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Invariant and Reducing Subspaces of Multiplication and Composition Operators

Virender Pal Singh[1*], B.S. Komal [2]

[1] *Department of Mathematics, University of Jammu, Jammu-180006, INDIA*
[2] *Department of Engineering Mathematics, MIET, Kot Bhalwal, Jammu-180006, INDIA*

Abstract. The invariant and reducing subspaces of composition operators, multiplication operators and weighted composition operators are studied in this paper.

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1. Introduction

Let X and Y be two non-empty sets. Let $F(X)$ and $F(Y)$ denote the locally convex topological vector spaces of complex valued functions on X and Y respectively. The mapping $T : Y \to X$ such that $f \in F(X) \Rightarrow f o T \in F(Y)$ gives rise to a linear transformation $C_T : F(X) \to F(Y)$ defined by

$$C_T f = f o T$$

for every $f \in F(X)$. In case C_T is continuous, we call it a *composition operator* induced by T. Similarly a mapping $\theta : X \to C$ such that $f \in F(X) \Rightarrow \theta.f \in F(X)$ can induce a multiplication transformation $M_\theta : F(X) \to F(Y)$ defined by

$$M_\theta f = \theta.f$$

Again if M_θ is continuous, we call it a *multiplication operator* induced by θ. The multiplication operator have its roots in spectral theory and Teoplitz operators. The importance of multiplication operators lies in the fact that every normal operator is unitarily equivalent to a multiplication operator. A continuous linear operator $M_{\theta,T} : F(X) \to F(Y)$ defined by

$$M_{\theta,T} f = \theta.f \circ T$$

*Corresponding author.

Email addresses: virender_ju@yahoo.co.in (V. Singh), bskomal2@yahoo.co.in (B.S. Komal)

is known as a *weighted composition operator*. It evident that $M_{\theta,T} = M_\theta C_T$. That is, a weighted composition operator is a product of a multiplication operator and a composition operator. Thus the multiplication operators and composition operators are special types of weighted composition operators.Weighted translation operators are the examples of weighted composition operators .

The Hilbert space

$$\ell^2(\mathscr{Z}) = \{\{x_n\} : \sum_{n=-\infty}^{\infty} |x_n|^2 < \infty\}$$

is widely studied for the unilateral shift operator, bilateral shift [5], weighted shift [9] and composition operator [2, 10, 11].

Let $T : Z \to Z$ be a mapping. Two integers m and n are said to be in the same orbit of T if there exist two non- negative integers r and s such that $T^r(m) = T^s(n)$. Here and elsewhere, T^r denote the composition of T with itself r times. If $n \in Z$, then

$$O_T(n) = \{m \in Z : T^r(m) = T^s(n) \text{ for some } r, s \in Z_+\},$$

where $Z_+ = N \cup \{0\}$ is called the orbit of n with respect to T. A mapping $T : Z \to Z$ is said to be antiperiodic at n, if $T^m(k) \neq k$ for every $m \in N$ and $k \in O_T(n)$. If T is antiperiodic at every $n \in Z$, then we say that T is purely antiperiodic. If for any integer n, there exists $m \in N$ such that $T^m(n) = n$, then T is called periodic at n. If T is periodic at every $n \in Z$, we say that T is *purely periodic*. If T is a bounded linear operator on a Hilbert space H, then a closed subspace M of H is called an *invariant subspace* of T if $TM \subset M$. By an invariant subspace of an operator we shall mean a non-trivial invariant subspace. If M is an invariant subspace of both T and its adjoint T^*, then we say that M is a reducing subspace of T. An operator T is called reductive if every invariant subspace of T reduces T. The symbol $B(H)$ denotes the Banach algebra of all bounded linear operators on a Hilbert space H.

During the past several decades, composition operators have been the subject matter of intensive study for several mathematicians. To mention, a few of them are [4–6, 8, 12], etc. One of the outstanding unsolved problem of the operator theory is the invariant subspace problem which is stated as follows: Does every bounded linear operator on a separable infinite dimensional Hilbert space have a non-trival invariant subspace? P. Enflo [3] and Read [7] settled the problem for Banach spaces.They obtained Banach spaces and operators without having invariant subspaces. Instead of making counter examples some people started characterizing the invariant subspaces of certain classes of operators. Beurling's theorem [1] is the first significant step in this direction. Our interest in this paper is to characterize the invariant and reducing subspaces of multiplication and composition operators.

