\mu}(y)) = [\alpha, \beta]$. Then $\widetilde{\mu}(x) = [\alpha, \beta]$ and

$$
\widetilde{\mu}(y) = [\alpha, \beta] \Longrightarrow \widetilde{\mu}(xy) \geq [\alpha, \beta] = Max^i(\widetilde{\mu}(x), \widetilde{\mu}(y))
$$

and $\widetilde{\mu}(x + y) \geq [\alpha, \beta] = Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$.
Case 2: Let $Max^i(\widetilde{\mu}(x), \widetilde{\mu}(y)) = [\gamma, \delta]$. Then $\widetilde{\mu}(x) = [\gamma, \delta]$ or $\widetilde{\mu}(y) = [\gamma, \delta] \Longrightarrow x \in I$ or $y \in I \Longrightarrow xy \in I$ (since I is an ideal of S) $\Longrightarrow \widetilde{\mu}(xy) = [\gamma, \delta] = Max^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$. Now $Max^i(\widetilde{\mu}(x), \widetilde{\mu}(y)) = [\gamma, \delta] \Longrightarrow Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y)) = [\alpha, \beta]$ or $[\gamma, \delta]$.
$Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y)) = [\alpha, \beta] \Longrightarrow \widetilde{\mu}(x + y) \geq [\alpha, \beta] = Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$. Again $Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y)) = [\gamma, \delta] \Longrightarrow \widetilde{\mu}(x) = [\gamma, \delta]$ and $\widetilde{\mu}(y) = [\gamma, \delta] \Longrightarrow x \in I$ and $y \in I \Longrightarrow x + y \in I$ (since I is an ideal of S) $\Longrightarrow$ $widetilde e\mu(x + y) = [\gamma, \delta] = Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$.
Thus in all cases we find that $\widetilde{\mu}(x + y) \geq Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$ and $\widetilde{\mu}(xy) \geq Max^i(\widetilde{\mu}(x), \widetilde{\mu}(y))$. Consequently, $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of S.

Lemma 5. *Let $\widetilde{\mu}$ be an $i.v.$ fuzzy ideal of a semiring S. Then the set $\widetilde{\mu}_0 = \{x \in S : \widetilde{\mu}(x) = \widetilde{\mu}(0_S)\}$ is an ideal of S.*

Proof. Since $0_S \in \widetilde{\mu}_0$, $\widetilde{\mu}_0$ is non-empty. Let $x, y \in \widetilde{\mu}_0$. Then $\widetilde{\mu}(x) = \widetilde{\mu}(0_S) = \widetilde{\mu}(y)$. Now, since $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of S, we have $\widetilde{\mu}(x + y) \geq Min^i(\widetilde{\mu}(x), \widetilde{\mu}(y)) = \widetilde{\mu}(0_S)$. Also by Remark 1, we have $\widetilde{\mu}(0_S) \geq \widetilde{\mu}(x+y)$. Thus $\widetilde{\mu}(x+y) = \widetilde{\mu}(0_S)$. So $x+y \in \widetilde{\mu}_0$. Let $s \in S$ and $t \in \widetilde{\mu}_0$. Then $\widetilde{\mu}(t) = \widetilde{\mu}(0_S)$. Now since $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of S, we have $\widetilde{\mu}(st) \geq Max^i(\widetilde{\mu}(s), \widetilde{\mu}(t)) = Max^i(\widetilde{\mu}(s), \widetilde{\mu}(0_S)) = \widetilde{\mu}(0_S)$ (since $\widetilde{\mu}(0_S) \geq \widetilde{\mu}(s)$, by Remark 1). Also since $\widetilde{\mu}(0_S) \geq \widetilde{\mu}(st)$, we have $\widetilde{\mu}(st) = \widetilde{\mu}(0_S)$. Thus $st \in \widetilde{\mu}_0$. Similarly, we can show that $ts \in \widetilde{\mu}_0$. Consequently, $\widetilde{\mu}_0$ is an ideal of S.

Definition 18. *An $i.v.$ fuzzy ideal $\widetilde{\mu}$ of a semiring S is said to be an interval-valued fuzzy prime ideal of S if $\widetilde{\mu}$ is not a constant function (i.e. $|Im\widetilde{\mu}| \geq 2$) and for any two $i.v.$ fuzzy ideals $\widetilde{\mu_1}$ and $\widetilde{\mu_2}$ of S, $\widetilde{\mu_1} \circ \widetilde{\mu_2} \subseteq \widetilde{\mu}$ implies that either $\widetilde{\mu_1} \subseteq \widetilde{\mu}$ or $\widetilde{\mu_2} \subseteq \widetilde{\mu}$.*

Theorem 1. *Let I be a prime ideal of a semiring S and $[\alpha, \beta] \in D[0,1] \setminus \{\widetilde{1}\}$. Let $\widetilde{\mu}$ be an $i.v.$ fuzzy subset of S, defined by :*

$$\widetilde{\mu}(x) = \begin{cases} \widetilde{1} & \textit{when } x \in I, \\ [\alpha, \beta] & \textit{otherwise}. \end{cases}$$

Then $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S.

Proof. By Lemma 4, it follows that $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of S. Clearly, $\widetilde{\mu}$ is non-constant. Let $\widetilde{\mu_1}$ and $\widetilde{\mu_2}$ be two $i.v.$ fuzzy ideals of S such that $(\widetilde{\mu_1} \circ \widetilde{\mu_2}) \subseteq \widetilde{\mu}$. We have to prove that either $\widetilde{\mu_1} \subseteq \widetilde{\mu}$ or $\widetilde{\mu_2} \subseteq \widetilde{\mu}$. If possible, let $\widetilde{\mu_1} \nsubseteq \widetilde{\mu}$ and $\widetilde{\mu_2} \nsubseteq \widetilde{\mu}$. Then there exist $x, y \in S$ such that $\widetilde{\mu_1}(x) \nleq \widetilde{\mu}(x)$ and $\widetilde{\mu_2}(y) \nleq \widetilde{\mu}(y)$. Now according to our assumption, any two interval numbers in $D[0,1]$ are comparable. So we find that $\widetilde{\mu_1}(x) > \widetilde{\mu}(x)$ and $\widetilde{\mu_2}(y) > \widetilde{\mu}(y)$. This implies that $\widetilde{\mu}(x) \neq \widetilde{1}$ and $\widetilde{\mu}(y) \neq \widetilde{1}$. Hence $\widetilde{\mu}(x) = \widetilde{\mu}(y) = [\alpha, \beta]$. So $x \notin I$ and $y \notin I$. Since I is a prime ideal of S, there exists $s \in S$ such that $xsy \notin I$, by Proposition 1. Therefore $\widetilde{\mu}(xsy) = [\alpha, \beta]$. Now,

$$\begin{aligned}(\widetilde{\mu_1} \circ \widetilde{\mu_2})(xsy) = & \sup_{xsy=pq} \left\{ Min^i \left(\widetilde{\mu_1}(p), \widetilde{\mu_2}(q) \right) \right\} \\ \geq & Min^i \left(\widetilde{\mu_1}(x), \widetilde{\mu_2}(sy) \right) \\ \geq & Min^i \left(\widetilde{\mu_1}(x), \widetilde{\mu_2}(y) \right) \text{ (Since } \widetilde{\mu}_2 \text{ is an } i.v. \text{ fuzzy ideal of } S\text{)} \\ > & Min^i \left(\widetilde{\mu}(x), \widetilde{\mu}(y) \right) \\ = & [\alpha, \beta] = \widetilde{\mu}(xsy).\end{aligned}$$

