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Some Decompositions Of Continuity In Ideal Topological Spaces

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Abstract. In this paper, we introduce and investigate the notions of pre-$\mathscr{I}$-regular sets and β-$\mathscr{I}$-regular sets in ideal topological spaces. Also we introduce the notions of weak $AB_{\mathscr{I}}$-sets and $SC_{\mathscr{I}}$-sets and to obtain decompositions of continuity. Further we have obtained a decomposition of $\mathscr{I}$-R-continuity.

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Key Words and Phrases: pre-$\mathscr{I}$-regular set, β-$\mathscr{I}$-regular set, $SC_{\mathscr{I}}$-set, weak $AB_{\mathscr{I}}$-set, gsp-$\mathscr{I}$-closed set, pre-gsp-$\mathscr{I}$-closed set, α-gsp-$\mathscr{I}$-closed set, $SC_{\mathscr{I}}$-countinuous, weak $AB_{\mathscr{I}}$-countinuous, β-$\mathscr{I}$-perfectly continuous, contra α-gsp-$\mathscr{I}$-continuous, contra pre-gsp-$\mathscr{I}$-continuous

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1. Introduction and Preliminaries

An ideal $\mathscr{I}$ on a topological space (X,τ) is a non-empty collection of subsets of X satisfying the following properties:(1) $A\in\mathscr{I}$ and $B\subseteq A$ imply $B\in\mathscr{I}$ (heredity); (2) $A\in\mathscr{I}$ and $B\in\mathscr{I}$ imply $A\cup B\in\mathscr{I}$ (finite additivity). A topological space (X,τ) with an ideal $\mathscr{I}$ on X is called an *ideal topological space* and is denoted by $(X,\tau,\mathscr{I})$. For a subset $A\subseteq X$, $A^*(\mathscr{I})=\{x\in X : U\cap A\notin\mathscr{I}$ for every $U\in\tau(x)\}$, is called the *local function* [6] of A with respect to $\mathscr{I}$ and τ. We simply write A^* in case there is no chance for confusion. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathscr{I})$ called the $*$-topology finer than τ is defined by $cl^*(A)=A\cup A^*$ [11]. Let (X,τ) denote a topological space on which no separation axioms are assumed unless explicitly stated. In a topological space (X,τ), the closure and the interior of any subset A of X will be denoted by $cl(A)$ and $int(A)$, respectively.

Definition 1. *A subset A of an ideal space $(X,\tau,\mathscr{I})$ is called*

1. *α-$\mathscr{I}$-open [4] if $A\subseteq int(cl^*(int(A)))$.*
2. *semi-$\mathscr{I}$-open [4] if $A\subseteq cl^*(int(A))$.*

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3. *pre-$\mathscr{I}$-open [3] if* $A \subseteq int(cl^*(A))$.
4. *β-$\mathscr{I}$-open [4] if* $A \subseteq cl(int(cl^*(A)))$.
5. *t-$\mathscr{I}$-set [4] if* $int(cl^*(A)) = int(A)$.
6. *α^*-$\mathscr{I}$-set [4] if* $int(cl^*(int(A))) = int(A)$.
7. *S-$\mathscr{I}$-set [4] if* $cl^*(int(A)) = int(A)$.
8. *$C_{\mathscr{I}}$-set [4] if $A = C \cap D$ where $C \in \tau$ and D is an α^*-$\mathscr{I}$-set.*
9. *semi-$\mathscr{I}$-regular set [8] if A is both a t-$\mathscr{I}$-set and a semi-$\mathscr{I}$-open set.*
10. *$AB_{\mathscr{I}}$-set [8] if $A = C \cap D$ where $C \in \tau$ and D is a semi-$\mathscr{I}$-regular set.*
11. *$B_{\mathscr{I}}$-set [4] if $A = C \cap D$ where $C \in \tau$ and D is a t-$\mathscr{I}$-set.*
12. *regular-$\mathscr{I}$-closed set [7] if* $A = (int(A))^*$.
13. *$A_{\mathscr{I}}$-set [7] if $A = C \cap D$ where $C \in \tau$ and D is a regular-$\mathscr{I}$-closed set.*
14. *$\mathscr{I}$-R-open set [12] if* $A = int(cl^*(A))$.
15. *$\mathscr{I}$-locally closed set [3] (briefly, $\mathscr{I}$-LC set) if $A = U \cap V$ where U is open and V is $*$-perfect.*
16. *weakly-$\mathscr{I}$-locally closed set [9] (briefly, weakly-$\mathscr{I}$-LC set) if $A = U \cap V$ where U is open and V is $*$-closed.*