2. Invariant and Reducing Subspaces of Multiplication Operators

In this section, we shall study invariant and reducing subspaces of multiplication operators.

Theorem 1. *Let $M_\theta \in B(\ell^2(\mathscr{Z}))$. Then $\overline{RanM_\theta}$ is a proper invariant subspace of M_θ if and only if $\theta(n) = 0$ for some $n \in \mathscr{Z}$.*

Proof. Suppose $\overline{RanM_\theta}$ is a proper closed subspace of $\ell^2(\mathscr{Z})$ invariant under M_θ. Then there exists $\beta \in \ell^2(\mathscr{Z})$ such that

$$\beta \perp \overline{RanM_\theta} \text{and} ||\beta|| = 1$$

Now

$$M_\theta M_{\overline{\theta}}\beta \in RanM_\theta.$$

Therefore

$$\langle \beta, M_{|\theta|^2}\beta \rangle = 0.$$

implies that

$$\sum_{n\in\mathscr{Z}} |\theta(n)|^2|\beta(n)|^2 = 0.$$

This implies that

$$|\theta(n)||\beta(n)| = 0 \text{ for every } \in \mathscr{Z}.$$

But $||\beta|| = 1$. Therefore $\theta(n) = 0$ for some $n \in \mathscr{Z}$.

Conversely suppose that $\theta(n) = 0$ for some $n \in \mathscr{Z}$. Then $\overline{RanM_\theta}$ is a proper closed subspace of $\ell^2(\mathscr{Z})$. If $f \in \overline{RanM_\theta}$, then

$$M_\theta f \in RanM_\theta \subset \overline{RanM_\theta}.$$

Hence $\overline{RanM_\theta}$ is an invariant subspace of $\ell^2(\mathscr{Z})$. □

Theorem 2. *Let $\theta : \mathscr{N} \to \mathscr{C}$ be an injective map such that $M_\theta \in B(\ell^2(\mathscr{Z}))$. Then a closed subspace M of $\ell^2(\mathscr{Z})$ is reducing under M_θ if and only if $M = \chi_E\ell^2(\mathscr{Z}) = \{\chi_E f : f \in \ell^2(\mathscr{Z})\}$, where $E \subset \mathscr{Z}$.*

Proof. Suppose $M = \chi_E\ell^2(\mathscr{Z})$. Let $f \in M$. Then

$$f = \sum_{n\in E} f_n e_n$$

which implies

$$M_\theta f = \sum_{n\in E} \theta_n f_n e_n \in M.$$

Hence M is invariant under M_θ. Clearly,

$$M_\theta^* f \in M$$

for every $f \in M$. Therefore M is reducing subspace of M_θ.

Conversely, suppose that M is a reducing subspace of M_θ. Let P be a projection on M. For $k \in \mathscr{Z}$, we shall show that either $Pe_k = e_k$ in which case $e_k \in M$ or $Pe_k = 0$ in which case $e_k \in M^\perp$. Let

$$Pe_k = \sum_{n\in\mathscr{Z}} \alpha_n e_n.$$

Then

$$M_\theta Pe_k = PM_\theta e_k$$

or

$$M_\theta Pe_k = \theta(k)Pe_k$$

or

$$\sum_{n\in\mathscr{Z}} \alpha_n(\theta_n - \theta_k)e_n = 0.$$

This implies that $\alpha_n = 0$ for every $n \neq k$. Thus $Pe_k = \alpha_k e_k$.

Now $P^2 e_k = Pe_k$ implies that $\alpha_k^2 e_k = \alpha_k e_k$.

Therefore, we infer that $\alpha_k = 0$ or $\alpha_k = 1$.

If $\alpha_k = 10$, then $Pe_k = e_k \in M$ and if $\alpha_k = 0$, then $Pe_k = 0$, so that $e_k \in M^\perp$. Take $E = \{k : Pe_k = e_k\}$.