This contradicts the fact that $(\widetilde{\mu_1} \circ \widetilde{\mu_2}) \subseteq \widetilde{\mu}$. Consequently, either $\widetilde{\mu_1} \subseteq \widetilde{\mu}$ or $\widetilde{\mu_2} \subseteq \widetilde{\mu}$. Hence $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S.

From the above theorem, we can easily produce an example of an $i.v.$ fuzzy prime ideal of a semiring as follows:

Example 2. *Let $\widetilde{\mu}$ be an $i.v.$ fuzzy subset of the set of non-negative integers $\mathbb{N}_0$ defined by:*

$$\widetilde{\mu}(x) = \begin{cases} \widetilde{1} & \textit{when } x \in 3\mathbb{N}_o, \\ [0.5, 0.6] & \textit{otherwise}. \end{cases}$$

Then $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of $\mathbb{N}_0$.

Theorem 2. *If $\widetilde{\mu}$ be an $i.v.$ fuzzy prime ideal of a semiring S, then*

(i) $\widetilde{\mu}(0_S) = \widetilde{1}$.

(ii) $|Im\widetilde{\mu}| = 2$.

(iii) $\widetilde{\mu}_0 = \left\{x \in S : \widetilde{\mu}(x) = \widetilde{\mu}(0_S)\right\}$ *is a prime ideal of S.*

Proof. Let $\widetilde{\mu}$ be an $i.v.$ fuzzy prime ideal of a semiring S.

(i) If possible, let $\widetilde{\mu}(0_S) \neq \widetilde{1}$. Since any two interval numbers in $D[0,1]$ are comparable, it follows that $\widetilde{\mu}(0_S) < \widetilde{1}$. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, $\widetilde{\mu}$ is non-constant. Also $\widetilde{\mu}(0_S) \geq \widetilde{\mu}(x)$ for all $x \in S$, by Remark 1. Since $\widetilde{\mu}$ is non-constant, there exists $s \in S$ such that $\widetilde{\mu}(0_S) > \widetilde{\mu}(s)$. Now we construct two $i.v.$ fuzzy subsets $\widetilde{\mu_1}$ and $\widetilde{\mu_2}$ of S as follows :

$$\widetilde{\mu_1}(x) = \begin{cases} \widetilde{1} & \text{when } x \in \widetilde{\mu}_0, \\ \widetilde{0} & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{\mu_2}(x) = \widetilde{\mu}(0_S) \text{ for all } x \in S.$$

Since $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of S, we have $\widetilde{\mu}_0$ is an ideal of S, by Lemma 5. Then by Lemma 4, we have $\widetilde{\mu_1}$ is an $i.v.$ fuzzy ideal of S. Also $\widetilde{\mu_2}$ is an $i.v.$ fuzzy ideal of S, since it is a constant function. Let $x \in S$ be such that $\widetilde{\mu}(x) = \widetilde{\mu}(0_S)$. This implies that $\widetilde{\mu_1}(x) = \widetilde{1}$. Then for any $y \in S$,
$Min^i\left(\widetilde{\mu_1}(x), \widetilde{\mu_2}(y)\right) = \widetilde{\mu_2}(y) = \widetilde{\mu}(0_S) = \widetilde{\mu}(xy)$. (Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, it is an $i.v.$ fuzzy ideal of S. So $\widetilde{\mu}(0_S) \geq \widetilde{\mu}(xy)$, by Remark 1 and
$\widetilde{\mu}(xy) \geq \widetilde{\mu}(x) = \widetilde{\mu}(0_S)$. So we get $\widetilde{\mu}(xy) = \widetilde{\mu}(0_S)$). Now, let $x \in S$ be such that $\widetilde{\mu}(x) \neq \widetilde{\mu}(0_S)$. This implies that $\widetilde{\mu_1}(x) = \widetilde{0}$. Then for any $y \in S$,
$Min^i\left(\widetilde{\mu_1}(x), \widetilde{\mu_2}(y)\right) = \widetilde{0} \leq \widetilde{\mu}(xy)$. Thus for any $z \in S$,
$\sup_{z=xy}\left\{Min^i\left(\widetilde{\mu_1}(x), \widetilde{\mu_2}(y)\right)\right\} \leq \widetilde{\mu}(z)$ i.e. $(\widetilde{\mu_1} \circ \widetilde{\mu_2})(z) \leq \widetilde{\mu}(z)$. This implies that $\widetilde{\mu_1} \circ \widetilde{\mu_2} \subseteq \widetilde{\mu}$. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, it follows that either $\widetilde{\mu_1} \subseteq \widetilde{\mu}$ or $\widetilde{\mu_2} \subseteq \widetilde{\mu}$. But $\widetilde{\mu_1}(0_S) = \widetilde{1} > \widetilde{\mu}(0_S)$ and $\widetilde{\mu_2}(s) = \widetilde{\mu}(0_S) > \widetilde{\mu}(s)$. Thus we arrive at a contradiction. Consequently, $\widetilde{\mu}(0_S) = \widetilde{1}$.

