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Convergence of Prolate Spheroidal Wavelets in a Generalized Sobolev Space and Frames

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Abstract. It has been noticed that prolate spheroidal wave functions and associated wavelets have many remarkable properties leading to new applications in electrical engineering and mathematics. In this paper we have studied the modified wavelets at different scales to retain a constant energy concentration interval and shown that the sequence of these wavelet bases form a frame (more general than Riesz basis) for V_m (Paley -Wiener spaces in which the sinc function replaced by prolate spheroidal wave functions from wavelet basis). Also, the convergence of associated approximations have been studied in generalized sobolev space which contains the Schwartz space as a particular case and our space is generalized the spaces studied by Pathak [15] and Hormander [9].

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1. Introduction

It is known as that the use of Fourier analysis is of paramount importance in electrical engineering. Communication engineers are much concerned with the class of functions they called signals. These are real functions $r(t)$, defined everywhere on a real line that they called time and these functions are square integrable on real line,

$$E = \int_{-\infty}^{\infty} r^2(t)dt < \infty \tag{1}$$

where E is called the energy of the signal $r(t)$. In fact, $r(t)$ will represent the voltage difference at time t between points in an electrical circuit or device or in other words the voltage difference between the terminals of a microphone. The signal space S, is the set of all signals, $r(t)$. It is nothing but the space $L^2(-\infty, \infty)$.

The Fourier transform of the signal is known as the amplitude spectrum of the signal and the inverse Fourier transform allow us to think of the signal $r(t)$ as a sum of sinusoids of

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different frequencies. The engineer often work more with the Fourier transform of a signal rather than the original signal.

The most natural classes of input signals shows that they too have amplitude spectra of finite support. For example, Fourier analysis of recorded male speech gives an amplitude spectrum that is zero for frequencies higher than 8,000 hertz (cycles/second). Conventional orchestral music has no frequencies higher than 20,000 hertz, while the output of a television camera (vidiocon) has an amplitude spectrum vanishing for $|f| > 2 \times 10^6$ hertz.

Thus, due to the frequency limiting nature of the devices and the nature of the signals, the communication engineer's led to think the space B_w of band limited signals i.e., the square integrable signals whose amplitude spectra vanish for $|f| > w$. Each member of B_w can be written as a finite Fourier transform

$$r(t) = \int_{-w}^{w} e^{2\pi i f t} \widehat{f} df. \tag{2}$$

Here $w > 0$ and $\widehat{f}$ denotes the Fourier transform of f. The $r(t)$ is said to be time limited if for some $T > 0, r(t)$ vanishes for all $|t| > T/2$. It can be seen from (2) that band limited signals are extremely smooth and $r(t)$ is an entire function of the complex variable t. It has no singularities in the finite $t-$plane is infinitely differentiable everywhere and has a Taylor series about every point with infinite radius of convergence. It follows that a nontrivial bandlimited signal can not vanish on any interval of the $t-$axis. If it possible then it and all its derivatives would be zero at some interior point of the interval, and a Taylor expansion would require it to be the trivial everywhere zero signal. The only signal that is both band limited and time limited is the trivial always-zero signal. If the signals are such that they are some how concentrated in both the time and frequency domains then purpose will be solved.

The continuous prolate spheroidal wave functions ($PSWFs$) are those that are most highly localized simultaneously in both the time and frequency domain. This fact was discovered by Slepian and his collaborators and was presented in a series of papers [12, 13, 6, 17, 18]. Since then the study of $PSWFs$ has been an active area of research in both electrical engineering and mathematics. The $PSWFs$ had previously known as solutions of Sturm - Liouville problem, from which many of their properties could be derived. The associated prolate spheroidal wavelets (PS wavelets) have been introduced by G.G. Walter and Xiaoping Shen [11]. Pollak and Landau [12] discovered the connection between $PSWFs$ and the energy concentration problem during the 1960's, the $PSWFs$ were shown to be an important tool for analyzing some problems raised in signal processing and telecommunications [14].

Now we shall discuss some properties of $PSWFs$ which are well knwon and found in a number of places (Landau [10], Landau and Widom [11], Landau and Pollak [12, 13], Papoulis [14], Slepian and Pollak [16]).