If

$$f = \sum_{n\in\mathscr{Z}} \alpha_n e_n \in M,$$

then

$$f = Pf = \sum_{n\in E} \alpha_n e_n.$$

Hence,

$$M = \chi_E \ell^2(\mathscr{Z}).$$

□

Theorem 3. *Let $\mathscr{M}$ be the set of all multiplication operators on $L^2(\lambda)$. A closed subspace M of $L^2(\lambda)$ is reducing under $\mathscr{M}$ if and only if there exists a measurable subset E of X of positive measure such that*

$$M = \chi_E L^2(\lambda).$$

Proof. Suppose $M = \chi_E L^2(\lambda)$ for a measurable subset E of X. We prove that M is reducing subspace of $\mathscr{M}$. Suppose that $f \in M$ and $M_\theta \in \mathscr{M}$. Now $f = \chi_E g$, where $g \in L^2(\lambda)$ and $M_\theta f = \chi_E M_\theta g$. Therefore

$$M_\theta f \in M.$$

which shows that M is an invariant subset of M_θ. Also

$$M_\theta^* f = M_{\overline{\theta}} f = \chi_E M_{\overline{\theta}} g \in M.$$

Thus it is invariant under M_θ^*. Since this is true for every $M_\theta \in \mathscr{M}$. Hence M is reducing subspace of $\mathscr{M}$.

Conversely suppose M is an reducing subspace of $\mathscr{M}$. Let P be a projection on M. Then for every $A \in \mathscr{M}$, we have $AP = PA$. But $\mathscr{M}$ is maximal abelian subalgebra of $B(L^2(\lambda))$, it follows that P is also a multiplication operator. That is, $P = M_\psi$ for some bounded function ψ. But $P^2 = P$ implies that $\psi^2 = \psi$ a.e. or $\psi(x)(\psi(x) - 1) = 0$ a.e. Hence, either $\psi(x) = 0$ a.e. or $\psi(x) = 1$ a.e.. If $\psi(x) = 0$ a.e. then $Pf = 0$ for every $f \in M$ which shows that

$M = \{0\}$. In case $\psi(x) = 1$ a.e., we have $P = I$ which implies that $M = L^2(\lambda)$. But M is a proper invariant subspace of $L^2(\lambda)$. Therefore, $E = \{x \in X : \psi(x) = 1 \text{ a.e.}\}$ is a measurable subset of X and $E \neq X$ a.e.. Thus

$$M = ranP = ranM_{\psi} = M_{\psi}L^2(\lambda) = \chi_E L^2(\lambda).$$

□

Corollary 1. *Let $\mathcal{M}$ be the set of all multiplication operators on $\ell^2(\mathscr{X})$. A closed subspace M of $\ell^2(\mathscr{X})$ is reducing under $\mathcal{M}$ if and only if there exists a subset E of $\mathscr{X}$ such that*

$$M = \chi_E \ell^2(\mathscr{X}).$$

3. Reducing Subspaces of Composition Operators

This section deals with reducing subspaces of composition operators.

Theorem 4. *Let $C_T \in B(\ell^2(\mathscr{X}))$. Then $T : \mathscr{X} \to \mathscr{X}$ is purely periodic if and only if C_T is reductive.*

Proof. We first assume that T is purely periodic. Suppose M is an invariant subspace of C_T. We shall prove that M is a reducing subspace of C_T. Let $f \in M$ be such that $f(n_1) \neq 0$ for some $n_1 \in \mathcal{N}$. Consider

$$E_{n_1} = O_T(n_1) = \{n_1, \ldots, n_k\}$$

such that

$$T^k(n_k) = n_1.$$

If $suppf \subset E_{n_1}$, then it is easy to show that $suppf = E_{n_1}$. Now

$$f = \sum_{j=1}^{k} f_{n_j} e_{n_j}$$

implies that

$$\sum_{j=0}^{k-1} C_T^j f = \left(\sum_{j=1}^{k} f_{n_j} \right) \sum_{j=1}^{k} e_{n_j} \in M_n \tag{1}$$

Now

$$\mathcal{N} = \bigcup_{i=1}^{\infty} O_T(n_i)$$

Therefore,

$$\ell^2(\mathcal{N}) = \sum_{i=1}^{\infty} \oplus \ell^2(O_T(n_i))$$

Also,

$$C_T = \sum_{i=1}^{\infty} \oplus C_{T_i},$$

where $T_i = T|_{O_T(n_i)}$ and the invariant subspace M can be written as

$$M = \sum_{i=1}^{\infty} \oplus M_i,$$

where M_i is an invariant subspace of C_{T_i}. Hence, either

$$M_i = span \sum_{p=1}^{k} e_p$$

or

$$M_i = \sum_{p \in O_T(n_i)} f_p e_p = 0.$$

In each case $C_T^* M_i \subset M_i$. Thus

$$C_T^* M = \sum_{i=1}^{\infty} \oplus C_{T_i}^* M_i \subset \sum_{i=1}^{\infty} \oplus M_i = M.$$

This proves that M is a reducing subspace of C_T.