(ii) Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, it follows that $\widetilde{\mu}$ is non-constant. So $|Im\widetilde{\mu}| \geq 2$. If possible, let $|Im\widetilde{\mu}| > 2$. Let $a, b \in S$ be such that $\widetilde{1} > \widetilde{\mu}(a) > \widetilde{\mu}(b)$. Now we construct two $i.v.$ fuzzy subsets $\widetilde{\mu_1}$ and $\widetilde{\mu_2}$ of S as follows :

$$\widetilde{\mu_1}(x) = \begin{cases} \widetilde{1} & \text{when } x \in <a> \\ \widetilde{0} & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{\mu_2}(x) = \widetilde{\mu}(a) \text{ for all } x \in S.$$

By Lemma 4, we have $\widetilde{\mu_1}$ is an $i.v.$ fuzzy ideal of S. Also $\widetilde{\mu_2}$ is an $i.v.$ fuzzy ideal of S, since it is a constant function. Now suppose $x \in <a>$ for $x \in S$. Then $\widetilde{\mu_1}(x) = \widetilde{1}$. So for any $y \in S$, $Min^i\left(\widetilde{\mu_1}(x), \widetilde{\mu_2}(y)\right) = \widetilde{\mu_2}(y) = \widetilde{\mu}(a) \leq \widetilde{\mu}(xy)$. (Since $x \in <a>$, we have $\widetilde{\mu}(x) \geq \widetilde{\mu}(a)$, by Lemma 3. Also since $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of S, we have $\widetilde{\mu}(xy) \geq \widetilde{\mu}(x)$ which implies that $\widetilde{\mu}(xy) \geq \widetilde{\mu}(a)$). Now let $x \in S$ be such that $x \notin <a>$. This implies

that $\widetilde{\mu_1}(x)=\widetilde{0}$. Then for any $y\in S$,
$Min^i\left(\widetilde{\mu_1}(x),\widetilde{\mu_2}(y)\right)=\widetilde{0}\leq\widetilde{\mu}(xy)$. Therefore for any $z\in S$,
$\sup_{z=xy}\left\{Min^i\left(\widetilde{\mu_1}(x),\ widetilde e\mu_2(y)\right)\right\}\leq\widetilde{\mu}(z)$ i.e. $(\widetilde{\mu_1}\circ\widetilde{\mu_2})(z)\leq\widetilde{\mu}(z)$. This implies that $\widetilde{\mu_1}\circ\widetilde{\mu_2}\subseteq\widetilde{\mu}$. Now since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, we find that either $\widetilde{\mu_1}\subseteq\widetilde{\mu}$ or $\widetilde{\mu_2}\subseteq\widetilde{\mu}$. But $\widetilde{\mu_1}(a)=\widetilde{1}>\widetilde{\mu}(a)$ and $\widetilde{\mu_2}(b)=\widetilde{\mu}(a)>\widetilde{\mu}(b)$. This contradicts the fact that either $\widetilde{\mu_1}\subseteq\widetilde{\mu}$ or $\widetilde{\mu_2}\subseteq\widetilde{\mu}$. Hence $|Im\widetilde{\mu}|=2$.

(iii) Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, $\widetilde{\mu}_0$ is an ideal of S, by Lemma 5. Also $\widetilde{\mu}$ is non-constant. So $\widetilde{\mu}_0$ is a proper ideal of S. Now let A,B be any two ideals of S such that $AB\subseteq\widetilde{\mu}_0$. Since A,B are ideals of S, $\widetilde{\chi}_A$ and $\widetilde{\chi}_B$ are $i.v.$ fuzzy ideals of S, by Lemma 1. Let $x\in S$. If $(\widetilde{\chi}_A\circ\widetilde{\chi}_B)(x)=\widetilde{0}$, we find that $(\widetilde{\chi}_A\circ\widetilde{\chi}_B)(x)=\widetilde{0}\leq\widetilde{\chi}_{\widetilde{\mu}_0}(x)$. If $(\widetilde{\chi}_A\circ\widetilde{\chi}_B)(x)\neq\widetilde{0}$, $(\widetilde{\chi}_A\circ\widetilde{\chi}_B)(x)=\sup_{x=pq}\left\{Min^i\left(\widetilde{\chi}_A(p),\widetilde{\chi}_B(q)\right)\right\}\neq\widetilde{0}$. This shows that $\sup_{x=pq}\left\{Min^i\left(\widetilde{\chi}_A(p),\widetilde{\chi}_B(q)\right)\right\}=\widetilde{1}$. Then $Min^i\left(\widetilde{\chi}_A(p),\widetilde{\chi}_B(q)\right)=\widetilde{1}$ for some $p,q\in S$ such that $x=pq$, where $\widetilde{\chi}_A(p)=\widetilde{1}$ and $\widetilde{\chi}_B(q)=\widetilde{1}$. This implies that $p\in A$ and $q\in B$ i.e. $pq\in AB$. Since $AB\subseteq\widetilde{\mu}_0$ we obtain that $x=pq\in\widetilde{\mu}_0$. Thus $\widetilde{\chi}_{\widetilde{\mu}_0}(x)=\widetilde{1}$. Then $(\widetilde{\chi}_A\circ\widetilde{\chi}_B)(x)=\widetilde{\chi}_{\widetilde{\mu}_0}(x)$. Consequently, $(\widetilde{\chi}_A\circ\widetilde{\chi}_B)(x)\leq\widetilde{\chi}_{\widetilde{\mu}_0}(x)$ for all $x\in S$. So we get that $\widetilde{\chi}_A\circ\widetilde{\chi}_B\subseteq\widetilde{\chi}_{\widetilde{\mu}_0}$. Let $y\in S$. If $\widetilde{\chi}_{\widetilde{\mu}_0}(y)=\widetilde{0}$ we find that $\widetilde{\chi}_{\widetilde{\mu}_0}(y)\leq\widetilde{\mu}(y)$. If $\widetilde{\chi}_{\widetilde{\mu}_0}(y)=\widetilde{1}$, it follows that $y\in\widetilde{\mu}_0$. Then we get that $\widetilde{\mu}(y)=\widetilde{\mu}(0_S)$. Again since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, we find that $\widetilde{\mu}(0_S)=\widetilde{1}$, by Theorem 2($i$). Thus $\widetilde{\chi}_{\widetilde{\mu}_0}(y)=\widetilde{\mu}(y)$. So we see that $\widetilde{\chi}_{\widetilde{\mu}_0}(y)\leq\widetilde{\mu}(y)$ for all $y\in S$. Consequently, $\widetilde{\chi}_{\widetilde{\mu}_0}\subseteq\widetilde{\mu}$. Thus we get $\widetilde{\chi}_A\circ\widetilde{\chi}_B\subseteq\widetilde{\chi}_{\widetilde{\mu}_0}\subseteq\widetilde{\mu}$ i.e. $\widetilde{\chi}_A\circ\widetilde{\chi}_B\subseteq\widetilde{\mu}$. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, we find that either $\widetilde{\chi}_A\subseteq\widetilde{\mu}$ or $\widetilde{\chi}_B\subseteq\widetilde{\mu}$. Suppose $\widetilde{\chi}_A\subseteq\widetilde{\mu}$. Let $z\in A$. Then $\widetilde{\chi}_A(z)=\widetilde{1}$. Also since $\widetilde{\chi}_A\subseteq\widetilde{\mu}$, we obtain that $\widetilde{\chi}_A(z)\leq\widetilde{\mu}(z)$. This implies that $\widetilde{\mu}(z)\geq\widetilde{1}$ i.e. $\widetilde{\mu}(z)=\widetilde{1}$. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, we have $\widetilde{\mu}(0_S)=\widetilde{1}$, by Theorem 2($i$). Thus $\widetilde{\mu}(z)=\widetilde{\mu}(0_S)$. Consequently, $z\in\widetilde{\mu}_0$. So we find that $A\subseteq\widetilde{\mu}_0$. Similarly, we can show that $B\subseteq\widetilde{\mu}_0$ whenever $\widetilde{\chi}_B\subseteq\widetilde{\mu}$. Thus we see that for any two ideals A,B of S, $AB\subseteq\widetilde{\mu}_0\Longrightarrow$ either $A\subseteq\widetilde{\mu}_0$ or $B\subseteq\widetilde{\mu}_0$. Hence $\widetilde{\mu}_0$ is a prime ideal of S.