The prolate spheroidal wave functions ($PSWFs$) $\left\{\varphi_{n,\sigma,\tau}(t)\right\}$, constitute an orthonormal basis of the space of $\sigma-$band limited functions on the realline, i.e., functions whose Fourier transforms have support on theinterval $[-\sigma, \sigma]$. The $PSWFs$ are maximally concentrated on an interval $[-\tau, \tau]$ and depend on parameters σ and τ. $PSWFs$ are characterized as the eigenfunctions of an integral operator with kernel arising from the sinc function

$S(t) = \sin(\pi t)/\pi t$:

$$\frac{\sigma}{\pi}\int_{-\tau}^{\tau}\varphi_{n,\sigma,\tau}(x)S\left(\frac{\sigma}{\pi}(t-x)\right)dx = \lambda_{n,\sigma,\tau}\varphi_{n,\sigma,\tau}(t), |t| \leq \tau. \quad (3)$$

It is easy to show that the symmetric kernel $S\left(\frac{\sigma}{\pi}(t-x)\right)$ is positive definite, so that from [2] we know that (3) has solutions in $L^2(-\tau,\tau)$ only for a discrete set of real positive values of $\lambda_{n,\sigma,\tau}$,say $\lambda_{0,\sigma,\tau} \geq \lambda_{1,\sigma,\tau} \leq \cdots$ and that as $n \to \infty, \lim \lambda_{n\sigma,0} = 0$. The variational problem that let to (3) only requires that equation to hold for $|t| \leq \tau$. With $\varphi_{n,\sigma,\tau}(x)$ on the left of (3) gives for $|x| \leq \tau$, however, the left is well defined for all t. We use this to extend the range of definition of the $\varphi_{n,\sigma,\tau}s$ and so define

$$\varphi_{n,\sigma,\tau}^{(t)} = \frac{\sigma}{\pi\lambda_{n,\sigma,\tau}}\int_{-\tau}^{0}\varphi_{n,\sigma,\tau}(x)S\left(\frac{\sigma}{\pi}(t-x)\right)dx, |t| > \tau.$$

The eigenfunctions $\varphi_{n,\sigma,\tau}$ are now defined for all t. In addition to the equation (3), the $\left\{\varphi_{n,\sigma,\tau}\right\}$ satisfy an integral equation over $(-\infty,\infty)$

$$\frac{\sigma}{\pi}\int_{-\infty}^{\infty}\varphi_{n,\sigma,\tau}(x)S\left(\frac{\sigma}{\pi}(t-x)\right)dx = \left(\varphi_{n,\sigma,\tau} * S_{\sigma}\right)(t) = \varphi_{n,\sigma,\tau}^{(t)} \quad (4)$$

with the same kernel. This leads to a dual orthogonality

$$\begin{aligned}\int_{-\tau}^{\tau}\varphi_{n,\sigma,\tau}(x)\varphi_{m,\sigma,\tau}(x)dx &= \lambda_{n,\sigma,\tau}\delta_{nm},\\ \int_{-\infty}^{\infty}\varphi_{n,\sigma,\tau}(x)\varphi_{m,\sigma,\tau}(x)dx &= \delta_{nm},\end{aligned}$$

and the fact that they constitute an orthogonal basis of $L^2(-\tau,\tau)$, as well as an orthogonal basis of the subspace B_σ of $L^2(-\infty,\infty)$, the Paley-Wiener space of all $\sigma-$bandlimited functions.

$PSWFs$ are also characterizing as:

(i) the eigenfunctions of a differential operator arising from a Helmholtz equation on a prolate spheroid:

$$\left(\tau^2 - t^2\right)\frac{d^2\varphi_{n,\sigma,\tau}}{dt^2} - 2t\frac{d\varphi_{n,\sigma,\tau}}{dt} - \sigma^2 t^2\varphi_{n,\sigma,\tau} = \mu_{n,\sigma,\tau}\varphi_{n,\sigma,\tau}.$$

(ii) the maximum energy concentration of a $\sigma-$bandlimited function on the interval $[-\tau,\tau]$; that is $\varphi_{0,\sigma,\tau}$ is the function of total energy $1\left(=\|\varphi_{0,\sigma,\tau}\|^2\right)$ such that $\int_{-\tau}^{\tau}|f(t)|^2dt$ is maximized, $\varphi_{1,\sigma,\tau}$ is the function with the maximum energy concentration among those functions orthogonal to $\varphi_{0,\sigma,\tau}$ etc.

