EUROPEAN JOURNAL OF MATHEMATICAL SCIENCES
Vol. 2, No. 1, 2013, 1-16
ISSN 2147-5512 – www.ejmathsci.com

Lower Semicontinuous Quantum Stochastic Differential Inclusions

Michael O. Ogundiran[1,*] and Ezekiel O. Ayoola[2]

[1] *Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria.*
[2] *Department of Mathematics, University of Ibadan, Ibadan, Nigeria*

Abstract. In this paper we established the existence of solutions of Lower semicontinuous quantum stochastic differential inclusions(QSDI). The existence of a continuous selection of a predefined integral operator was established. This selection which is an adapted stochastic process is a solution of the Lower semicontinuous quantum stochastic differential inclusions.

2010 Mathematics Subject Classifications: 81S25

Key Words and Phrases: Annihilation, creation and gauge processes, Lower semicontinuous multi-functions.

1. Introduction

The theory of quantum stochastic differential inclusions is a multivalued analogue of quantum stochastic calculus of Hudson and Parthasarathy formulation [9]. The theory of differential inclusions has vast applications and one of its motivations is the application in the study of control theory. In [5] the existence of solutions of quantum stochastic differential inclusions with Lipschitzian coefficients lying in certain locally convex spaces was established. A further study of this quantum stochastic differential inclusions was done in [6] with hypermaximal monotone type and in [7] for evolution type. The topological properties of solution sets and existence of continuous selections of the solution sets for the Lipschitzian quantum stochastic differential inclusions were established in [2] and [3].
For a classical differential inclusion the existence of solutions of discontinuous cases, upper and lower semicontinuous differential inclusions were established in [1] and [4]. These weaker forms of regularity of the coefficients are also applicable in the study of optimal quantum stochastic control theory [10]. The aim of this work is to establish the existence of solution of Lower semicontinuous quantum stochastic differential inclusions. We first define

*Corresponding author.

Email addresses: adeolu74113@yahoo.com (M. Ogundiran), eoayoola@googlemail.com (E. Ayoola)

an integral operator which is a mapping consisting of adapted stochastic processes and established the existence of a continuous map which is a selection of the mapping. Hence we established the existence of at least a solution of the Lower semicontinuous quantum stochastic differential inclusions. This is a generalization of the result in [1] to our non commutative setting. This will lead in a later work to further applications of quantum stochastic calculus to quantum stochastic differential equations with discontinuous coefficients, solutions of pertinent quantum stochastic control problems and quantum optics. In sequel the work shall be arranged as follows: section 2 shall be for preliminaries on notations and definitions while section 3 shall be for our main results.

2. Preliminaries

In this section we state the definitions and notations which shall be employed in the sequel.

2.1. Notations

In what follows, if U is a topological space, we denote by clos(U), the collection of all non-empty closed subsets of U.
To each pair (D,H) consisting of a pre-Hilbert space D and its completion H, we associate the set $L_w^+(D,H)$ of all linear maps x from D into H, with the property that the domain of the operator adjoint contains D. The members of $L_w^+(D,H)$ are densely-defined linear operators on H which do not necessarily leave D invariant and $L_w^+(D,H)$ is a linear space when equipped with the usual notions of addition and scalar multiplication.
To H corresponds a Hilbert space $\Gamma(H)$ called the boson Fock space determined by H. A natural dense subset of $\Gamma(H)$ consists of linear space generated by the set of exponential vectors(Guichardet, [8]) in $\Gamma(H)$ of the form

$$e(f)=\bigoplus_{n=0}^{\infty}(n!)^{-\frac{1}{2}}\bigotimes^{n} f, \quad f\in H,$$

where $\bigotimes^0 f=1$ and $\bigotimes^n f$ is the n-fold tensor product of f with itself for $n\geq 1$.
In what follows, $\mathbb{D}$ is some pre-Hilbert space whose completion is $\mathscr{R}$ and γ is a fixed Hilbert space.
$L^2_\gamma(\mathbb{R}_+)$(resp. $L^2_\gamma([0,t))$, resp. $L^2_\gamma([t,\infty))$ $t\in\mathbb{R}_+$) is the space of square integrable γ-valued maps on $\mathbb{R}_+$ (resp .$[0,t)$, resp. $[t,\infty)$).
The inner product of the Hilbert space $\mathscr{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle .,. \rangle$ and $\|\ .\ \|$ the norm induced by $\langle .,. \rangle$.
Let $\mathbb{E},\mathbb{E}_t$ and $\mathbb{E}^t$, $t>0$ be linear spaces generated by the exponential vectors in Fock spaces $\Gamma(L^2_\gamma(\mathbb{R}_+)),\Gamma(L^2_\gamma([0,t)))$ and $\Gamma(L^2_\gamma([t,\infty)))$ respectively ;

$$\mathscr{A}\equiv L_w^+(\mathbb{D}\underline{\otimes}\mathbb{E},\mathscr{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+)))$$
$$\mathscr{A}_t\equiv L_w^+(\mathbb{D}\underline{\otimes}\mathbb{E}_t,\mathscr{R}\otimes\Gamma(L^2_\gamma([0,t))))\otimes\mathbb{I}^t$$
$$\mathscr{A}^t\equiv \mathbb{I}_t\otimes L_w^+(\mathbb{E}^t,\Gamma(L^2_\gamma([t,\infty)))),\ t>0$$

where $\underline{\otimes}$ denotes algebraic tensor product and $\mathbb{I}_t$(resp.$\mathbb{I}^t$) denotes the identity map on $\mathscr{R} \otimes \Gamma(L^2_\gamma([0,t))))$(resp .$\Gamma(L^2_\gamma([t,\infty))))$, $t > 0$ For every $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ define

$$\| x \|_{\eta\xi} = | \langle \eta, x\xi \rangle |, \quad x \in \mathscr{A}$$

then the family of seminorms

$$\{\| \,.\, \|_{\eta\xi} : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$$

generates a topology τ_w , weak topology .
The completion of the locally convex spaces $(\mathscr{A}, \tau_w)$, $(\mathscr{A}_t, \tau_w)$ and $(\mathscr{A}^t, \tau_w)$ are respectively denoted by $\widetilde{\mathscr{A}}$, $\widetilde{\mathscr{A}_t}$ and $\widetilde{\mathscr{A}^t}$.
We define the Hausdorff topology on clos($\widetilde{\mathscr{A}}$) as follows:
For $x \in \widetilde{\mathscr{A}}$, $\mathscr{M}, \mathscr{N} \in$ clos($\widetilde{\mathscr{A}}$) and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, define