Definition 2. *An ideal space $(X, \tau, \mathscr{I})$ is said to be $\mathscr{I}$-submaximal [1] if every subset of X is $\mathscr{I}$-locally closed.*

Definition 3. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma)$ is said to be α-$\mathscr{I}$-continuous [4] (resp. semi-$\mathscr{I}$-continuous [4]) if $f^{-1}(V)$ is a α-$\mathscr{I}$-closed set (resp. semi-$\mathscr{I}$-closed set) in $(X, \tau, \mathscr{I})$ for each closed set V of (Y, σ).*

Definition 4. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma)$ is said to be contra β-$\mathscr{I}$-continuous [2] if $f^{-1}(V)$ is β-$\mathscr{I}$-closed in $(X, \tau, \mathscr{I})$ for every open set V of (Y, σ).*

2. Pre-$\mathscr{I}$-regular Sets

Definition 5. *A subset A of an ideal space $(X, \tau, \mathscr{I})$ is said to be pre-$\mathscr{I}$-regular if A is both an S-$\mathscr{I}$-set and a pre-$\mathscr{I}$-open set.*

Remark 1. *The concepts of S-$\mathscr{I}$-sets and pre-$\mathscr{I}$-open sets are independent.*

Example 1. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{a, d\}, \{a, b, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{b\}\}$. Then,*

1. *$A = \{c\}$ is an S-$\mathscr{I}$-set but not a pre-$\mathscr{I}$-open set.*

2. *$A = \{a, d\}$ is a pre-$\mathscr{I}$-open set but not an S-$\mathscr{I}$-set.*

Proposition 1. *Let $(X, \tau, \mathscr{I})$ be an ideal space and $A \subset X$. Then the following hold:*

1. *If A is pre-$\mathscr{I}$-regular, then A is a pre-$\mathscr{I}$-open set;*
2. *If A is pre-$\mathscr{I}$-regular, then A is an S-$\mathscr{I}$-set.*

The converse of Proposition 1 need not be true as seen from the following example.

Example 2. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{a, d\}, \{a, b, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{b\}\}$. Then,*

1. *$A = \{c\}$ is an S-$\mathscr{I}$-set but not a pre-$\mathscr{I}$-regular set.*
2. *$A = \{a, d\}$ is a pre-$\mathscr{I}$-open set but not a pre-$\mathscr{I}$-regular set.*

Remark 2.

1. *Semi-$\mathscr{I}$-open sets and pre-$\mathscr{I}$-regular sets are independent concepts.*
2. *α-$\mathscr{I}$-open sets and pre-$\mathscr{I}$-regular sets are independent concepts.*

Example 3. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{a, d\}, \{a, b, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{b\}\}$. Then,*

1. *$A = \{a\}$ is a pre-$\mathscr{I}$-regular set but not an α-$\mathscr{I}$-open set and a semi-$\mathscr{I}$-open set.*
2. *$A = \{a, d\}$ is an α-$\mathscr{I}$-open set and a semi-$\mathscr{I}$-open set but not a pre-$\mathscr{I}$-regular set.*

Definition 6. *A subset A of an ideal space $(X, \tau, \mathscr{I})$ is said to be an $SC_{\mathscr{I}}$-set if $A = C \cap D$ where $C \in \tau$ and D is a pre-$\mathscr{I}$-regular set.*

Proposition 2. *In an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

1. *Every open set is an $SC_{\mathscr{I}}$-set.*
2. *Every α-$\mathscr{I}$-open set is an $SC_{\mathscr{I}}$-set.*
3. *Every pre-$\mathscr{I}$-regular set is an $SC_{\mathscr{I}}$-set.*
4. *Every $SC_{\mathscr{I}}$-set is a $C_{\mathscr{I}}$-set.*

Proof. This is obvious.