The Fourier transform of $\varphi_{n,\sigma,\tau}$ is given by

$$\widehat{\varphi}_{n,\sigma,\tau}(w) = (-1)^n \sqrt{\frac{2\pi\tau}{\sigma\lambda_{n,\sigma,\tau}}} \varphi_{n,\sigma,\tau}\left(\frac{\tau w}{\sigma}\right) \chi_\sigma(w), \tag{5}$$

where $\chi_\sigma(w)$ is the characteristic function of $[-\sigma, \sigma]$. The inverse Fourier transform is given by

$$\varphi_{n,\sigma,\tau} = (-1)^n \sqrt{\frac{2\pi\tau}{\sigma\lambda_{n,\sigma,\tau}}} \frac{1}{2\sigma} \int_{-\sigma}^{\sigma} \varphi_{n,\sigma,\tau}\left(\frac{\tau w}{\sigma}\right) e^{iwt} dw. \tag{6}$$

By change of variables, (6) converted into

$$\int_{-\tau}^{\tau} \varphi_{n,\sigma,\tau}(x) e^{i\sigma w x/\tau} dx = \gamma_{n,\sigma,\tau} \varphi_{n,\sigma,\tau}(w). \tag{7}$$

As we have seen that (3) holds for all $t \in R$, we find that both $\varphi_{n,\sigma\tau,1}(x)$ and $\sqrt{\tau}\varphi_{n,\sigma,\tau}(\tau x)$ are solutions of the same eigen value problem. Since each of the eigenvalues has multiplicity one, it follows that each is a multiple of other, and after normalization, we get

$$\varphi_{n,\sigma,\tau,1}(x) = \sqrt{\tau}\varphi_{n,\sigma,\tau}(\tau x),$$

and

$$\varphi_{0,\sigma,\tau}(2x) = \frac{1}{\sqrt{2}} \varphi_{0,2\sigma,\tau/2}(x).$$

We are interested mainly in $\varphi_{0,\sigma,\tau}$ whose concentration on the interval $[-\tau, \tau]$ is maximum. Since $\varphi_{k,\sigma,\tau}$ has exactly k zeros in the interval $[-\tau, \tau]$, so $\varphi_{0,\sigma,\tau}$ $(PSWFs)$ are entire functions and therefore can not vanish on any interval, they can be made uniformly small outside of $[-\tau, \tau]$ for τ or σ sufficiently large, so that computationally they behave like functions with compact support.

To construct *PS* wavelets, the scaling function $\phi = \varphi_{0,\pi,\tau}$, where τ is any positive number, was introduced by [19], and obtained a basis composed of its translates. The integer translates of ϕ form a Riesz basis of a space $V_0 \subset (L^2(R))$ which turns out to be the Paley-Wiener space B_π of $\pi-$bandlimited functions. A multi-resolution analysis (*MRA*) are then based on this construction. The other spaces are obtained by dilations by factors of two and consist of the Paley-Wiener spaces $V_m = B_{2^m\pi}$. The sinc function $S(t) = \sin \pi t / \pi t$ is the standard scaling function of this *MRA*. It is well known that sinc function has very good frequency localization, but not very good time localization. It follows that this wavelet basis has limited use in compact support in the time domain. However, $PSWFs$ are superior to sinc function and they are similar to Doubechies wavelets for practical computations.

Using the standard wavelet approach in which dilations of $\varphi_{0,\pi,\tau}(2^m t)$ are used to get the basis $\left\{\varphi_{0,\pi,\tau}(2^m t - n)\right\}$ of V_m we get

$$\phi(2^m t) = \varphi_{0,\pi,\tau}(2^m t) = 2^{m/2} \varphi_{0,2^m\pi,2^{-m}\pi\tau^{(t)}},$$

which show that the concentration interval becomes smaller as m increases. So we have to fix the concentration interval by taking $\left\{\varphi_{0,2^m\pi,\tau}(t-2^{-m}n)\right\}$ instead as a possible Riesz basis of V_m. Thus our new basis for V_0 and V_m are different from the standard wavelet basis for V_0 and V_m consisting of translates of the sinc function.

2. Frames

The use of Riesz basis may have some serious drawbacks. One important problem is the lack of flexibility which is in some sense a consequence of the uniqueness of the representation. Therefore, why not using a slightly weaker concept and allowing some redundancy, i.e., why not working with frames ? In many cases the wavelet experts prefer to work with frames instead of Riesz basis. The frame concept has been introduced by Duffin and Schafer [3]. However, the starting point of the modern frame theory was the fundamental Feichtinger/Gröchenig theory which has been developed since 1980 in a series of articles [4, 5, 6, 7, 8].

Definition 1. *The family $\{\phi_i\}_{i\in N}$ is a frame if there exist constants $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_i |<f,\phi_i>|^2 \leq B\|f\|^2, textforallf \in L^2(R).$$

The constants A and B are called frame bounds.