$$\rho_{\eta\xi}(\mathscr{M}, \mathscr{N}) \equiv \max(\delta_{\eta\xi}(\mathscr{M}, \mathscr{N}), \delta_{\eta\xi}(\mathscr{N}, \mathscr{M})),$$

where

$$\delta_{\eta\xi}(\mathscr{M}, \mathscr{N}) \equiv \sup_{x \in \mathscr{M}} \mathbf{d}_{\eta\xi}(x, \mathscr{N}),$$

and

$$\mathbf{d}_{\eta\xi}(x, \mathscr{N}) \equiv \inf_{y \in \mathscr{N}} \| x - y \|_{\eta\xi} .$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H , is generated by the family of pseudometrics $\{\rho_{\eta\xi}(.) : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. Moreover, if $\mathscr{M} \in clos(\widetilde{\mathscr{A}})$, then $\| \mathscr{M} \|_{\eta\xi}$ is defined by

$$\| \mathscr{M} \|_{\eta\xi} \equiv \rho_{\eta\xi}(\mathscr{M}, \{0\});$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
For $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, define

$$\mathbf{d}(x, B) \equiv \inf_{y \in B} | x - y |,$$

$$\delta(A, B) \equiv \sup_{x \in A} \mathbf{d}(x, B),$$

and

$$\rho(A, B) \equiv \max(\delta(A, B), \delta(B, A)).$$

Then ρ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space. We also define:

$$d_{\eta\xi}((t, x), (t_0, x_0)) = \max\{| t - t_0 |, \| x - x_0 \|_{\eta\xi}\}.$$

Let $I \subseteq \mathbb{R}_+$. A *stochastic process* indexed by I is an $\widetilde{\mathscr{A}}$-valued measurable map on I.
A stochastic process X is called *adapted* if $X(t) \in \widetilde{\mathscr{A}_t}$ for each $t \in I$.
We write Ad($\widetilde{\mathscr{A}}$) for the set of all adapted stochastic processes indexed by I.

Definition 1. *A member X of $Ad(\widetilde{\mathscr{A}})$ is called*

(i) weakly absolutely continuous if the map $t \mapsto \langle \eta, X(t)\xi \rangle$, $t \in I$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

(ii) locally absolutely p-integrable if $\| X(.) \|^p_{\eta\xi}$ is Lebesgue -measurable and integrable on $[0,t) \subseteq I$ for each $t \in I$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

We denote by $\mathrm{Ad}(\widetilde{\mathscr{A}})_{wac}$(resp.$L^p_{loc}(\widetilde{\mathscr{A}})$) the set of all weakly, absolutely continuous(resp. locally absolutely p-integrable) members of $\mathrm{Ad}(\widetilde{\mathscr{A}})$.
Stochastic integrators: Let $L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ [resp.$L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$] be the linear space of all measurable , locally bounded functions from $\mathbb{R}_+$ to γ [resp. to B(γ) , the Banach space of bounded endomorphisms of γ]. If $f \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$, then πf is the member of $L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ given by $(\pi f)(t) = \pi(t)f(t)$, $t \in \mathbb{R}_+$.
For $f \in L^2_{\gamma}(\mathbb{R})_+$ and $\pi \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$; the annihilation , creation and gauge operators, $a(f), a^+(f)$ and $\lambda(\pi)$ in $L^+_w(\mathbb{D}, \Gamma(L^2_{\gamma}(\mathbb{R})_+))$ respectively, are defined as:

$$\begin{aligned} a(f)\mathbf{e}(g) &= \langle f, g\rangle_{L^2_{\gamma}(\mathbb{R}_+)}\mathbf{e}(g), \\ a^+(f)\mathbf{e}(g) &= \frac{d}{d\sigma}\mathbf{e}(g+\sigma f)\mid_{\sigma=0}, \\ \lambda(\pi)\mathbf{e}(g) &= \frac{d}{d\sigma}\mathbf{e}(e^{\sigma\pi}f)\mid_{\sigma=0} \end{aligned}$$

for all $g \in L^2_{\gamma}(\mathbb{R}_+)$.
For arbitrary $f \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$, they give rise to the operator-valued maps A_f, A^+_f and Λ_{π} defined by:

$$\begin{aligned} A_f(t) &\equiv a(f\chi_{[0,t)}), \\ A^+_f(t) &\equiv a^+(f\chi_{[0,t)}), \\ \Lambda_{\pi}(t) &\equiv \lambda(\pi\chi_{[0,t)}) \end{aligned}$$

for all $t \in \mathbb{R}_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$. The maps A_f, A^+_f and Λ_{π} are stochastic processes , called annihilation, creation and gauge processes, respectively, when their values are identified with their ampliations on $\mathscr{R} \otimes \Gamma(L^2_{\gamma}(\mathbb{R}_+))$. These are the stochastic integrators in Hudson and Parthasarathy[9] formulation of boson quantum stochastic integration.
For processes $p, q, u, v \in L^2_{loc}(\widetilde{\mathscr{A}})$, the quantum stochastic integral:

$$\int_{t_0}^{t} \big(p(s)d\Lambda_{\pi}(s) + q(s)dA_f(s) + u(s)dA^+_g(s) + v(s)ds\big), \quad t_0, t \in \mathbb{R}_+$$

is interpreted in the sense of Hudson-Parthasarathy[9].

2.2. Quantum Stochastic Differential Inclusions

Definition 2. *(a) By a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$ we mean a multifunction on I with values in $clos(\widetilde{\mathscr{A}})$.*

(b) If Φ is a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, then a selection of Φ is a stochastic process $X : I \to \widetilde{\mathscr{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.

(c) A multivalued stochastic process Φ will be called (i) adapted if $\Phi(t) \subseteq \widetilde{\mathscr{A}_t}$ for each $t \in \mathbb{R}_+$; (ii) measurable if $t \mapsto \mathbf{d}_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \widetilde{\mathscr{A}}, \eta, \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})$

(d) locally absolutely p-integrable if $t \mapsto \| \Phi(t) \|_{\eta\xi}$, $t \in \mathbb{R}_+$ lie in $L^p_{loc}(I)$ for arbitrary $\eta, \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})$.