Example 4. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{a, d\}, \{a, b, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{b\}\}$. Then,*

1. *$A = \{a\}$ is an $SC_{\mathscr{I}}$-set but not an open set.*
2. *$A = \{a\}$ is an $SC_{\mathscr{I}}$-set but not an α-$\mathscr{I}$-open set.*
3. *$A = \{a, d\}$ is an $SC_{\mathscr{I}}$-set but not a pre-$\mathscr{I}$-regular set.*
4. *$A = \{c\}$ is a $C_{\mathscr{I}}$-set but not an $SC_{\mathscr{I}}$-set.*

Theorem 1. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following are equivalent:*

1. *A is an open set;*

2. *A is a semi-$\mathscr{I}$-open set and an $SC_{\mathscr{I}}$-set.*

Proof. (1)⇒(2): This is obvious.
(2)⇒(1): Let A be a semi-$\mathscr{I}$-open set and an $SC_{\mathscr{I}}$-set. Then we have $A \subset cl^*(int(A))$. Also since A is an $SC_{\mathscr{I}}$-set we have $A = U \cap V$, where U is open and V is a pre-$\mathscr{I}$-regular set. Further since cl^* is a Kuratowski closure operator,

$$A \subset cl^*(int(A)) = cl^*(int(U \cap V)) = cl^*(int(U) \cap int(V)) \subseteq cl^*(int(U)) \cap cl^*(int(V)) \rightarrow (1).$$

Additionally, Since V is a pre-$\mathscr{I}$-regular set, V is also an S-$\mathscr{I}$-set. Thus $int(V) = cl^*(int(V))$. Using this in (1), we have $A \subset cl^*(int(U)) \cap int(V)$. Since $A \subset U$, we have

$$\begin{aligned} A &= U \cap A \subset U \cap (cl^*(int(U)) \cap int(V)) \\ &= (U \cap (cl^*(int(U))) \cap int(V) \\ &= U \cap int(V) \text{ and } \subset U \cap int(V). \end{aligned}$$

Since U is an open set, we have $A \subset U \cap int(V) = int(U \cap V) = int(A)$. Thus $A \in \tau$.

Remark 3. *The notions of semi-$\mathscr{I}$-open sets and $SC_{\mathscr{I}}$-sets are independent.*

Example 5. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{a, d\}, \{a, b, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{b\}\}$. Then,*

1. *$A = \{a\}$ is an $SC_{\mathscr{I}}$-set but not a semi-$\mathscr{I}$-open set.*

2. *$A = \{a, c, d\}$ is a semi-$\mathscr{I}$-open set but not an $SC_{\mathscr{I}}$-set.*

3. β-$\mathscr{I}$-regular Sets

Definition 7. *A subset A of an ideal space $(X, \tau, \mathscr{I})$ is said to be β-$\mathscr{I}$-regular if A is both a β-$\mathscr{I}$-open set and an α^*-$\mathscr{I}$-set.*

Remark 4. *β-$\mathscr{I}$-open sets and α^*-$\mathscr{I}$-sets are independent concepts.*

Example 6. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{c\}, \{a, c\}, X\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. Then,*

1. *$A = \{a, c\}$ is a β-$\mathscr{I}$-open set but not an α^*-$\mathscr{I}$-set.*

2. *$A = \{a\}$ is an α^*-$\mathscr{I}$-set but not a β-$\mathscr{I}$-open set.*

Proposition 3. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

1. *Every semi-$\mathscr{I}$-regular set is β-$\mathscr{I}$-regular.*

2. *Every pre-$\mathscr{I}$-regular set is β-$\mathscr{I}$-regular.*

3. *Every β-$\mathscr{I}$-regular set is a β-$\mathscr{I}$-open set.*

4. *Every β-$\mathscr{I}$-regular set is an α^*-$\mathscr{I}$-set.*

Proof.

1. Since every semi-$\mathscr{I}$-open set is β-$\mathscr{I}$-open and every t-$\mathscr{I}$-set is an α^*-$\mathscr{I}$-set [4], this is obvious.

2. Since every pre-$\mathscr{I}$-open set is β-$\mathscr{I}$-open and every S-$\mathscr{I}$-set is an α^*-$\mathscr{I}$-set [4], this is obvious.

3., 4. The proof follows from their definitions.

The converse of Proposition 3 need not be true as seen from the following examples.