We are interested in the family of functions $\{2^{m/2}\phi_m(t-n2^{-m}) = 2^{m/2}\varphi_0, 2^m\pi, \tau\ (t-n2^{-n}):$ $m,n\in Z\}$ to form a frame in $L^2(R)$.

Theorem 1. *Let $V_m = B_{2^m\pi}$, then $\left\{\phi_m(t-n2^{-m})\right\}_{n\in Z}$ is a frame of V_m.*

Proof. Suppose $f \in L^2(R)$. Then

$$\sum_{m,n=-\infty}^{\infty}\left|\left\langle f, 2^{m/2}\phi_m\left(t-n2^{-m}\right)\right\rangle\right|^2 = \sum_{m,n=-\infty}^{\infty} 2^m \left|\int_{-\infty}^{\infty}\widehat{f}(w)\overline{\widehat{\phi}_m(w)}e^{inw2^{-m}}dw\right|^2.$$

Since for any $s > 0$ the integral

$$\int_{-\infty}^{\infty} g(t)dt = \sum_{l=-\infty}^{\infty}\int_0^s g(t+ls)dt,$$

by taking $s = \frac{2\pi}{2^{-m}}$, we obtain

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty} 2^m &\left|\sum_{l=-\infty}^{\infty}\int_0^s e^{2\pi inwls}\widehat{f}(w+ls)\overline{\widehat{\phi}_m(w+ls)}dw\right|^2 \\ &= \sum_{m,n=-\infty}^{\infty} 2^m \left|\int_0^s e^{2\pi inwls}\left(\sum_{l=-\infty}^{\infty}\widehat{f}(w+ls)\right)\overline{\widehat{\phi}_m(w+ls)}dw\right|^2\end{aligned}$$

$$= \sum_{m,n=-\infty}^{\infty} 2^m s \int_0^s \left| \sum_{l=-\infty}^{\infty} \widehat{f}(w+ls)\overline{\widehat{\phi}_m(w+ls)} \right|^2 dw.$$

Using the Parsewal's formula for trigonometric Fourier series, we have

$$\sum_{m,n=-\infty}^{\infty} \frac{2\pi}{2^{-2m}} \int_0^s \left(\sum_{l=-\infty}^{\infty} \widehat{f}(w+ls)\overline{\widehat{\phi}_m(w+ls)} \right) \left(\sum_{j=-\infty}^{\infty} \overline{\widehat{f}(w+js)}\widehat{\phi}_m(w+js) \right)$$

since $F(w) = \sum_{j=-\infty}^{\infty} \overline{\widehat{f}(w+js)}\widehat{\phi}_m(w+js)$ is a periodic function with a period of s, we have

$$\sum_{m,n=-\infty}^{\infty} \frac{2\pi}{2^{-2m}} \left(\sum_{l=-\infty}^{\infty} \widehat{f}(w+ls)\overline{\widehat{\phi}_m(w+ls)} \right) F(w)dw$$

$$= \sum_{m,j=-\infty}^{\infty} \frac{2\pi}{2^{-2m}} \int_{-\infty}^{\infty} \widehat{f}(w)\overline{\widehat{\phi}_m(w)\widehat{f}(w+js)}\widehat{\phi}_m(w+js)dw$$

$$= \int_{-\infty}^{\infty} \left|\widehat{f}(w)\right|^2 \sum_{j=-\infty}^{\infty} \frac{2\pi}{2^{-2m}} \left|\widehat{\phi}_m(w)\right|^2 dw + \sum_{\substack{m,j=-\infty \\ j\neq 0}}^{\infty} \frac{2\pi}{2^{-2m}} \int_{-\infty}^{\infty} \widehat{f}(w).$$

$$\cdot \overline{\widehat{f}(w+js)\widehat{\phi}_m(w)}.\widehat{\phi}_m(w+js)dw = I + II.$$

To find a bound on the second summation, we use the Schwartz inequality

$$|II| \le \sum_{\substack{m,j=-\infty \\ j\neq 0}}^{\infty} \frac{2\pi}{2^{-2m}} \left(\int_{-\infty}^{\infty} \left|\widehat{f}(w)\right|^2 \left|\widehat{\phi}_m(w)\right| \left|\widehat{\phi}_m(w+js)\right| dw \right)^{1/2}$$

$$\left(\int_{-\infty}^{\infty} \left|\widehat{f}(w+js)\right|^2 \left|\widehat{\phi}_m(w)\right| \left|\widehat{\phi}_m(w+js)\right| dw \right)^{1/2}$$

$$= \sum_{\substack{m,j=-\infty \\ j\neq 0}}^{\infty} \frac{2\pi}{2^{-2m}} \left(\int_{-\infty}^{\infty} \left|\widehat{f}(w)\right|^2 \left|\widehat{\phi}_m(w)\right| \left|\widehat{\phi}_m(w+js)\right| dw \right)^{1/2}$$

$$\left(\int_{-\infty}^{\infty} \left|\widehat{f}(w)\right|^2 \left|\widehat{\phi}_m(w-js)\right| \left|\widehat{\phi}_m(w)\right| dw \right)^{1/2}.$$