For $p \in (0, \infty)$ and $I \subseteq \mathbb{R}_+$, the set of all locally absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\widetilde{\mathscr{A}})_{mvs}$. Denote by $L^p_{loc}(I \times \widetilde{\mathscr{A}})_{mvs}$ the set of maps $\Phi : I \times \widetilde{\mathscr{A}} \to clos(\widetilde{\mathscr{A}})$ such that $t \mapsto \Phi(t, X(t))$, $t \in I$, lies in $L^p_{loc}(\widetilde{\mathscr{A}})_{mvs}$ for every $X \in L^p_{loc}(\widetilde{\mathscr{A}})$.
Moreover, if $\Phi \in L^p_{loc}(I \times \widetilde{\mathscr{A}})_{mvs}$, then we denote by

$$L_p(\Phi) \equiv \{\phi \in L^p(\widetilde{\mathscr{A}}) : \phi \text{ is a selection of } \Phi\}.$$

Let $f, g \in L^2_\gamma(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, $\mathbb{I}$, the identity map on $\mathscr{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$, and M is any of the stochastic processes A_f, A^+_g, Λ_π and $s \mapsto s\mathbb{I}$, $s \in \mathbb{R}_+$.
We introduce the *stochastic integral*{resp. *differential*} expressions as follows:
If $\Phi \in L^2_{loc}(I \times \widetilde{\mathscr{A}})_{mvs}$ and $(t, X) \in I \times L^2_{loc}(\widetilde{\mathscr{A}})$, then

$$\int_{t_0}^t \Phi(s, X(s)) dM(s) \equiv \left\{ \int_{t_0}^t \phi(s) dM(s) : \phi \in L_2(\Phi) \right\}.$$

This leads to the following definition:

Definition 3. *Let $E, F, G, H \in L^2_{loc}(I \times \widetilde{\mathscr{A}})$ and (t_0, x_0) be a fixed point of $I \times \widetilde{\mathscr{A}}$. Then a relation of the form*

$$\begin{aligned} dX(t) \in\ & E(t, X(t)) d\Lambda_\pi(t) + F(t, X(t)) dA_f(t) \\ & + G(t, X(t)) dA^+_g(t) + H(t, X(t)) dt \text{ almost all } t \in I, \\ X(t_0) =\ & x_0 \end{aligned} \tag{1}$$

is called Quantum stochastic differential inclusions(QSDI) with coefficients E, F, G, H and initial data (t_0, x_0).
Equation(1) is understood in the integral form:

$$\begin{aligned} X(t) \in x_0 + \int_{t_0}^t & \big(E(s, X(s)) d\Lambda_\pi(s) + F(s, X(s)) dA_f(s) \\ & + G(s, X(s)) dA^+_g(s) + H(s, X(s)) ds\big), \text{ almost all } t \in I \end{aligned}$$

called a stochastic integral inclusion with coefficients E, F, G, H and initial data (t_0, x_0)

An equivalent form of (1) has been established in [5], Theorem 6.2 as follows:
For $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\alpha, \beta \in L^2_\gamma(\mathbb{R}_+)$ with $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$, define the following complex-valued functions:

$$\mu_{\alpha\beta},\ \nu_\beta,\ \sigma_\alpha : I \to \mathbb{C}, \quad I \subset \mathbb{R}_+$$

by

$$\begin{aligned}\mu_{\alpha\beta}(t) &= \langle \alpha(t), \pi(t)\beta(t)\rangle_\gamma,\\ \nu_\beta(t) &= \langle f(t), \beta(t)\rangle_\gamma,\\ \sigma_\alpha(t) &= \langle \alpha(t), g(t)\rangle_\gamma,\end{aligned}$$

$t \in I$, $f, g \in L^2_{\gamma,loc}(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma),loc}$. To these functions we associate the maps μE, νF, σG, $\mathbb{P}$ from $I \times \widetilde{\mathscr{A}}$ into the set of sesquilinear forms on $\mathbb{D}\underline{\otimes}\mathbb{E}$ defined by :

$$\begin{aligned}(\mu E)(t,x)(\eta,\xi) &= \{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\},\\ (\nu F)(t,x)(\eta,\xi) &= \{\langle \eta, \nu_\beta(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\},\\ (\sigma G)(t,x)(\eta,\xi) &= \{\langle \eta, \sigma_\alpha(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\},\\ \mathbb{P}(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (\nu F)(t,x)(\eta,\xi)\\ &\quad + (\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi),\\ H(t,x)(\eta,\xi) &= \{v(t,x)(\eta,\xi) : v(.,X(.))\end{aligned} \tag{2}$$

is a selection of

$$H(.,X(.))\forall X \in L^2_{loc}(\widetilde{\mathscr{A}})\}. \tag{3}$$

Then, Problem (1) is equivalent to

$$\begin{aligned}\frac{d}{dt}\langle \eta, X(t)\xi\rangle &\in \mathbb{P}(t,X(t))(\eta,\xi),\\ X(t_0) &= x_0\end{aligned} \tag{4}$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, almost all $t \in I$.
The notion of solution of (1) or equivalently (3) is defined as follows:

Definition 4. *By a solution of (1) or equivalently* (3)*, we mean a stochastic process $\varphi \in Ad(\widetilde{\mathscr{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathscr{A}})$ such that*

$$\begin{aligned}d\varphi(t) \in\ & E(t,\varphi(t))d\Lambda_\pi(t) + F(t,\varphi(t))dA_f(t)\\ & + G(t,\varphi(t))dA^+_g(t) + H(t,\varphi(t))dt \text{ almost all } t \in I,\\ \varphi(t_0) =\ & \varphi_0\end{aligned}$$

or equivalently

$$\begin{aligned}\frac{d}{dt}\langle \eta, \varphi(t)\xi\rangle &\in \mathbb{P}(t,\varphi(t))(\eta,\xi),\\ \varphi(t_0) &= \varphi_0\end{aligned}$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, almost all $t \in I$.