From the above two theorems we have the following result:

Theorem 3. *If $\widetilde{\mu}$ be an $i.v.$ fuzzy subset of a semiring S, then $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S if and only if $Im\widetilde{\mu}=\{\widetilde{1},[\alpha,\beta]\}$ where $[\alpha,\beta]\in D[0,1]\setminus\{\widetilde{1}\}$ and $\widetilde{\mu}_0$ is a prime ideal of S.*

Theorem 4. *Let $I(\neq S)$ be a subset of a semiring S. Then I is a prime ideal of S if and only if $\widetilde{\chi}_I$ is an $i.v.$ fuzzy prime ideal of S.*

Proof. Let I be a prime ideal of S. Now in Theorem 1, if we replace $[\alpha,\beta]$ by $\widetilde{0}$, we find that $\widetilde{\chi}_I$ is an $i.v.$ fuzzy prime ideal of S.
Conversely, let $\widetilde{\chi}_I$ be an $i.v.$ fuzzy prime ideal of S. Then
$(\widetilde{\chi}_I)_0=\left\{x\in S:\widetilde{\chi}_I(x)=\widetilde{\chi}_I(0_S)\right\}=\left\{x\in S:\widetilde{\chi}_I(x)=\widetilde{1}\right\}=I$. Hence I is a prime ideal of S, by Theorem 2(iii).

Lemma 6. *Let $\widetilde{\mu}$ be an $i.v.$ fuzzy prime ideal of a semiring S. Then for any $a, b \in S$; $inf\left\{\widetilde{\mu}(asb) : s \in S\right\} = Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right)$.*

Proof. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, we have $Im\widetilde{\mu} = \{\widetilde{1}, [\alpha, \beta]\}$, where $[\alpha, \beta] \in D[0,1] \setminus \{\widetilde{1}\}$ and $\widetilde{\mu}_0$ is a prime ideal of S, by Theorem 3.
<u>Case 1</u>: Let $Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right) = \widetilde{1}$. This implies that either $\widetilde{\mu}(a) = \widetilde{1}$ or $\widetilde{\mu}(b) = \widetilde{1}$ i.e. $\widetilde{\mu}(a) = \widetilde{\mu}(0_S)$ or $\widetilde{\mu}(b) = \widetilde{\mu}(0_S)$, by Theorem 2($i$). Therefore either $a \in \widetilde{\mu}_0$ or $b \in \widetilde{\mu}_0$. This implies that $asb \in \widetilde{\mu}_0$ for all $s \in S$, by Proposition 1. Therefore $\widetilde{\mu}(asb) = \widetilde{\mu}(0_S) = \widetilde{1}$ for all $s \in S$, by Theorem 2(i). This shows that $inf\left\{\widetilde{\mu}(asb) : s \in S\right\} = \widetilde{1} = Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right)$.
<u>Case 2</u>: Let $Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right) = [\alpha, \beta]$. Then $\widetilde{\mu}(a) = [\alpha, \beta]$ and $\widetilde{\mu}(b) = [\alpha, \beta]$, since $Im\widetilde{\mu} = \{\widetilde{1}, [\alpha, \beta]\}$. This implies that $a \notin \widetilde{\mu}_0$ and $b \notin \widetilde{\mu}_0$. Since $\widetilde{\mu}_0$ is a prime ideal of S, there exists $s_0 \in S$ such that $as_0b \notin \widetilde{\mu}_0$, by Proposition 1. Then $\widetilde{\mu}(as_0b) = [\alpha, \beta]$. Consequently, $inf\left\{\widetilde{\mu}(asb) : s \in S\right\} = Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right)$.

Theorem 5. *Let $\widetilde{\mu}$ be an $i.v.$ fuzzy prime ideal of a semiring S and a, b be any two elements of S. Then the following are equivalent :*

(i) $\widetilde{\mu}(ab) = Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right)$.

(ii) $\widetilde{\mu}(ab) = \widetilde{\mu}(ba)$.

Proof. <u>$(i) \Longrightarrow (ii)$</u> : Let $\widetilde{\mu}(ab) = Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right)$. Then
$\widetilde{\mu}(ab) = Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right) = Max^i\left(\widetilde{\mu}(b), \widetilde{\mu}(a)\right) = \widetilde{\mu}(ba)$.
<u>$(ii) \Longrightarrow (i)$</u> : Let $\widetilde{\mu}(ab) = \widetilde{\mu}(ba)$ for any two elements a, b of S. Then $\widetilde{\mu}(asb) = \widetilde{\mu}(bas)$ for all $s \in S$. Also since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S,
$\widetilde{\mu}(asb) = \widetilde{\mu}(bas) \geq Max^i(\widetilde{\mu}(ba), \widetilde{\mu}(s)) \geq \widetilde{\mu}(ba) = \widetilde{\mu}(ab)$ for all $s \in S$. Hence
$inf\left\{\widetilde{\mu}(asb) : s \in S\right\} \geq \widetilde{\mu}(ab)$. Now by Lemma 6, we have
$Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right) = inf\left\{\widetilde{\mu}(asb) : s \in S\right\}$. So $Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right) \geq \widetilde{\mu}(ab)$. Again
$\widetilde{\mu}(ab) \geq Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right)$. Consequently, $\widetilde{\mu}(ab) = Max^i\left(\widetilde{\mu}(a), \widetilde{\mu}(b)\right)$.