Using Holder's inequality we get

$$\le \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \left(\int_{-\infty}^{\infty} \left|\widehat{f}(w)\right|^2 \sum_{m=-\infty}^{\infty} \frac{2\pi}{2^{-2m}} \left|\widehat{\phi}_m(w)\right| \left|\widehat{\phi}_m(w+js)\right| dw \right)^{1/2}$$

$$\left(\int_{-\infty}^{\infty}\left|\widehat{f}(w)\right|^{2}\sum_{m=-\infty}^{\infty}\frac{2\pi}{2^{-2m}}\left|\widehat{\phi}_m(w-js)\right|\left|\widehat{\phi}_m(w)\right|dw\right)^{1/2}.$$

Now let us define

$$\beta(\xi)=\sup_{|w|\in 2^m\pi}\sum_{m=-\infty}^{\infty}\frac{2\pi}{2^{-2m}}\left|\widehat{\phi}_m(w)\right|\left|\widehat{\phi}_m(w+\xi)\right|$$

then the above expression can be written as

$$\|f\|^2\sum_{\substack{j=-\infty\\ j\neq 0}}^{\infty}\left[\beta(js)\beta(-js)\right]^{1/2}=\|f\|^2\sum_{\substack{j=-\infty\\ j\neq 0}}^{\infty}\left[\beta\left(\frac{2\pi j}{2^{-m}}\right)\beta\left(\frac{-2\pi j}{2^{-m}}\right)\right].$$

If we denote

$$A=\sup_{|w|\leq 2^m\pi}\sum_{m=-\infty}^{\infty}\left|\widehat{\phi}_m(w)\right|^2-\sum_{\substack{j=-\infty\\ j\neq 0}}^{\infty}\left[\beta\left(\frac{2\pi j}{2^{-m}}\right)\beta\left(\frac{-2\pi j}{2^{-m}}\right)^{1/2}\right]$$

and

$$B=\sup_{|w|\leq 2^m\pi}\sum_{m=-\infty}^{\infty}\left|\widehat{\phi}_m(w)\right|^2+\sum_{\substack{j=-\infty\\ j\neq 0}}^{\infty}\left[\beta\left(\frac{2\pi j}{2^{-m}}\right)\beta\left(\frac{-2\pi j}{2^{-m}}\right)^{1/2}\right]$$

we get

$$A\|f\|^2\leq\sum_{m,n=-\infty}^{\infty}\left|\left\langle f,\phi_m\left(t-2^{-m}n\right)\right\rangle\right|^2\leq B\|f\|^2. \tag{8}$$

If we take $\beta(\xi)\leq C(1+|\xi|)^{-(1+\varepsilon)}$, we get

$$\begin{aligned}
\sum_{\substack{j=-\infty\\ j\neq 0}}^{\infty}\left[\beta\left(\frac{2\pi j}{2^{-m}}\right)\beta\left(\frac{-2\pi j}{2^{-m}}\right)\right]^{1/2} &= 2\sum_{j=1}^{\infty}\left[\beta\left(\frac{2\pi j}{2^{-m}}\right)\beta\left(\frac{-2\pi j}{2^{-m}}\right)\right]^{1/2}\\
&\leq 2C\sum_{j=1}^{\infty}\left(1+\frac{2\pi j}{2^{-m}}\right)^{-(1+\varepsilon)}\\
&\leq 2C\int_0^{\infty}\left(1+\frac{2\pi j}{2^{-m}}\right)^{-(1+\varepsilon)}dt\\
&= \frac{C2^{-m}}{\pi\varepsilon},\ \text{since}\frac{C2^{-m}}{\pi\varepsilon}\to 0 \text{ for large } m.
\end{aligned}$$

In order to complete the proof we need only to show that

(i) $\inf_{|w|\leq 2^m\pi}\sum_{m=-\infty}^{\infty}\left|\widehat{\phi}_m(w)\right|^2>0,$
and