The existence of solution of (1) implies the existence of solution of (3) and vice-versa. As explained in [5], for the map $\mathbb{P}$:

$$\mathbb{P}(t,x)(\eta,\xi) \neq \widetilde{\mathbb{P}}(t,\langle\eta,x\xi\rangle)$$

for some complex-valued multifunction $\widetilde{\mathbb{P}}$ defined on $I \times \mathbb{C}$ for $t \in I$, $x \in \widetilde{\mathscr{A}}$, $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

2.3. Lower Semicontinuous Multivalued Maps

Definition 5. *(a) Let $\mathscr{N} \in clos(\widetilde{\mathscr{A}})$ be non-empty and $I \subseteq \mathbb{R}_+$.*
A multifunction $\Phi : I \times \mathscr{N} \to clos(\widetilde{\mathscr{A}})$ will be said to be lower semicontinuous at a point $(t_0,x_0) \in I \times \mathscr{N}$, if for every $\epsilon > 0, \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ there exists $\delta_{\eta\xi} = \delta_{\eta\xi}((t_0,x_0),\epsilon) > 0$ such that $\forall x \in \mathscr{N}$, $t \in I$ if

$$d_{\eta\xi}\big((t,x),(t_0,x_0)\big) < \delta_{\eta\xi} \text{ then } \Phi(t_0,x_0) \subset \Phi(t,x) + B_{\eta\xi,\epsilon}(0).$$

If Φ is lower semicontinuous(lsc) at every point $(t_0,x_0) \in I \times \mathscr{N}$, then it will be said to be lower semicontinuous on $I \times \mathscr{N}$.

(b) Analogously if Φ is a sesquilinear form valued multifunction, then the map $\Phi : I \times \mathscr{N} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ will be said to be lower semicontinuous at a point $(t_0,x_0) \in I \times \mathscr{N}$, if for every $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, \epsilon > 0$ there exists $\delta_{\eta\xi} = \delta_{\eta\xi}((t_0,x_0),\epsilon) > 0$ such that $\forall x \in \mathscr{N}$, $t \in I$, if

$$d_{\eta\xi}\big((t,x),(t_0,x_0)\big) < \delta_{\eta\xi} \text{ then } \Phi(t_0,x_0)(\eta,\xi) \subset \Phi(t,x)(\eta,\xi) + B_\epsilon(0).$$

In what follows, a map shall be called lower semicontinuous on a domain if it is so at every point of the domain.

The next result shows that, if μE, νF, σG, H are lower semicontinuous then $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ is lower semicontinuous.

Proposition 1. *Assume that the following holds:*

(i) The coefficients E,F,G,H appearing in (1) belongs to the space $L^2_{loc}(I \times \widetilde{\mathscr{A}})_{mvs}$.

(ii) For an arbitrary elements $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the maps μE, νF, σG, H defined by equation (2) are lower semicontinuous on $I \times \widetilde{\mathscr{A}}$.

Then, the map $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ is lower semicontinuous on $I \times \widetilde{\mathscr{A}}$.

Proof. For an arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, since $\mu E, \nu F, \sigma G, H$ are lower semicontinuous $I \times \widetilde{\mathscr{A}}$. Then for any point $(t_0,x_0) \in I \times \widetilde{\mathscr{A}}$, given $\epsilon > 0$, there exist $\delta_{\eta\xi,E}, \delta_{\eta\xi,F}, \delta_{\eta\xi,G}, \delta_{\eta\xi,H} > 0$, such that for each $M \in \{\mu E, \nu F, \sigma G, H\}$,

$$M(t_0,x_0)(\eta,\xi) \subset M(t,x)(\eta,\xi) + B_\epsilon(0) \;\; \forall x \in \mathscr{N}, \text{ almost all } t \in I \text{ and } d_{\eta\xi}\big((t,x),(t_0,x_0)\big) < \delta_{\eta\xi,M}.$$

Hence the proposition follows from the relation:

$$\begin{aligned}\mathbb{P}(t_0,x_0)(\eta,\xi) &= (\mu E)(t_0,x_0)(\eta,\xi)+(\nu F)(t_0,x_0)(\eta,\xi)\\ &\quad +(\sigma G)(t_0,x_0)(\eta,\xi)+H(t_0,x_0)(\eta,\xi)+B_\epsilon(0)\\ &\subset \mathbb{P}(t,x)(\eta,\xi)+B_{5\epsilon}(0).\end{aligned}$$

3. Main Results

In this subsection under some assumptions, we prove an existence theorem for lower semi-continuous quantum stochastic differential inclusions by using a predefined integral operator.

Definition 6. *Let $C(I)$ be the space of continuous maps from I to $sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$. For all $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$; $X,Z\in Ad(\widetilde{\mathscr{A}})_{wac}\cap L^2_{loc}(\widetilde{\mathscr{A}})$, we define the set:*

$$\mathscr{K}_{\eta\xi}=\{\langle\eta,X(t)\xi\rangle\in C(I):\exists\lambda\in\mathbb{R}_+;|\langle\eta,(X(t)-X(s))\xi\rangle|<\lambda\,|t-s|,t,s\in I \text{ and } X(t_0)=x_0\}.$$

Moreover, the integral operator $\mathscr{F}_{\eta\xi}$ is defined as

$$\mathscr{F}_{\eta\xi}(X)=\{\langle\eta,Z(t)\xi\rangle\in\mathscr{K}_{\eta\xi}:\frac{d}{dt}\langle\eta,Z(t)\xi\rangle\in\mathbb{P}(t,X(t))(\eta,\xi)\text{ a.e. } t\in I\}.$$

We also define the following sets as applicable in the subsequent result.
For any $(t,x),(t_0,x_0)\in I\times\widetilde{\mathscr{A}},\lambda_{\eta\xi}>0$, a real number; $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$.

$$Q_{(t_0,x_0),\lambda_{\eta\xi}}=\{(t,x)\in I\times\widetilde{\mathscr{A}}:d_{\eta\xi}((t,x),(t_0,x_0))\le\lambda_{\eta\xi}\},$$

where

$$\begin{aligned}d_{\eta\xi}((t,x),(t_0,x_0)) &= \max\{|t-t_0|,\|x-x_0\|_{\eta\xi}\},\\ Q_{x_0,\lambda_{\eta\xi}} &= \{x\in\widetilde{\mathscr{A}}:\|x-x_0\|_{\eta\xi}<\lambda_{\eta\xi}\},\\ Q_{\lambda_{\eta\xi}} &= \{x\in\widetilde{\mathscr{A}}:\|x\|_{\eta\xi}<\lambda_{\eta\xi}\},\end{aligned}$$

and set

$$Q_\epsilon(\eta,\xi)=\{\langle\eta,x\xi\rangle:x\in Q_\epsilon\}.$$

In what follows, we make the following assumptions:
$I=[t_0,T]$, $\lambda_{\eta\xi}>0$ and $\Omega\subset I\times\widetilde{\mathscr{A}}$, open, such that:

(i) $I\times Q_{x_0,\frac{T}{2}\lambda_{\eta\xi}}\subseteq\Omega$,

(ii) $\exists\,\lambda_{M,\eta\xi}>0\ \forall\ M\in\{E,F,G,H\}$ with $\max_M\lambda_{M,\eta\xi}<\lambda_{\eta\xi}$ and

(iii) $\|M(t,x)\|_{\eta\xi}\le\lambda_{M,\eta\xi}$ for each M on $I\times Q_{x_0,\frac{T}{2}\lambda_{\eta\xi}}$.