2.2. Interval-Valued Fuzzy Completely Prime Ideal of a Semiring

Definition 19. *Let S be a semiring and $x \in S$. Let $\widetilde{a} \in D[0,1] \setminus \{\widetilde{0}\}$. Then an $i.v.$ fuzzy subset $x_{\widetilde{a}}$ of S is called an interval-valued fuzzy point of S if*

$$x_{\widetilde{a}}(y) = \begin{cases} \widetilde{a} & \text{if } x = y, \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

An interval-valued fuzzy point $x_{\widetilde{a}}$ is said to be contained in an $i.v.$ fuzzy subset $\widetilde{\mu}$ of S, denoted by $x_{\widetilde{a}} \in \widetilde{\mu}$, if $\widetilde{a} \leq \widetilde{\mu}(x)$.

Remark 2. *Let $x_{\widetilde{a}}$ be an $i.v.$ fuzzy point of a semiring S and $\widetilde{\mu}$ be an $i.v.$ fuzzy subset of S. Then $x_{\widetilde{a}} \in \widetilde{\mu}$ if and only if $x_{\widetilde{a}} \subseteq \widetilde{\mu}$.*

Remark 3. *Let $x_{\widetilde{a}}$ and $y_{\widetilde{b}}$ be two $i.v.$ fuzzy points of a semiring S. Then*
$(x_{\widetilde{a}} \circ y_{\widetilde{b}}) = \Big(xy\Big)_{Min^i(\widetilde{a},\widetilde{b})}$.

Definition 20. *A non-constant $i.v.$ fuzzy ideal $\widetilde{\mu}$ of a semiring S is said to be an interval-valued fuzzy completely prime ideal of S if for any two $i.v.$ fuzzy points $x_{\widetilde{a}}$ and $y_{\widetilde{b}}$ of S, $x_{\widetilde{a}} \circ y_{\widetilde{b}} \in \widetilde{\mu} \Longrightarrow$ either $x_{\widetilde{a}} \in \widetilde{\mu}$ or $y_{\widetilde{b}} \in \widetilde{\mu}$.*

Theorem 6. *Every $i.v.$ fuzzy completely prime ideal of a semiring S is an $i.v.$ fuzzy prime ideal of S.*

Proof. Let $\widetilde{\mu}$ be an $i.v.$ fuzzy completely prime ideal of S. Then $\widetilde{\mu}$ is non-constant. Let $\widetilde{\mu_1}$ and $\widetilde{\mu_2}$ be any two $i.v.$ fuzzy ideals of S such that $\widetilde{\mu_1} \circ \widetilde{\mu_2} \subseteq \widetilde{\mu}$. Let $\widetilde{\mu_1} \nsubseteq \widetilde{\mu}$. This implies that there exists $x \in S$ such that $\widetilde{\mu_1}(x) \nleq \widetilde{\mu}(x)$. Now since according to our assumption, any two interval numbers of $D[0,1]$ are comparable, we have $\widetilde{\mu_1}(x) > \widetilde{\mu}(x)$. This shows that $x_{\widetilde{\mu_1}(x)} \notin \widetilde{\mu}$. Now for any $y \in S$ and $w \in S$

$$\begin{aligned}(x_{\widetilde{\mu_1}(x)} \circ y_{\widetilde{\mu_2}(y)})(w) &= \Big(xy\Big)_{Min^i(\widetilde{\mu_1}(x),\widetilde{\mu_2}(y))}(w) \text{ (by Remark 3)} \\ &= \begin{cases} Min^i(\widetilde{\mu_1}(x),\widetilde{\mu_2}(y)) & \text{when } w = xy, \\ \widetilde{0} & \text{otherwise.}\end{cases}\end{aligned}$$

Now $\widetilde{\mu}(w) \geq (\widetilde{\mu_1} \circ \widetilde{\mu_2})(w) = \sup_{w=xy}\Big\{Min^i(\widetilde{\mu_1}(x),\widetilde{\mu_2}(y))\Big\} \geq Min^i(\widetilde{\mu_1}(x),\widetilde{\mu_2}(y))$ (when $w = xy$) $= (x_{\widetilde{\mu_1}(x)} \circ y_{\widetilde{\mu_2}(y)})(w)$. Also when $w \neq xy$, we have $\widetilde{\mu}(w) \geq (x_{\widetilde{\mu_1}(x)} \circ y_{\widetilde{\mu_2}(y)})(w)$. This shows that $(x_{\widetilde{\mu_1}(x)} \circ y_{\widetilde{\mu_2}(y)}) \subseteq \widetilde{\mu}$ i.e. $(x_{\widetilde{\mu_1}(x)} \circ y_{\widetilde{\mu_2}(y)}) \in \widetilde{\mu}$, by Remark 2. Now since $\widetilde{\mu}$ is an $i.v.$ fuzzy completely prime ideal of S and $x_{\widetilde{\mu_1}(x)} \notin \widetilde{\mu}$, we have $y_{\widetilde{\mu_2}(y)} \in \widetilde{\mu}$. This implies that $\widetilde{\mu_2}(y) \leq \widetilde{\mu}(y) \Longrightarrow \widetilde{\mu_2} \subseteq \widetilde{\mu}$. Consequently, $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S.

In general, the converse of the Theorem 6 for an arbitrary semiring is not true.

Example 3. *Consider the semiring $M_2(\mathbb{R}_0^+)$, where $M_2(\mathbb{R}_0^+)$ denotes the set of all 2×2 matrices with non-negative real entries.*
Let $\widetilde{\mu}$ be an $i.v.$ fuzzy subset of $M_2(\mathbb{R}_0^+)$, defined by :

$$\widetilde{\mu}(A) = \begin{cases} \widetilde{1} & \text{when } A \text{ is the null matrix,} \\ \widetilde{0} & \text{otherwise.}\end{cases}$$

Then it can be shown that $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of $M_2(\mathbb{R}_0^+)$ but $\widetilde{\mu}$ is not an $i.v.$ fuzzy completely prime ideal of $M_2(\mathbb{R}_0^+)$.