Example 7. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{a, d\}, \{a, b, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{b\}\}$. Then,*

1. *$A = \{a\}$ is a β-$\mathscr{I}$-regular set but not a semi-$\mathscr{I}$-regular set.*

2. *$A = \{a, c\}$ is a β-$\mathscr{I}$-regular set but not a pre-$\mathscr{I}$-regular set.*

3. *$A = \{c\}$ is an α^*-$\mathscr{I}$-set but not a β-$\mathscr{I}$-regular set.*

Example 8. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{c\}, \{a, c\}, X\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. Then, $A = \{a, c\}$ is a β-$\mathscr{I}$-open set but not a β-$\mathscr{I}$-regular set.*

Remark 5. *For the sets defined above, we have the following implications:*

$$\begin{array}{ccc}
\text{Regular-}\mathscr{I}\text{-closed} \Rightarrow *\text{-perfect} \Rightarrow & & \tau^*\text{-closed} \\
\Downarrow & & \Downarrow \\
\text{semi-}\mathscr{I}\text{-regular} & \Longrightarrow & \text{t-}\mathscr{I}\text{-set} \\
\Downarrow & & \Downarrow \\
\beta\text{-}\mathscr{I}\text{-regular} & \Longrightarrow & \alpha^*\text{-}\mathscr{I}\text{-set} \\
\Downarrow & & \\
\beta\text{-}\mathscr{I}\text{-open} & &
\end{array}$$

Definition 8. *A subset A of an ideal space $(X, \tau, \mathscr{I})$ is said to be a weak $AB_{\mathscr{I}}$-set if $A = U \cap V$ where U is open and V is β-$\mathscr{I}$-regular.*

Proposition 4. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

1. *Every open set is a weak $AB_{\mathscr{I}}$-set.*

2. *Every β-$\mathscr{I}$-regular set is a weak $AB_{\mathscr{I}}$-set.*

3. *Every $AB_{\mathscr{I}}$-set is a weak $AB_{\mathscr{I}}$-set.*

4. *Every weak $AB_{\mathscr{I}}$-set is a $C_{\mathscr{I}}$-set.*

Proof. The proof is obvious.

The converse of Proposition 4 need not be true as seen from the following example.

Example 9. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{a, d\}, \{a, b, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{b\}\}$. Then,*

1. *$A = \{a\}$ is a weak $AB_{\mathscr{I}}$-set but not an open set.*
2. *$A = \{a, b, d\}$ is a weak $AB_{\mathscr{I}}$-set but not a β-$\mathscr{I}$-regular set.*
3. *$A = \{a\}$ is a weak $AB_{\mathscr{I}}$-set but not an $AB_{\mathscr{I}}$-set.*
4. *$A = \{c\}$ is a $C_{\mathscr{I}}$-set but not a weak $AB_{\mathscr{I}}$-set.*

Remark 6. *We have the following diagram:*

$$\begin{array}{ccccccc} \text{open set} \Rightarrow & A_{\mathscr{I}}\text{-set} & \Rightarrow & \mathscr{I}\text{-LC set} \Rightarrow & \text{weakly } \mathscr{I}\text{-LC set} \\ & \Downarrow & & & \Downarrow \\ & AB_{\mathscr{I}}\text{-set} & \Rightarrow & \text{weak } AB_{\mathscr{I}}\text{-set} \Rightarrow & C_{\mathscr{I}}\text{-set} \\ & & & \Uparrow & \\ & & & \beta\text{-}\mathscr{I}\text{-regular set} & \end{array}$$

Lemma 1. *[5] Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$.*

1. *If $U \in \tau$, then $U \cap cl^*(A) \subset cl^*(U \cap A)$.*
2. *If $A \subset S \subset X$, then $cl^*_S(A) = cl^*(A) \cap S$.*

Proposition 5. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. If A is a weak $AB_{\mathscr{I}}$-set then A is β-$\mathscr{I}$-open.*

Proof. Let A be a weak $AB_{\mathscr{I}}$-set. Then $A = U \cap V$, where U is open and V is a β-$\mathscr{I}$-regular set. Hence V is also a β-$\mathscr{I}$-open set. Since V is β-$\mathscr{I}$-open,

$$\begin{aligned} A =& U \cap V \subset U \cap cl(int(cl^*(V))) \subset cl^*(U) \cap cl(int(cl^*(V))) \\ &\subset cl(U \cap int(cl^*(V))) \subset cl(int(U) \cap int(cl^*(V))) \\ =& cl(int(U \cap cl^*(V))) \subset cl(int(cl^*(U \cap V))), \end{aligned}$$

by Lemma 1. Hence A is β-$\mathscr{I}$-open.