Lemma 1. *Suppose that $K \subseteq I \times \widetilde{\mathscr{A}}$ is compact.*
For arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, suppose that the multivalued map

$$(t,x) \to M(t,x)(\eta,\xi)$$

is lower semicontinuous for each $M \in \{\mu E, \nu F, \sigma G, H\}$.
For $\epsilon > 0$, set

$$\omega_{\eta\xi,\epsilon}(t,x) = \sup\{\omega_{\eta\xi} : \bigcap_{(\tau,\zeta)\in Q_{(t,x),\omega_{\eta\xi}}} M(\tau,\zeta)(\eta,\xi) + Q_\epsilon(\eta,\xi) \neq \emptyset\}. \tag{5}$$

Then

(a) *for some $\omega_\epsilon > 0$ we have*

$$\omega_{\eta\xi,\epsilon}(t,x) \geq \omega_\epsilon \text{ for all } (t,x) \in I \times \widetilde{\mathscr{A}}, \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E},$$

(b) *for every continuous u , $(t,u(t)) \in K$, there exists a measurable map $t \to v(t)(\eta,\xi)$, such that*

$$d_{\eta\xi}((t,x),(t,u(t))) < \omega_\epsilon,$$

implies

$$d(v(t)(\eta,\xi), M(t,x)(\eta,\xi)) \leq \epsilon.$$

Proof. (a) The definition of lower semicontinuity implies that the set inside brackets in (4) is non-empty , so that $\omega_{\eta\xi,\epsilon}(t,x)$ is positive.
We claim that it is a continuous function.
Fix $\sigma > 0$ arbitrarily, and remark that whenever $d_{\eta\xi}((\tau_1,\zeta_1),(\tau_2,\zeta_2)) < \frac{\sigma}{3}$,

$$Q^1 = Q_{(\tau_1,\zeta_1),\omega_\epsilon((\tau_2,\zeta_2)-\frac{2\sigma}{3})} \subset Q_{(\tau_2,\zeta_2),\omega_\epsilon((\tau_2,\zeta_2)-\frac{\sigma}{3})} = Q^2$$

that is,

$$\bigcap_{(\tau,\zeta)\in Q^2} M(\tau,\zeta)(\eta,\xi) + Q_\epsilon(\eta,\xi) \neq \emptyset \Rightarrow \bigcap_{(\tau,\zeta)\in Q^1} M(\tau,\zeta)(\eta,\xi) + Q_\epsilon(\eta,\xi) \neq \emptyset.$$

Whenever $d_{\eta,\xi}((t,x),(t^*,x^*)) < \frac{\sigma}{3}$, setting $(t,x) = (\tau_1,\zeta_1)$, $(t^*,x^*) = (\tau_2,\zeta_2)$ we obtain

$$\omega_{\eta\xi,\epsilon}(t,x) \geq \omega_{\eta\xi,\epsilon}(t^*,x^*) - \frac{2\sigma}{3}$$

while interchanging (t,x) and (t^*,x^*), we have

$$\omega_{\eta\xi,\epsilon}(t,x) \geq \omega_{\eta\xi,\epsilon}(t^*,x^*) - \frac{2\sigma}{3}.$$

Hence $(t,x) \to \omega_{\eta\xi,\epsilon}(t,x)$ is a continuous and positive map defined on a compact set.
(b) We define the map Φ, given by

$$(t,x) \to \Phi(t,x)(\eta,\xi) = \bigcap_{(\tau,\zeta)\in Q_{(t,x),\omega_\epsilon}} M(\tau,\zeta)(\eta,\xi) + Q_\epsilon(\eta,\xi). \tag{6}$$

Then Φ is lower semicontinuous . In fact let y^* be in $\Phi(t^*,x^*)(\eta,\xi)$, so that for every (τ,ζ) in $Q_{(t^*,x^*),\omega_\epsilon}$,

$$d(y^*, M(\tau,\zeta)(\eta,\xi)) = \epsilon - \omega_{\eta\xi,\epsilon}(\tau,\zeta), \quad \omega_{\eta\xi,\epsilon}(\tau,\zeta) > 0$$

or equivalently , there exists $y_{\eta,\xi}(\tau,\zeta)$ in $M(\tau,\zeta)(\eta,\xi)$ so that $| y^* - y_{\eta\xi}(\tau,\zeta) | \leq \epsilon - \frac{\sigma}{2}$.
By the lower semicontinuity of M, there exists $\delta = \delta(\tau,\zeta)$ so that (τ',ζ') in $Q_{(\tau,\zeta),\delta}$ implies $d(y_{\eta\xi}(\tau,\zeta), M(\tau',\zeta')(\eta,\xi)) < \frac{\sigma}{2}$, hence, in particular

$$d(y^*, M(\tau',\zeta')(\eta,\xi)) < \epsilon.$$

The open set

$$\mathscr{U} = \bigcup_{(\tau,\zeta)\in Q_{(t^*,x^*),\omega_\epsilon}} Q_{(\tau,\zeta),\delta(\tau,\zeta)}$$

contains the compact set $Q_{(t^*,x^*),\omega_\epsilon}$, hence, whenever

$$d_{\eta\xi}((t,x),(t^*,x^*)) < \rho,$$

sufficiently small

$$Q_{(t,x),\omega_\epsilon} \subset \mathscr{U},$$

and thus

$$d(y^*, M(\tau,\zeta)(\eta,\xi)) < \epsilon \text{ or } y^* \in \Phi(t,x)(\eta,\xi).$$

Since the map $t \to \Phi(t,x)(\eta,\xi)$ is lower semicontinuous and has closed values, then by Theorem 2.14.2 [1] there exists a measurable selection $v(t)(\eta,\xi)$ of $M(t,x)(\eta,\xi)$, which is the required selection.