But in particular, for commutative semiring we have the following result :

Theorem 7. *Every $i.v.$ fuzzy prime ideal of a commutative semiring S is an $i.v.$ fuzzy completely prime ideal of S.*

Proof. Let S be a commutative semiring and $\widetilde{\mu}$ be an $i.v.$ fuzzy prime ideal of S. Then $\widetilde{\mu}$ is non-constant. Let $x_{\widetilde{a}}$ and $y_{\widetilde{b}}$ be any two $i.v.$ fuzzy points of S such that $x_{\widetilde{a}} \circ y_{\widetilde{b}} \in \widetilde{\mu}$. Then by Remark 3, we get $(xy)_{Min^i(\widetilde{a},\widetilde{b})} \in \widetilde{\mu}$. So we have $Min^i(\widetilde{a},\widetilde{b}) \leq \widetilde{\mu}(xy)$.
Now we construct two $i.v.$ fuzzy subsets $\widetilde{\mu_3}$ and $\widetilde{\mu_4}$ of S as follows:

$$\widetilde{\mu_3}(z) = \begin{cases} \widetilde{a} & \text{when } z \in < x >, \\ \widetilde{0} & \text{otherwise;} \end{cases} \quad \text{and} \quad \widetilde{\mu_4}(z) = \begin{cases} \widetilde{b} & \text{when } z \in < y >, \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

Then from Lemma 4, it follows that $\widetilde{\mu_3}$ and $\widetilde{\mu_4}$ are $i.v.$ fuzzy ideals of S. Now $(\widetilde{\mu_3} \circ \widetilde{\mu_4})(z) = \sup_{z=uv} \left\{ Min^i(\widetilde{\mu_3}(u), \widetilde{\mu_4}(v)) \right\}$. When z can not be expressed as $z = uv$, where $u \in < x >$ and $v \in < y >$, then $(\widetilde{\mu_3} \circ \widetilde{\mu_4})(z) = \widetilde{0} \leq \widetilde{\mu}(z)$. Let $z = uv$ for some $u, v \in S$, where $u \in < x >$ and $v \in < y >$. Then $(\widetilde{\mu_3} \circ \widetilde{\mu_4})(z) = \sup_{z=uv} \left\{ Min^i(\widetilde{a},\widetilde{b}) \right\} = Min^i(\widetilde{a},\widetilde{b})$. Since S is a commutative semiring with identity, $u \in < x >$ and $v \in < y > \Longrightarrow u = s_1 x$ and $v = s_2 y$ for some $s_1, s_2 \in S$. Then we have

$$\widetilde{\mu}(z) = \widetilde{\mu}(uv) = \widetilde{\mu}(s_1 x s_2 y) = \widetilde{\mu}(s_1 s_2 xy) \geq \widetilde{\mu}(xy) \geq Min^i(\widetilde{a},\widetilde{b}) = (\widetilde{\mu_3} \circ \widetilde{\mu_4})(z).$$

This implies that $(\widetilde{\mu_3} \circ \widetilde{\mu_4}) \subseteq \widetilde{\mu}$. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, we have either $\widetilde{\mu_3} \subseteq \widetilde{\mu}$ or $\widetilde{\mu_4} \subseteq \widetilde{\mu}$. This implies that $\widetilde{\mu_3}(x) \leq \widetilde{\mu}(x)$ or $\widetilde{\mu_4}(y) \leq \widetilde{\mu}(y) \Longrightarrow \widetilde{a} \leq \widetilde{\mu}(x)$ or $\widetilde{b} \leq \widetilde{\mu}(y) \Longrightarrow x_{\widetilde{a}} \in \widetilde{\mu}$ or $y_{\widetilde{b}} \in \widetilde{\mu}$. Consequently, $\widetilde{\mu}$ is an $i.v.$ fuzzy completely prime ideal of S.

Proposition 4. *Let $\widetilde{\mu}$ be an $i.v.$ fuzzy completely prime ideal of a semiring S. Then $\widetilde{\mu}_0 = \{x \in S : \widetilde{\mu}(x) = \widetilde{\mu}(0_S)\}$ is a completely prime ideal of S.*

Proof. Let $\widetilde{\mu}$ be an $i.v.$ fuzzy completely prime ideal of a semiring S. Then $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, by Theorem 6. So $\widetilde{\mu}_0$ is a prime ideal of S, by Theorem 2(iii) and hence a proper ideal of S. Let $x, y \in S$ be such that $xy \in \widetilde{\mu}_0$. Then $\widetilde{\mu}(xy) = \widetilde{\mu}(0_S) = \widetilde{1}$, by Theorem 2$(i)$. This implies that $(xy)_{\widetilde{1}} \in \widetilde{\mu}$. Thus by Remark 3, we get $(x_{\widetilde{1}} \circ y_{\widetilde{1}}) \in \widetilde{\mu}$. This implies that either $x_{\widetilde{1}} \in \widetilde{\mu}$ or $y_{\widetilde{1}} \in \widetilde{\mu}$, since $\widetilde{\mu}$ is an $i.v.$ fuzzy completely prime ideal of S. Thus it follows that either $\widetilde{1} \leq \widetilde{\mu}(x)$ or $\widetilde{1} \leq \widetilde{\mu}(y)$.
$\Longrightarrow \widetilde{\mu}(x) = \widetilde{1}$ or $\widetilde{\mu}(y) = \widetilde{1}$ i.e. $\widetilde{\mu}(x) = \widetilde{\mu}(0_S)$ or $\widetilde{\mu}(y) = \widetilde{\mu}(0_S)$.
$\Longrightarrow x \in \widetilde{\mu}_0$ or $y \in \widetilde{\mu}_0$.
Hence $\widetilde{\mu}_0$ is a completely prime ideal of S.

Theorem 8. *Let $\widetilde{\mu}$ be an $i.v.$ fuzzy subset of a semiring S such that $Im\widetilde{\mu} = \{\widetilde{1}, [\alpha, \beta]\}$; where $[\alpha, \beta] \in D[0,1] \setminus \{\widetilde{1}\}$ and $\widetilde{\mu}_0 = \{x \in S : \widetilde{\mu}(x) = \widetilde{\mu}(0_S)\}$ is a completely prime ideal of S. Then $\widetilde{\mu}$ is an $i.v.$ fuzzy completely prime ideal of S.*

Proof. Let $\widetilde{\mu}$ be an $i.v.$ fuzzy subset of a semiring S such that $Im\widetilde{\mu} = \{\widetilde{1}, [\alpha, \beta]\}$; where $[\alpha, \beta] \in D[0,1] \setminus \{\widetilde{1}\}$ and $\widetilde{\mu}_0 = \{x \in S : \widetilde{\mu}(x) = \widetilde{\mu}(0_S)\}$ is a completely prime ideal of S. Since $\widetilde{\mu}_0$ is a completely prime ideal of S, it is a prime ideal of S. Then by Theorem 3, we get that $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal and hence a non-constant $i.v.$ fuzzy ideal of S. Let $x_{\widetilde{a}}$ and $y_{\widetilde{b}}$ be any two arbitrary $i.v.$ fuzzy points of S such that $x_{\widetilde{a}} \circ y_{\widetilde{b}} \in \widetilde{\mu}$. Then by Remark 3, we find that