(ii) $sup_{|w|\leq 2^m\pi}\sum_{m=-\infty}^{\infty}\left|\widehat{\phi}_m(w)\right|^2<\infty.$

Let us consider (i), we have

$$\begin{aligned}\sum_{m=-\infty}^{\infty}\left|\widehat{\phi}_m(w)\right|^2 &= \frac{2\tau}{2^m\lambda_{0,2^m\pi,\tau}}\sum_{m=-\infty}^{\infty}\left|\varphi_{0,2^m\pi,\tau}\left(\frac{w\tau}{2^m\pi}\right)\right|^2\chi_{2^m\pi}(w)\\ &\geq \inf_{|w|\leq 2^m\pi}\frac{2\tau}{2^m\lambda_{0,2^m\pi,\tau}}\left|\varphi_{0,2^m\pi,\tau}\left(\frac{w\tau}{2^m\pi}\right)\right|^2>0\end{aligned}$$

by (5). Similarly to prove (ii) we see that

$$\sum_{m=-\infty}^{\infty}\left|\widehat{\phi}_m(w)\right|^2\leq \sup_{|w|\leq 2^m\pi}\frac{2\tau}{2^m\lambda_{0,2^m\pi,\tau}}\left|\varphi_{0,2^m\pi,\tau}\left(\frac{w\tau}{2^m\pi}\right)\right|^2<\infty.$$

Hence there exist α such that $A>0$ for any $2^{-m}\in(0,\alpha)$. Also we have $B<\infty$ for all $2^{-m}\in(0,\alpha)$. Thus $\phi_m(t-n2^{-m})$ constitute a frame for all such 2^{-m}.This completes the proof of the theorem.

A prolate spheroidal system is a family of functions $\{\phi_m(t-n2^{-m}):m,n\in Z\}\subset(L^2)$. For any frame there exists a dual frame which play a similar role in frame theory as the bi-orthogonal system of a Riesz basis. It is possible to find a dual Riesz basis for $\phi_m(t-n^{-m})$. In view of [19, Proposition 1] and define the Fourier transform of the dual function $\widetilde{\varphi}_{0,2^m\pi,\tau}(t)$ as

$$\widehat{\widetilde{\varphi}}_{0,2^m\pi,\tau}(w)=\frac{\widehat{\varphi}_{0,2^m\pi,\tau}(w)}{2^{m+1}\pi\sum_n\left|\widehat{\varphi}_{0,2^m\pi,\tau}(-2^{m+1}\pi n)\right|^2}=\frac{\chi_{2^m\pi}(w)}{2^{m+1}\pi\widehat{\varphi}_{0,2^m\pi,\tau}(w)}$$

or

$$\sum_n\widehat{\widetilde{\varphi}}_{0,2^m\pi,\tau}(w-2^{m+1}\pi n)\overline{\left(\widehat{\varphi}_{0,2^m\pi,\tau}\left(w-2^{m+1}\pi n\right)\right)}=1,$$

we see that $\left\{\widetilde{\varphi}_{0,2^m\pi,\tau}(t-2^{-m}\pi)\right\}$ is bi-orthogonal to $\left\{\varphi_{0,2^m\pi,\tau}(t-2^{-m}n)\right\}$ because $\widehat{\widetilde{\varphi}}_{0,2^m\pi,\tau}(w)$ is positive on $[-2^m\pi,2^m\pi]$, it follows that $\left\{\widetilde{\varphi}_{0,2^m\pi,\tau}(t-n2^{-m}\pi)\right\}$ is a Riesz basis of $B_{2^m\pi}$. The dual frame is given by $\left\{S^{*-1}\varphi_{0,2^m\pi,\tau}(t-n2^{-m})\right\}$, where S^* denotes the frame operator

$$S^*f=\sum_{m,n\in Z}\left\langle f,\phi_m(t-n2^{-m})\right\rangle\phi_m(t-n2^{-m}),f\in L^2.$$

We note that S^* is bounded and invertible if and only if the frame condition (8) holds. The special structure of prolate spheroidal frames yields that the dual frame is again a prolate spheroidal frame i.e., its elements are of the form $S^{*-1}\phi_m(t-n2^{-m})=\widetilde{\phi}_m(t-n2^{-m})$. The function $\widetilde{\phi}_{0,\pi,\tau}\in L^2$ is called the (canonical) dual prolate spheroidal window and it is the focus of many results and questions in prolate spheroidal wavelets analysis.