Theorem 2. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following are equivalent:*

1. *A is β-$\mathscr{I}$-regular.*
2. *A is α^*-$\mathscr{I}$-set and a weak $AB_{\mathscr{I}}$-set.*

Proof. (1)⇒(2): This is obvious, since every β-$\mathscr{I}$-regular set is an α^*-$\mathscr{I}$-set and a weak $AB_{\mathscr{I}}$-set.
(2)⇒(1): Let A be an α^*-$\mathscr{I}$-set and a weak $AB_{\mathscr{I}}$-set. The proof follows from the fact that weak $AB_{\mathscr{I}}$-set is β-$\mathscr{I}$-open.

Corollary 1. *Let $(X,\tau,\mathscr{I})$ be an $\mathscr{I}$-submaximal ideal topological space. Then weak $AB_{\mathscr{I}}(X) = \beta\mathscr{I}O(X)$.*

Proof. Since X is $\mathscr{I}$-submaximal, then by [10], every strong β-$\mathscr{I}$-open set of X is an $AB_{\mathscr{I}}$-set and hence a weak $AB_{\mathscr{I}}$-set. Conversely, since every weak $AB_{\mathscr{I}}$-set is β-$\mathscr{I}$-open, the proof is complete.

Theorem 3. *For a subset A of an ideal topological space $(X,\tau,\mathscr{I})$, the following are equivalent:*

1. *A is an open set.*
2. *A is a weak $AB_{\mathscr{I}}$-set and an α-$\mathscr{I}$-open set.*

Proof. (1)⇒(2): This is obvious.
(2)⇒(1): Let A be a weak $AB_{\mathscr{I}}$-set and an α-$\mathscr{I}$-open set. Since A is α-$\mathscr{I}$-open, we have $A \subseteq int(cl^*(int(A)))$. Furthermore, because A is a weak $AB_{\mathscr{I}}$-set we have $A = U \cap V$, where U is open and V is β-$\mathscr{I}$-regular. Now

$$\begin{aligned} A &\subseteq int(cl^*(int(A))) = int(cl^*(int(U\cap V))) = int(cl^*(int(U)\cap int(V))) \\ &\subset int[cl^*((int(U))\cap cl^*(int(V))] = int(cl^*((int(U)))\cap int(cl^*(int(V))). \end{aligned}$$

Additionally, since V is a β-$\mathscr{I}$-regular set, V is also an α^*-$\mathscr{I}$-set. Thus $int(cl^*(int(V))) = int(V)$. Therefore $A \subset int(cl^*((int(U)))\cap int(V)$. Besides, because $A \subset U$, we have

$$\begin{aligned} A &= U\cap A \subset U\cap(int(cl^*((int(U)))\cap int(V)) = (U\cap int(cl^*((int(U))))\cap int(V) \\ &= U\cap int(V). \end{aligned}$$

Since U is an open set, $A \subset U\cap int(V) = int(U\cap V) = int(A)$. Thus $A\in\tau$.

Remark 7. *The notions of α-$\mathscr{I}$-open sets and weak $AB_{\mathscr{I}}$-sets are independent.*

Example 10. *Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{c\},\{a,c\},X\}$ and $\mathscr{I} = \{\emptyset,\{c\}\}$. Then,*

1. *$A = \{a,c\}$ is an α-$\mathscr{I}$-open set but not a weak $AB_{\mathscr{I}}$-set.*
2. *$A = \{b\}$ is a weak $AB_{\mathscr{I}}$-set but not an α-$\mathscr{I}$-open set.*

4. Decompositions of Continuity

Definition 9. *A function $f : (X, \tau, \mathscr{I}) \rightarrow (Y, \sigma)$ is said to be $SC_{\mathscr{I}}$-continuous (resp. weakly $AB_{\mathscr{I}}$-continuous) if $f^{-1}(V)$ is an $SC_{\mathscr{I}}$-set (resp. weak $AB_{\mathscr{I}}$-set) in $(X, \tau, \mathscr{I})$ for each open set V of (Y, σ).*

Remark 8. 1. *Every continuous function is α-$\mathscr{I}$-continuous* [4] but not conversely.

2. Every continuous function is semi-$\mathscr{I}$-continuous [4] but not conversely.
3. Every continuous function is $SC_{\mathscr{I}}$-continuous but not conversely.
4. Every continuous function is weakly $AB_{\mathscr{I}}$-continuous but not conversely.

Theorem 4. *For a function $f : (X, \tau, \mathscr{I}) \rightarrow (Y, \sigma)$, the following are equivalent:*

1. *f is continuous.*
2. *f is semi-$\mathscr{I}$-continuous and $SC_{\mathscr{I}}$-continuous.*

Proof. Follows from Theorem 1.