$(xy)_{Min^i(\widetilde{a},\widetilde{b})} \in \widetilde{\mu}$ i.e. $Min^i(\widetilde{a},\widetilde{b}) \leq \widetilde{\mu}(xy)$. We have to prove that either $x_{\widetilde{a}} \in \widetilde{\mu}$ or $y_{\widetilde{b}} \in \widetilde{\mu}$. If possible, let $x_{\widetilde{a}} \notin \widetilde{\mu}$ and $y_{\widetilde{b}} \notin \widetilde{\mu}$. Since according to our assumption any two interval numbers are comparable, it follows that $\widetilde{a} > \widetilde{\mu}(x)$ and $\widetilde{b} > \widetilde{\mu}(y)$. Therefore $\widetilde{\mu}(x) = [\alpha,\beta] = \widetilde{\mu}(y)$, since $Im\widetilde{\mu} = \{\widetilde{1},[\alpha,\beta]\}$. This implies that $x \notin \widetilde{\mu}_0$ and $y \notin \widetilde{\mu}_0$. Now since $\widetilde{\mu}_0$ is a completely prime ideal of S, we have $xy \notin \widetilde{\mu}_0$. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, $\widetilde{\mu}(0_S) = \widetilde{1}$, by Theorem 2$(i)$. So $xy \notin \widetilde{\mu}_0 \Longrightarrow \widetilde{\mu}(xy) \neq \widetilde{\mu}(0_S) = \widetilde{1}$. Consequently, $\widetilde{\mu}(xy) = [\alpha,\beta]$, since $Im\widetilde{\mu} = \{\widetilde{1},[\alpha,\beta]\}$. Now
$[\alpha,\beta] = \widetilde{\mu}(xy) \geq Min^i(\widetilde{a},\widetilde{b}) > Min^i(\widetilde{\mu}(x),\widetilde{\mu}(y)) = [\alpha,\beta]$. Thus we arrive at a contradiction. Hence either $x_{\widetilde{a}} \in \widetilde{\mu}$ or $y_{\widetilde{b}} \in \widetilde{\mu}$. Thus $\widetilde{\mu}$ is an $i.v.$ fuzzy completely prime ideal of S.

Theorem 9. *Let $\widetilde{\mu}$ be an $i.v.$ fuzzy ideal of a semiring S with $Im\widetilde{\mu} = \{\widetilde{1},[\alpha,\beta]\}$; where $[\alpha,\beta] \in D[0,1] \setminus \{\widetilde{1}\}$. Then $\widetilde{\mu}$ is an $i.v.$ fuzzy completely prime ideal of S if and only if its only proper level ideal $\overline{U}(\widetilde{\mu},\widetilde{1})$ is a completely prime ideal of S.*

Proof. Since $Im\widetilde{\mu} = \{\widetilde{1},[\alpha,\beta]\}$, where $[\alpha,\beta] \in D[0,1] \setminus \{\widetilde{1}\}$, we have $\overline{U}(\widetilde{\mu},[\alpha,\beta]) = S$. So only proper level ideal of $\widetilde{\mu}$ is $\overline{U}(\widetilde{\mu},\widetilde{1})$, by Lemma 2. Let $\widetilde{\mu}$ be an $i.v.$ fuzzy completely prime ideal of S. Then $\widetilde{\mu}_0$ is a completely prime ideal of S, by Proposition 4. Now since $\widetilde{\mu}$ is an $i.v.$ fuzzy completely prime ideal of S, we find that $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, by Theorem 6. Then $\widetilde{\mu}(0_S) = \widetilde{1}$, by Theorem 2$(i)$. Also we find that
$\overline{U}(\widetilde{\mu},\widetilde{1}) = \{x \in S : \widetilde{\mu}(x) \geq \widetilde{1}\} = \{x \in S : \widetilde{\mu}(x) = \widetilde{1}\} = \{x \in S : \widetilde{\mu}(x) = \widetilde{\mu}(0_S)\} = \widetilde{\mu}_0$. Hence $\overline{U}(\widetilde{\mu},\widetilde{1})$ is a completely prime ideal of S.
Conversely, let $\overline{U}(\widetilde{\mu},\widetilde{1})$ be a completely prime ideal of S. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy ideal of S, it follows that $\widetilde{\mu}(0_S) \geq \widetilde{\mu}(x)$ for all $x \in S$, by Remark 1. Again since $Im\widetilde{\mu} = \{\widetilde{1},[\alpha,\beta]\}$, where $[\alpha,\beta] \in D[0,1] \setminus \{\widetilde{1}\}$, we find that $\widetilde{\mu}(0_S) = \widetilde{1}$. Thus
$\overline{U}(\widetilde{\mu},\widetilde{1}) = \{x \in S : \widetilde{\mu}(x) \geq \widetilde{1}\} = \{x \in S : \widetilde{\mu}(x) = \widetilde{1}\} = \{x \in S : \widetilde{\mu}(x) = \widetilde{\mu}(0_S)\} = \widetilde{\mu}_0$. So $\widetilde{\mu}_0$ is a completely prime ideal of S. Then by Theorem 8, we have $\widetilde{\mu}$ is an $i.v.$ fuzzy completely prime ideal of S.

Theorem 10. *Let $\widetilde{\mu}$ be an $i.v.$ fuzzy prime ideal of a semiring S. Then $\widetilde{\mu}$ is an $i.v.$ fuzzy completely prime ideal of a semiring S if and only if for any two $i.v.$ fuzzy points $x_{\widetilde{a}}$ and $y_{\widetilde{b}}$ of S, $x_{\widetilde{a}} \circ y_{\widetilde{b}} \in \widetilde{\mu} \Longrightarrow y_{\widetilde{b}} \circ x_{\widetilde{a}} \in \widetilde{\mu}$.*

Proof. Let $\widetilde{\mu}$ be an $i.v.$ fuzzy completely prime ideal of S. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, we find that $Im\widetilde{\mu} = \{\widetilde{1},[\alpha,\beta]\}$, where $[\alpha,\beta] \in D[0,1] \setminus \{\widetilde{1}\}$ and $\widetilde{\mu}_0$ is a prime ideal of S, by Theorem 3. In fact we find that $\widetilde{\mu}_0$ is a completely prime ideal of S, since $\widetilde{\mu}$ is an $i.v.$ fuzzy completely prime ideal of S, by Proposition 4. Let $x_{\widetilde{a}}$ and $y_{\widetilde{b}}$ be any two $i.v.$ fuzzy points of S such that $x_{\widetilde{a}} \circ y_{\widetilde{b}} \in \widetilde{\mu}$. Then we have $\left(xy\right)_{Min^i(\widetilde{a},\widetilde{b})} \in \widetilde{\mu}$, by Remark 3. This implies that $Min^i(\widetilde{a},\widetilde{b}) \leq \widetilde{\mu}(xy)$. Now
<u>Case 1</u>: Let $xy \in \widetilde{\mu}_0$. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, we find that $\widetilde{\mu}(0_S) = \widetilde{1}$, by Theorem 2$(i)$. Then $xy \in \widetilde{\mu}_0$ implies that $\widetilde{\mu}(xy) = \widetilde{\mu}(0_S) = \widetilde{1}$. Now since the prime ideal $\widetilde{\mu}_0$ of S is also a completely prime ideal of S and $xy \in \widetilde{\mu}_0$, we find that $yx \in \widetilde{\mu}_0$, by Proposition 2. This shows that $\widetilde{\mu}(yx) = \widetilde{\mu}(0_S) = \widetilde{1}$, by Theorem 2$(i)$. Therefore, $Min^i(\widetilde{a},\widetilde{b}) \leq \widetilde{\mu}(xy) = \widetilde{\mu}(yx)$. This implies that $(yx)_{Min^i(\widetilde{a},\widetilde{b})} \in \widetilde{\mu}$ i.e. $(yx)_{Min^i(\widetilde{b},\widetilde{a})} \in \widetilde{\mu}$. Then by Remark 3, it follows