Proposition 2. *Assume that the following holds*

(i) For arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, *the multivalued map* $(t,x) \to G(t,x)(\eta,\xi)$ *is lower semicontinuous.*

(ii) $g : I \times \widetilde{\mathscr{A}} \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ *is continuous single-valued map, and*

(iii) $\varepsilon : \widetilde{\mathscr{A}} \to \mathbb{R}_+$ *is lower semicontinuous.*

Then the map $(t,x) \to \Phi(t,x)(\eta,\xi)$ *defined by*

$$\Phi(t,x)(\eta,\xi) = B_{\varepsilon(x)}(g(t,x)(\eta,\xi)) \bigcap G(t,x)(\eta,\xi)$$

is lower semicontinuous on its domain.

Proof. Fix (t^*, x^*) in DomΦ, $y^*_{\eta\xi} \in \Phi(t^*, x^*)(\eta, \xi)$ and $\omega > 0$. For some $\sigma > 0$, $\mid y^*_{\eta\xi} - g(t^*, x^*)(\eta, \xi) \mid = \varepsilon(x^*) - \sigma$.
There exists δ_1 such that to any $(t, x) \in I \times \widetilde{\mathscr{A}}$ with $d_{\eta\xi}((t, x), (t^*, x^*)) < \delta_1$, we can associate $y(t, x)(\eta, \xi)$ in $G(t, x)(\eta, \xi)$ so that

$$\mid y_{\eta\xi,(t,x)} - y^*_{\eta\xi} \mid < \min\{\omega, \frac{\sigma}{3}\},$$

and δ_2 such that

$$d_{\eta\xi}((t, x), (t^*, x^*) < \delta_2$$

implies

$$\varepsilon(x) > \varepsilon(x^*) - \frac{\sigma}{3},$$

and δ_3 such that

$$d_{\eta\xi}((t, x), (t^*, x^*) < \delta_3$$

implies $\mid g(t^*, x^*)(\eta, \xi) - g(t, x)(\eta, \xi) \mid < \frac{\sigma}{3}$.
Then when $d_{\eta\xi}((t, x), (t^*, x^*) < \min\{\delta_1, \delta_2, \delta_3\}$,

$$\begin{aligned}\mid y(t, x)(\eta, \xi) - g(t, x)(\eta, \xi) \mid \leq & \mid y(t, x)(\eta, \xi) - y^*_{\eta\xi} \mid + \mid y^*_{\eta\xi} - g(t^*, x^*)(\eta, \xi) \mid \\ & + \mid g(t^*, x^*)(\eta, \xi) - g(t, x)(\eta, \xi) \mid \\ & < \frac{\sigma}{3} + \varepsilon(x^*) - \sigma + \frac{\sigma}{3} \\ = & \varepsilon(x^*) - \frac{\sigma}{3} < \varepsilon(x)\end{aligned}$$

that is $y(t, x)(\eta, \xi) \in \Phi(t, x)(\eta, \xi)$, and

$$\mid y^*(t, x)(\eta, \xi) - y(t, x)(\eta, \xi) \mid < \omega.$$

We now prove the existence of solution of Lower semicontinuous quantum stochastic differential inclusions.

Theorem 1. *Suppose that the following holds:*

(i) *For every $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the map $(t, x) \to \mathbb{P}(t, x)(\eta, \xi)$ is a non-empty compact and lower semicontinuous multifunction.*

(ii) *$(t_0, x_0) \in I \times \widetilde{\mathscr{A}}$, for all $(t, x) \in I \times Q_{x_0, \frac{T}{2}\lambda}$, $\lambda > 0$, such that $\mid \mathbb{P}(t, x)(\eta, \xi) \mid < \lambda$.*

Then there exists a set $\mathscr{K}_{\eta\xi}$ and a continuous map $\varphi : \mathscr{K}_{\eta\xi} \to L^1(I)$, a selection of $\mathscr{F}_{\eta\xi}$.

Proof. We shall first show the existence of a finite number $m(0)$ of measurable maps v_i from I into $Q_\lambda(\eta, \xi)$; of a continuous partition of I into $\mathscr{J}^0_i = [\tau^0_{i-1}, \tau^0_i]$ with characteristic functions χ_i such that setting

$$g^0(u)(t)(\eta, \xi) = \sum \chi^0_i(t) v_i(t)(\eta, \xi),$$

we have for every t:

$$d(g^0(u)(t)(\eta,\xi),\mathbb{P}(t,u(t))(\eta,\xi)) < 1. \tag{7}$$

In fact,set in Lemma(1), M to be $\mathbb{P}$, ϵ to be 1 and let ω_0 be the constant provided by (a).
Let $\mathscr{U}^i = Q_{u^i,\omega_0}$, we define $\mathscr{U}^i(\eta,\xi) = \{\langle\eta,x\xi\rangle : x \in \mathscr{U}^i\}$ a finite open covering of the compact $\mathscr{K}_{\eta\xi}$. Let $v_i(t)(\eta,\xi)$, be the corresponding measurable functions as provided by (b). Fix u and t ; where $|\ \chi_i^0(u)(t)(\eta,\xi)\ | > 0$, u is in Q_{u^i,ω_0}, $d(v_i(t)(\eta,\xi),\mathbb{P}(t,u(t))(\eta,\xi)) < 1$, and (6) holds.
We claim that for $n = 0,1,...$ we can define ; $m(n)$ measurable functions $v_i^{(n)}$ from I into $Q_\lambda(\eta,\xi)$, a continuous partition of I , $\mathscr{I}_i^{(n)} = [\tau_{i-1}^{(n)}(u),\tau_i^{(n)}(u)]$ having characteristic functions $\chi_i^{(n)}$ such that setting

$$g^{(n)}(u)(t)(\eta,\xi) = \sum \chi_i^{(n)}(t)v_i^{(n)}(t)(\eta,\xi),$$

we have

(i) for every t,

$$d(g^{(n)}(u)(t)(\eta,\xi),\mathbb{P}(t,u(t))(\eta,\xi)) < \frac{1}{2^n}$$

except on a finite number of intervals , having total length $\frac{1}{2^n}$,

(ii)

$$|\ g^{(n)}(u)(t)(\eta,\xi) - g^{(n-1)}(u)(t)(\eta,\xi)\ | < \frac{1}{2^{n+1}}, n \geq 1.$$

Assume the above to hold up to $n = \nu - 1$, we shall prove that it holds for $n = \nu$.