Conversely, suppose that, C_T is reductive. We prove that T is purely periodic. If possible, suppose that T is not periodic at some $n_0 \in \mathscr{N}$. Then

$$T^k(n_0) \neq n_0$$

for every $k \in \mathscr{N}$. Write

$$T^k(n_0) = n_k.$$

Let $M_1 = span\{e_p : T^l(p) = n_0 \text{ for some } l \in \mathscr{N} \cup \{0\}\}$. Clearly, M_1 is a closed subspace of $\ell^2(\mathscr{N})$ invariant under C_T. Now $e_{n_0} \in M_1$ because $T^0(n_0) = n_0$. But, $C_T^* e_{n_0} = e_{n_1} \notin M_1$. Hence M_1 cannot be a reducing subspace of C_T. This completes the proof. □

Corollary 2. *Let $C_T \in B(\ell^2(\mathscr{Z}))$ be surjective but not injective. Then C_T is not reductive.*

4. Reducing Subspaces of Weighted Composition Operators

In this section we study reducing subspaces of weighted composition operators on $\ell^2(\mathscr{Z})$.

Theorem 5. *Let $M_{\theta,T} \in B(\ell^2(\mathscr{Z}))$. Suppose $T : \mathscr{Z} \to \mathscr{Z}$ is an injection but not a surjection and $\theta : \mathscr{Z} \to \mathscr{C}$ is such that $\theta(n) \neq 0$ for every $n \in \mathscr{Z}$. Then $M_{\theta,T}$ has a reducing subspace if and only if there are two points in $\mathscr{Z}$ which are not in the same orbit of T.*

Proof. We first assume that $M_{\theta,T}$ has a reducing subspace, say M. We then show that there exists two points in $\mathscr{Z}$ which are not in the same orbit of T. If possible, suppose that there is $n \in \mathscr{Z}$ such that $O_T(n) = \mathscr{Z}$. Let

$$E_n = \{n \in \mathscr{Z} : n \notin T(\mathscr{Z})\}.$$

Since T is not surjective, so E_n is non-empty. Let P be a projection on M. If $card\ E_n = 1$, then for $n_0 \in E_n$, we have

$$0 = PM_{\theta,T}e_{n_0} = M_{\theta,T}Pe_{n_0}$$

which implies that

$$Pe_{n_0} \in KerM_{\theta,T}$$

which is one-dimensional. Hence, there exists $\alpha \in \mathscr{C}$ such that $Pe_{n_0} = \alpha e_{n_0}$. It is evident that $e_{n_0} \in M$. Hence $M^*_{\theta,T}e_{n_0} = \overline{\theta(n_0)}e_{T(n_0)} \in M$ which implies that $e_{T(n_0)} \in M$. Similarly, $M^*_{\theta,T}M^*_{\theta,T}\alpha e_{n_0} = \theta(n_0)\theta(T(n_0))e_{T^2(n_0)} \in M$ implies that $e_{T^2(n_0)} \in M$. In general $e_{T^k(n_0)} \in M$ for every $k \in \mathscr{N}$. Further, $M_{\theta,T}e_{n_0} \in M$ implies that $\chi_{T^{-1}(n_0)} \in M$. Again, $M_{\theta,T}M_{\theta,T}e_{n_0} \in M$ implies that $\chi_{(T^2)^{-1}(n_0)} \in M$. In general $\chi_{(T^k)^{-1}(n_0)} \in M$. Now $O_{T(n_0)} = \mathscr{Z}$, implies that $\ell^2(\mathscr{Z}) \subset M$. This contradicts the fact that M is a proper invariant subspace of C_T. Next, if $card\ E_n \geq 2$, then $n_1, n_2 \in E_n$ which clearly are not in the same orbit of T.