that $y_{\widetilde{b}} \circ x_{\widetilde{a}} \in \widetilde{\mu}$.
Case 2: Let $xy \notin \widetilde{\mu}_0$. This implies that $\widetilde{\mu}(xy) \neq \widetilde{\mu}(0_S)$ i.e. $\widetilde{\mu}(xy) \neq \widetilde{1}$, by Theorem 2($i$). Then $\widetilde{\mu}(xy) = [\alpha, \beta]$, since $Im\widetilde{\mu} = \{\widetilde{1}, [\alpha, \beta]\}$. So we have
$\widetilde{\mu}(yx) \geq [\alpha, \beta] = \widetilde{\mu}(xy) \geq Min^i(\widetilde{a}, \widetilde{b}) \Longrightarrow (yx)_{Min^i(\widetilde{a},\widetilde{b})} \in \widetilde{\mu} \Longrightarrow (yx)_{Min^i(\widetilde{b},\widetilde{a})} \in \widetilde{\mu}$. Then by Remark 3, we get $y_{\widetilde{b}} \circ x_{\widetilde{a}} \in \widetilde{\mu}$.
Conversely, let $\widetilde{\mu}$ be an $i.v.$ fuzzy prime ideal of S such that for any two $i.v.$ fuzzy points $x_{\widetilde{a}}$ and $y_{\widetilde{b}}$ of S, $x_{\widetilde{a}} \circ y_{\widetilde{b}} \in \widetilde{\mu} \Longrightarrow y_{\widetilde{b}} \circ x_{\widetilde{a}} \in \widetilde{\mu}$. Since $\widetilde{\mu}$ is an $i.v.$ fuzzy prime ideal of S, $Im\widetilde{\mu} = \{\widetilde{1}, [\alpha, \beta]\}$, where $[\alpha, \beta] \in D[0,1] \setminus \{\widetilde{1}\}$ and $\widetilde{\mu}_0$ is a prime ideal of S, by Theorem 3. Let $xy \in \widetilde{\mu}_0$. Then $\widetilde{\mu}(xy) = \widetilde{\mu}(0_S) = \widetilde{1}$, by Theorem 2($i$). So $(xy)_{\widetilde{1}} \in \widetilde{\mu} \Longrightarrow x_{\widetilde{1}} \circ y_{\widetilde{1}} \in \widetilde{\mu}$ (by Remark 3) $\Longrightarrow y_{\widetilde{1}} \circ x_{\widetilde{1}} \in \widetilde{\mu} \Longrightarrow (yx)_{\widetilde{1}} \in \widetilde{\mu}$ (by Remark 3)
$\Longrightarrow \widetilde{1} \leq \widetilde{\mu}(yx) \Longrightarrow \widetilde{\mu}(yx) = \widetilde{1} = \widetilde{\mu}(0_S)$ (by Theorem 2(i)) $\Longrightarrow yx \in \widetilde{\mu}_0$. Hence $widetilde\mu_0$ is a completely prime ideal of S, by Proposition 2. Thus $\widetilde{\mu}_0$ is a completely prime ideal of S and $Im\widetilde{\mu} = \{\widetilde{1}, [\alpha, \beta]\}$, where $[\alpha, \beta] \in D[0,1] \setminus \{\widetilde{1}\}$. Consequently, by Theorem 8 we find that $\widetilde{\mu}$ is an $i.v.$ fuzzy completely prime ideal of S.

ACKNOWLEDGEMENTS The third author is thankful to CSIR, India, for providing financial assistance.

References

[1] J Ahsan, K Saifullah, and F Khan. Fuzzy semirings. *Fuzzy Sets and systems*, 60:309–320, 1993.

[2] B Davvaz. Fuzzy ideals of near rings with interval-valued membership functions. *J. Sci. I. R. Iran*, 12(2):171–175, 2001.

[3] T K Dutta and B K Biswas. Fuzzy prime ideals of a semiring. *Bull. Malaysian Math Soc. (second series)*, 17:9–16, 1994.

[4] J S Golan. *Semirings and their applications.* Kluwer Academic Publishers, Dordrecht, Boston, 1999.

[5] I Grattan Guiness. Fuzzy membership mapped onto interval and many-valued quantities. *Z. Math. Logik. Grundladen Math.*, 22:149–160, 1975.

[6] H Hedayati. Generalized fuzzy k-ideals of semirings with interval-valued membership functions. *Bull. Malays. Math. Sci. Soc.*, 32(3):409–424, 2009.

[7] K U Jahn. Intervall-wertige mengen. *Math. Nach.*, 68:115–132, 1975.

[8] Y Jun and K Kim. Interval-valued fuzzy r-subgroups of near-rings. *Indian J. pure appl. Math.*, 33(1):71–80, 2002.

[9] N Kuroki. Fuzzy bi-ideals in semigroups. *Commentarii Mathematici Universitatis Sancti Pauli.*, 28(1):17–21, 1979.

[10] D Lee and C Park. Interval-valued (ϵ, $\epsilon \vee$ q)-fuzzy ideal in rings. *International Mathematical Forum.*, 4:623–630, 2009.

[11] A Rosenfeld. Fuzzy groups. *J. Math. Anal. Appl.*, 35:512–519, 1971.

[12] R Sambuc. *Fonctions ϕ-floues.* PhD thesis, Univ. Marseille, France, 1975.

[13] G Sun and Y Yin. Interval-valued fuzzy h-ideals of hemirings. *International Mathematical Forum*, 5:545–556, 2010.

[14] L Zadeh. Fuzzy sets. *Infor. and Contr.*, 8:338–353, 1965.

[15] L Zadeh. The concept of a linguistic variable and its application to approximate reasoning. *Inform. Sci.*, 8:199–249, 1975.