There exists an open set $\mathscr{S}^\nu$ such that all the maps $t \to v_i^{(\nu-1)}(t)(\eta,\xi)$ are continuous on $I \setminus \mathscr{S}^\nu$, and the measure of $\mathscr{S}^\nu$ is smaller than $\frac{1}{2^{\nu+1}}$.
Let $\delta > 0$ be such that $\|\ w - u\ \|_{\eta\xi} < \delta$ implies that for each i,
$|\ \tau_i^{(\nu-1)}(u) - \tau_i^{(\nu-1)}(w)\ | < (2^{-\nu}(4m(\nu-1)))^{-1}$. A finite number of $Q_{\hat{u}_j,\delta}$ covers $\mathscr{K}$. For each j call E_j the finite union of open intervals $|\ t - \tau_i(\hat{u}_j)\ | < (2^{-\nu}(4(\nu-1)))^{-1}$, $i = 1,...,m(\nu-1)$. Then whenever u is in $Q_{\hat{u}_j,\delta}$, when t is any of the closed intervals whose union is $I \setminus E_j$, $g^{\nu-1}(u)(t)(\eta,\xi) = g^{\nu-1}(\hat{u})(t)(\eta,\xi) = v_i^{\nu-1}(t)(\eta,\xi)$ for some i.
Hence when t belongs to the closed $(I \setminus E_j) \setminus \mathscr{S}^\nu$, the map $t \mapsto g^{\nu-1}u(t)(\eta,\xi)$ is continuous. Set $|\ \rho_j^\nu(t)(\eta,\xi)\ |$ to be 2λ on the open $(E_j \cup \mathscr{S}^\nu)$ and to be $\frac{1}{2^{\nu-1}}$ on the closed $I \setminus (E_j \cup \mathscr{S}^\nu)$. The map $(t,x) \to \mathbb{P}_j(t,x)(\eta,\xi)$ is defined by

$$\mathbb{P}_j(t,x)(\eta,\xi) = Q_{g^{\nu-1}(\hat{u}_j)(t)(\eta,\xi),|\rho_j^\nu(t)(\eta,\xi)|}(\eta,\xi) \bigcap \mathbb{P}(t,x)(\eta,\xi)$$

is strict for (t,x) in $Q_{(t,\hat{u}_j(t)),\delta}$.
In fact, when t is in $(E_j \cup \mathscr{S}^\nu)$. It is enough to remark that both $g^{\nu-1}$ and $\mathbb{P}$ take values in $Q_\lambda(\eta,\xi)$.
Let $(t,x) :\ t$ in $I \setminus (E_j \cup \mathscr{S}^\nu)$, $\|\ x - \hat{u}_j(t)\ \|_{\eta\xi} < \delta$. Then a translate $u(.)$ of $\hat{u}_j(.)$ is in $Q_{\hat{u}_j(.),\delta}$

and is such that $u(t) = x$. For this u, $g^{\nu-1}(u)(t)(\eta,\xi) = g^{\nu-1}(\hat{u}_j)(t)(\eta,\xi)$ and, by point (i) of the induction

$$d(g^{\nu-1}(u)(t)(\eta,\xi), \mathbb{P}(t,x)(\eta,\xi)) < \frac{1}{2^{\nu-1}}.$$

Since $t \to \rho^{\nu}(t)$ is lower semicontinuous and $\mathbb{P}_j$ is strict, proposition 2 implies that $(t,x) \to \mathbb{P}_j(t,x)(\eta,\xi)$ is lower semicontinuous. Set in Lemma 1, M to be $\mathbb{P}_j$, ϵ to be $\frac{1}{2^{\nu+1}}$ and call ω_j the constant provided by point (a). A finite number of $Q_{u^i_j,\omega_j}(\eta,\xi)$ covers the compact $\mathscr{K}_{\eta\xi} \cap Q_{\hat{u}_j,\delta}(\eta,\xi)$ By Lemma 1(b), there exists for each i a measurable $v^i_j(t)(\eta,\xi)$ such that $d_{\eta\xi}((t,x),(t,u^i_j(t))) < \omega_i$ implies

$$d(v^i_j(t)(\eta,\xi), \mathbb{P}_j(t,x)(\eta,\xi)) \leq \frac{1}{2^{\nu+1}} < \frac{1}{2^{\nu}}. \tag{8}$$

The collection of open sets $\mathscr{U}^i_j(\eta,\xi) = Q_{\hat{u}_j,\delta}(\eta,\xi) \cap Q_{u^i_j,\omega_j}(\eta,\xi)$ covers $\mathscr{K}_{\eta\xi}$. Let χ^i_j be the characteristic functions of the corresponding continuous partition $\{\mathscr{F}^j_{\eta\xi,i}\}$ of I. Set

$$g^{\nu}(u)(t)(\eta,\xi) = \sum_{i,j} \chi^i_j(t) v^i_j(t)(\eta,\xi).$$

We claim that the functions v^i_j and the map g^{ν} satisfy our induction assumptions.
Fix u and t. Whenever t belongs to $\mathscr{I}^j_i(u)$, $g^{\nu}(u)(t)(\eta,\xi) = v^i_j(t)(\eta,\xi)$ and u belongs to $Q_{u^i_j,w_j}$, and by (7),

$$\mathbf{d}(v^i_j(t)(\eta,\xi), \mathbb{P}_j(t,u(t))(\eta,\xi)) < \frac{1}{2^{\nu}}. \tag{9}$$