3. Approximation in Generalized Sobolev Space

To refresh our memory, we recall definitions and properties of certain function and distribution spaces given in [1]. Let M be the set of continuous and real valued functions ν on R satisfying the following conditions :

(1) $0 = \nu(0) \le \nu(\xi+\eta) \le \nu(\xi)+\nu(\eta) : \xi,\eta \in R$

(2) $\int_0^\infty \frac{\nu(\xi)d\xi}{(1+|\xi|)^{n+1}} < \infty,$

(3) $\nu(\xi) \ge a + b\log(1+|\xi|), \xi \in R$

for some real number a and positive real number b. We denote by M_c the set of all $\nu \in M$ satisfying condition $\nu(\xi) = \Omega(|\xi|)$ with a concave function Ω on $[0,\infty)$.

Let $\nu \in M_c$ and S_ν be the set of all function $\phi \in L^1(R)$ with the property that ϕ and $\widehat{\phi} \in C^\infty$ and for each index γ and each non-negative α^* we have

$$\begin{aligned} p_{\gamma,\alpha^*}(\phi) &= \sup_{x\in R} e^{\alpha^*\nu(x)}|D^\alpha \phi(x)| < \infty, \\ q_{\gamma,\alpha^*}(\phi) &= \sup_{\xi\in R} e^{\alpha^*\nu(\xi)}|D^\alpha \phi(\xi)| < \infty. \end{aligned}$$

The topology of S_ν is defined by the semi-norms p_{γ,α^*} and q_{γ,α^*}. The dual of S_ν is denoted by $S_\nu\prime$, the elements of which are called ultra-distributions. It is interesting to mention here that for $\nu(\xi) = \log(1+|\xi|), S_\nu$ is reduced to the Schwartz space.

We denote the space D_ν the set of all ϕ in $L^1(R)$ such that ϕ has compact support and $\|\phi\|_{\beta^*} < \infty$ for all $\beta^* > 0$ and

$$\|\phi\|_{\beta^*} = \int_R |\widehat{\phi}(\xi)| e^{\beta^*\nu(\xi)} d\xi.$$

Also K_ν is defined to be the set of positive functions k in R with

$$k(\xi+\eta) =\le e^{\beta^*\nu(-\xi)}k(\eta) \text{ for all } \xi,\eta \in R.$$

Let $\nu \in M_c, k \in K_\nu$ and $1 \le p < \infty$. Then generalized Sobolev space $H^\nu_{p,k}(R)$ is defined to be the space of all ultra-distributions $f \in S'_\nu$ such that

$$\|f\|_{p,k} = \left(\int_R \left| k(\xi)\widehat{f}(\xi) \right|^p d\xi \right)^{1/p} < \infty$$

and

$$\|f\|_{\infty,k} = ess\sup k(\xi) \left| \widehat{f}(\xi) \right|.$$

Remark 1. *The space $H^{\nu}_{p,k}(R)$ is a generalization of the Hormander space [9] and reduces to the space $H_{p,k}(R)$ for $\nu = \log(1+|\xi|)$. For $k(\xi) = e^{s\nu(\xi)}$ and $1 \le p < \infty, H^{\nu}_{p,k} = H^{s,p}_{\nu}$, the generalized Sobolev spce studied by Pathak [15].*

The approximation of a function in $L^2(R)$ by function in V_m is given by a series of the form

$$\sum_n b_n^m \phi_m\left(t - 2^{-m}n\right) \tag{9}$$

where the coefficients may be obtained either from the dual Riesz basis or by sampling. In former case the result is the projection f_m of a function f onto V_m, while in the latter case the result is locally a positive approximation. The coefficients for the projection are

$$b_n^m = \int_{-\infty}^{\infty} f(t)\widetilde{\phi}_m\left(t - 2^{-m}n\right)dt, \tag{10}$$

and the kernel of this projection is given

$$q_m(x,t) = \sum_n \phi_m\left(x - 2^{-m}n\right)\widetilde{\phi}_m\left(t - 2^{-m}n\right)$$

or

$$f_m(x) = \int_{-\infty}^{\infty} f(t)q_m(x,t)dt.$$

Now we prove

Theorem 2. *Let $f \in H^{\nu}_{p,k}(R)$, the generalized Sobolev space, for $1 \le p < \infty$, let the approximation f_m to f be given by a series of the form (9) where the coefficients are given by (10), then $f_m \to f$ in $H^{\nu}_{p,k}$ as $m \to \infty$.*