Conversely, suppose that there exists two numbers m_0 and n_0 such that m_0 and n_0 are not in the same orbit of T. Then clearly $\ell^2(O_T(n_0))$ is a proper closed reducing subspace of $M_{\theta,T}$. □

Theorem 6. *Let T be an antiperiodic bijection and $card\{n \in \mathscr{Z} : \theta(n) = 0\} = 1$. Then $M_{\theta,T}$ is reducible if and only if there exists two points in $\mathscr{Z}$ which are not in the same orbit of T.*

Proof. Suppose $n_0 \in \mathscr{Z}$ is such that $\theta(n_0) = 0$. Suppose M is a reducing subspace of $M_{\theta,T}$. Let P be a projection on M. Then $PM_{\theta,T} = M_{\theta,T}P$ or equivalently $M^*_{\theta,T}P = PM^*_{\theta,T}$. Now $M^*_{\theta,T}Pe_{n_0} = PM^*_{\theta,T}e_{n_0} = 0$. Hence $Pe_{n_0} \in kerM^*_{\theta,T}$. But $kerM^*_{\theta,T} = span\{e_{n_o}\}$. Therefore, $Pe_{n_0} = \alpha e_{n_0}$ for some $\alpha \in \mathscr{C}$. Now

$$\alpha e_{n_0} = Pe_{n_0} = PPe_{n_0} = \alpha^2 e_{n_0}$$

which implies that either $\alpha = 0$ or $\alpha = 1$. If $\alpha = 0$, then $e_{n_0} \in M^\perp$ and therefore,
$M_{\theta,T}e_{n_0} = \theta(T^{-1}(n_0))e_{T^{-1}(n_0)} \in M^\perp$ or $e_{T^{-1}(n_0)} \in M^\perp$. Again,

$$M_{\theta,T}M_{\theta,T}e_{n_0} = \theta(T^{-1}(n_0))\theta(T^{-1}(T^{-1}(n_0)))e_{(T^2)^{-1}(n_0)} \in M^\perp$$

which implies that $e_{(T^2)^{-1}(n_0)} \in M^\perp$. Similarly, we can prove that $e_{(T^k)^{-1}(n_0)} \in M^\perp$. Further, $M^*_{\theta,T}e_{n_0} = \theta(T(n_0))e_{T(n_0)} \in M^\perp$ and $M^*_{\theta,T}M^*_{\theta,T}e_{n_0} = \theta(T(n_0))\theta(T^2(n_0))e_{T^2(n_0)} \in M^\perp$. Hence $e_{T^2(n_0)} \in M^\perp$. Continuing in this manner, we can show that $e_{T^k(n_0)} \in M^\perp$ for every $k \in \mathscr{Z}$.

Thus we have shown that $\ell^2(O_T(n_0)) \subset M^\perp$. If $O_T(n_0) = \mathscr{Z}$, then $\ell^2(\mathscr{Z}) \subset M^\perp$ which implies that $M^\perp = \ell^2(\mathscr{Z})$, so that $M^{\perp\perp} = \{0\}$ or $M = \{0\}$ which is a contradiction. Hence

$$O_T(n_0) \neq \mathscr{Z}.$$

That is,

$$\mathscr{Z} - O_T(n_0) \neq \phi.$$

Choose $m_0 \in \mathscr{Z} - O_T(n_0)$. Therefore, m_0 and n_0 are not in the same orbit of T.

Conversely, if there exists two points m_0 and n_0 in $\mathscr{Z}$ which are not in the same orbit of T, then taking $M = \ell^2(O_T(n_0)) \neq \ell^2(\mathscr{Z})$, we can prove that M is a reducing subspace of $M_{\theta,T}$. □'

Corollary 3. *Let $M_{\theta,T} \in B(\ell^2(\mathscr{Z}))$. Suppose T is periodic at $n \in \mathscr{Z}$ and $\theta = \theta \circ T$. Then $M_{\theta,T}$ has a reducing subspace.*

Example 1. *Suppose $\theta : \mathscr{Z} \to \mathscr{C}$ be defined by $\theta(n) = \frac{1}{n}$. Let $T : \mathscr{Z} \to \mathscr{Z}$ be defined by*

$$T(n) = \begin{cases} n+1, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even} \end{cases}$$

Then T is periodic at every $n \in \mathscr{Z}$ of period 2. Clearly, $M_{2n+1} = \chi_{E_{2n+1}} \ell^2(\mathscr{Z})$ is reducing subspace of T for every $n \in \mathscr{Z}$, where $E_{2n+1} = [2n+1, 2n+2]$.

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