Since $\mathbb{P}_j(t,u(t))(\eta,\xi) \subset \mathbb{P}(t,u(t))(\eta,\xi)$, (8) check point (i).
To check point (ii) , assume t in $I \setminus (E_j \cup \mathscr{S}^{\nu})$. Then $\rho^{\nu}(t) = \frac{1}{2^{\nu-1}}$,

$$\mathbb{P}_j(t,x)(\eta,\xi)) \subset Q_{g^{\nu-1}(\hat{u}_j)(t)(\eta,\xi),\frac{1}{2^{\nu-1}}} = Q_{g^{\nu-1}(u)(t)(\eta,\xi),\frac{1}{2^{\nu-1}}}$$

hence

$$d(v^i_j(t)(\eta,\xi), Q_{g^{\nu-1}(u)(t)(\eta,\xi),\frac{1}{2^{\nu-1}}}(\eta,\xi)) < \frac{1}{2^{\nu}} \tag{10}$$

or

$$\mid g^{\nu}(u)(t)(\eta,\xi) - g^{\nu-1}(u)(t)(\eta,\xi) \mid < \frac{1}{2^{\nu+1}} \tag{11}$$

except on an open set $(E_j \cup \mathscr{S}^{\nu})$ with measure at most $\frac{1}{2^{\nu}}$. The sequence of measurable maps $\{g^n(u)(.)(\eta,\xi)\}$ is a Cauchy sequence converging to some measurable function that we denote by $g(u)(.)(\eta,\xi)$ and $g(u)(t)(\eta,\xi) \in \mathbb{P}(t,u(t))(\eta,\xi)$.
Let $K_{\eta\xi}$ be defined by:

$$K_{\eta\xi} = \{u(t)(\eta,\xi) = \langle \eta, u(t)\xi \rangle \in \mathscr{K}_{\eta\xi} : u(t)(\eta,\xi) \text{ is Lipschitzian and } u(t_0)(\eta,\xi) = x_0\}.$$

The continuous map $\varphi : K_{\eta\xi} \to K_{\eta\xi}$ defined by $\varphi(\langle \eta, u(t)\xi\rangle) = \langle \eta, \varphi(u)(t)\xi\rangle$

$$\langle \eta, \varphi(u)(t)\xi\rangle = x_0 + \int_{t_0}^{t} g(u)(s)(\eta,\xi)ds,$$

$\mid \langle \eta, \varphi(u)(t_2)\xi\rangle - \langle \eta, \varphi(u)(t_1)\xi\rangle \mid$ which gives

$$\mid \int_{t_1}^{t_2} g(u)(s)(\eta,\xi)ds \mid <\| g \|| \int_{t_1}^{t_2} u(s)(\eta,\xi)ds \mid$$
$$\in K_{\eta\xi}.$$

Moreover,

$$\frac{d}{dt}\langle \eta, \varphi(u)(t)\xi\rangle = g(u)(t)(\eta,\xi) \in \mathbb{P}(t,u(t))(\eta,\xi).$$

We are left to show the continuity of φ. In fact, we shall show directly that φ is uniformly continuous.

From a(ii) above, for every $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, u(t)(\eta,\xi) \in K_{\eta\xi}$

$$\begin{aligned}\int_I \mid g^{\nu+1}(u)(s)(\eta,\xi) - g^{\nu}(u)(s)(\eta,\xi) \mid ds &\le \frac{2T}{2^{\nu+1}} + \frac{2M}{2^{\nu+1}}\\ &= \frac{T+M}{2^{\nu}}\end{aligned}$$

so that

$$\begin{aligned}&\int_I \mid g^{n+1}(u)(s)(\eta,\xi) - g^{n}(u)(s)(\eta,\xi) \mid ds + \int_I \mid g^{n+2}(u)(s)(\eta,\xi) - g^{n+1}(u)(s)(\eta,\xi) \mid ds + ...\\ &\le (\frac{1}{2^n})(T+M)(1+\frac{1}{2}+\frac{1}{4}+...)\\ &= \frac{T+M}{2^{n-1}},\end{aligned}$$

and since $\int_I \mid g^{\nu}(u)(s)(\eta,\xi) - g^{(}u)(s)(\eta,\xi) \mid ds$ converges to 0,

$$\begin{aligned}\mid \varphi(\langle \eta, u(t)\xi\rangle) - \varphi(\langle \eta, w(t)\xi\rangle) \mid &\le \int_{t_0}^{t} \mid g(u)(s)(\eta,\xi) - g(w)(s)(\eta,\xi) \mid ds\\ \le \int_I \mid g^{n}(u)(s)(\eta,\xi) - g^{n}(w)(s)(\eta,\xi) \mid ds &+ \int_I \mid g^{n}(u)(s)(\eta,\xi) - g(u)(s)(\eta,\xi) \mid ds\\ &+ \int_I \mid g^{n}(w)(s)(\eta,\xi) - g(w)(s)(\eta,\xi) \mid ds\\ &\le \int_I \mid g^{n}(u)(s)(\eta,\xi) - g^{n}(w)(s)(\eta,\xi) \mid ds + \frac{4(T+M)}{2^n}.\end{aligned}$$

Also, by a(ii)

$$\int_I | g^n(u)(s)(\eta,\xi) - g^n(w)(s)(\eta,\xi) | \, ds \le \frac{2M}{2^{n+1}} = \frac{M}{2^n}.$$

Therefore $| u(t)(\eta,\xi) - w(t)(\eta,\xi) | \le \delta$ implies, for every $t \in I$,

$$| \langle \eta, \varphi(u)(t)\xi \rangle - \langle \eta, \varphi(w)(t)\xi \rangle | \le \frac{T+5M}{2^n} \le \epsilon.$$

Proving the continuity of φ. Hence φ is the required selection of $\mathscr{F}_{\eta\xi}$.

From above we have the following existence result.

Corollary 1. *For an arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, suppose that multivalued stochastic processes*

$$M : I \times \widetilde{\mathscr{A}} \to 2^{sesq.(\mathbb{D}\underline{\otimes}\mathbb{E})^2}, \quad M \in \{\mu E, \nu F, \sigma G, H\}$$

are compact-valued, lower semicontinuous multifunction.
Let $(t_0, x_0) \in I \times \widetilde{\mathscr{A}}$.
Then the problem

$$\begin{aligned} dX(t) &\in E(t,X(t))d\Lambda_\pi(t) + F(t,X(t))dA_f(t) \\ &\quad + G(t,X(t))dA_g^+(t) + H(t,X(t))dt \text{ almost all } t \in I, \\ X(t_0) &= x_0 \end{aligned} \tag{12}$$

has at least one solution defined on I lying in $Ad(\widetilde{\mathscr{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathscr{A}})$.

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