Proof. We have

$$\begin{aligned}
\|f_m - f\|^p_{p,k} &= \int_R \left|\left(\widehat{f}_m(\xi) - \widehat{f}(\xi)\right)k(\xi)\right|^p d\xi \\
&= \int_R \left|\frac{1}{2\pi}\int_R \widehat{q}_m(\xi,w)\widehat{f}(w)dw - \widehat{f}(\xi)\right|^p |k(\xi)|^p d\xi \\
&= \int_R \left|\frac{1}{2\pi}\int_R 2\pi\chi_{2^m\pi}(w)\delta(w-\xi)\widehat{f}(w)dw - \widehat{f}(\xi)\right|^p |k(\xi)|^p d\xi \\
&= \int \left|\left(\chi_{2^m\pi}(\xi) - 1\right)\right|^p \left|\widehat{f}(\xi)\right|^p |k(\xi)|^p d\xi \\
&= \int_R \left|k(\xi)\widehat{f}(\xi)\right|^p \left|\chi_{2^m\pi}(\xi) - 1\right|^p d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \left\{\int_{2^m\pi}^{\infty} + \int_{-\infty}^{-2^m\pi}\right\} \left|k(\xi)\widehat{f}(\xi)\right|^p \frac{(\xi^2+1)^p}{(xi^2+1)^p} d\xi \\
&\leq 2\int_{-\infty}^{\infty} \left|k(\xi)\widehat{f}(\xi)\right|^p \frac{(\xi^2+1)^p}{\left((2^m\pi)^2+1\right)^2} d\xi.
\end{aligned}
$$

Now, applying dominated convergence theorem, we get $f_m \to f$ in $H^{\nu}_{p,k}$. Similarly we can prove the theorem for $p = \infty$.

If the coefficients are given by the sampled values of the function, then the approximation in V_m is given by the hybrid series

$$
f_m^s(t) = \sum_n f(2^{-m}n) \frac{\phi_m(t - 2^{-m}n)}{2^m \widehat{\phi}_m(0)}. \tag{11}
$$

The kernel is given by

$$
k_m(x,t) = \sum_n \frac{\phi_m(x - 2^{-m}n)}{2^m \widehat{\phi}_m(0)} \delta(t - 2^{-m}n)
$$

i.e.,

$$
f_m^s(x) = \int f(t) k_m(x,t) dt.
$$

Now we shall show that the rate of convergence for the hybrid series is slower than for the projection series in generalized sobolev space $H^{\nu}_{p,k}$, but the former has other nice properties. First is in image approximations there is no Gibbs phenomenon due to local positiveness of kernel [19, Lemma 1], and second is the integration is avoided in the calculations of coefficients. The difference between our two approximations in $H^{\nu}_{p,k}$ is

$$
\begin{aligned}
\left\|f_m - f_m^s\right\|_{p,k}^p &= \int_R \left|\left(\widehat{f}_m(\xi) - f_m^s(\xi)\right) k(\xi)\right|^p \xi \\
&= \int_R \left|\frac{1}{2\pi}\int_R \left[\widehat{q}_m(\xi,w) - \widehat{K}_m(\xi,w)\right] \widehat{f}(w) k(\xi) dw\right|^p d\xi
\end{aligned}
$$

substituting the value of $\left[\widehat{q}_m(\xi,w) - \widehat{k}_m(\xi,w)\right]$ from [19, eq. 4.2], we get

$$
\begin{aligned}
&= \int_R \left|\frac{1}{2\pi}\int_R \left[\frac{(\tau\xi)^2}{6\phi_m(0)} \phi_m(\tau\xi/2^m\pi) 2\pi \chi_{2^m\pi}(\xi) \delta(\xi - w)\right] \widehat{f}(w) dw\right|^p |k(\xi)|^p d\xi \\
&= \int_{-2^m\pi}^{2^m\pi} \left|\frac{(\tau\xi)^2}{6\phi_m(0)} \phi_m(\tau\xi/2^m\pi) \widehat{f}(\xi)\right|^p |k(\xi)|^p d\xi \\
&\leq \int_{-2^m\pi}^{2^m\pi} \left|\frac{(\tau\xi)^2}{6} \widehat{f}(\xi) k(\xi)\right|^p \frac{(\xi^2+1)^p}{(\xi^2+1)^p} d\xi
\end{aligned}
$$

$$\leq \quad 2\int_{-\infty}^{\infty}\left|\frac{\tau^2}{6}\widehat{f}(\xi)k(\xi)\right|^p(\xi^2+1)^p d\xi$$

by definition this integral is finite for $1 \leq p < \infty$ and also for $p = \infty$. Hence we conclude that a better rate of convergence holds for f_m.

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