Theorem 5. *For a function $f : (X, \tau, \mathscr{I}) \rightarrow (Y, \sigma)$, the following are equivalent:*

1. *f is continuous.*
2. *f is weakly $AB_{\mathscr{I}}$-continuous and α-$\mathscr{I}$-continuous.*

Proof. Follows from Theorem 3.

5. Decompositions of $\mathscr{I}$-R-Continuity

Definition 10. *A subset A of an ideal space $(X, \tau, \mathscr{I})$ is said to be gsp-$\mathscr{I}$-closed if $\beta\mathscr{I}cl(A) \subset U$ whenever $A \subset U$ and U is open.*

Definition 11. *A subset A of an ideal space $(X, \tau, \mathscr{I})$ is said to be pre-gsp-$\mathscr{I}$-closed (resp. α-gsp-$\mathscr{I}$-closed) if $\beta\mathscr{I}cl(A) \subset U$ whenever $A \subset U$ and U is pre-open (resp. α-open).*

Proposition 6. *For an ideal space $(X, \tau, \mathscr{I})$, the following hold:*

1. *Every $\mathscr{I}$-R-open set is an open set.*
2. *Every $\mathscr{I}$-R-open set is a β-$\mathscr{I}$-regular set.*

Proof.

1. Let A be an $\mathscr{I}$-R-open set. Then $A = int(cl^*(A))$. Hence $cl^*(A) = int(cl^*(cl^*(A))) = int(cl^*(A))$. This shows that $cl^*(A)$ is open. Since $A \subseteq cl^*(A)$, it follows that A is open.

2. Suppose A be an $\mathscr{I}$-R-open set. Then $A = int(cl^*(A)) \subseteq cl(int(cl^*(A)))$. This shows that A is β-$\mathscr{I}$-open. Again A is $\mathscr{I}$-R-open implies that $int(A) = int(cl^*(int(A)))$, which shows that A is an α^*-$\mathscr{I}$-set. Thus every $\mathscr{I}$-R-open set A is β-$\mathscr{I}$-open as well as an α^*-$\mathscr{I}$-set. Hence A is β-$\mathscr{I}$-regular.

The converse of Proposition 6 need not be true as seen from the following example.

Example 11. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{a, d\}, \{a, b, d\}, X\}$ and $\mathscr{I} = \{\emptyset, \{b\}\}$. Then,*

1. *$A = \{a, b, d\}$ is an open set but not an $\mathscr{I}$-R-open set.*
2. *$A = \{a\}$ is a β-$\mathscr{I}$-regular set but not an $\mathscr{I}$-R-open set.*

Proposition 7. *The following statements hold for a subset A of an ideal topological space $(X, \tau, \mathscr{I})$:*

1. *If A is a β-$\mathscr{I}$-regular set, then A is β-$\mathscr{I}$-closed.*
2. *If A is a β-$\mathscr{I}$-closed set, then A is pre-gsp-$\mathscr{I}$-closed.*
3. *If A is a pre-gsp-$\mathscr{I}$-closed set, then A is α-gsp-$\mathscr{I}$-closed.*
4. *If A is a pre-gsp-$\mathscr{I}$-closed set, then A is gsp-$\mathscr{I}$-closed.*
5. *If A is an α-gsp-$\mathscr{I}$-closed set, then A is gsp-$\mathscr{I}$-closed.*

Theorem 6. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following conditions are equivalent:*

1. *A is $\mathscr{I}$-R-open.*
2. *A is open and β-$\mathscr{I}$-regular.*
3. *A is open and β-$\mathscr{I}$-closed.*
4. *A is open and pre-gsp-$\mathscr{I}$-closed.*
5. *A is α-$\mathscr{I}$-open and pre-gsp-$\mathscr{I}$-closed.*
6. *A is α-$\mathscr{I}$-open and α-gsp-$\mathscr{I}$-closed.*

Proof. (1)⇒(2): This follows from Proposition 6.

The implications (2)⇒(3), (3)⇒(4), (4)⇒(5), (5)⇒(6) are obvious from their definitions. (6)⇒(1): Let A be an α-$\mathscr{I}$-open set and an α-gsp-$\mathscr{I}$-closed set. Since A is an α-gsp-$\mathscr{I}$-closed set, we have $\beta\mathscr{I}cl(A) \subset A$ and hence A is β-$\mathscr{I}$-closed. Therefore we obtain $int(cl(int^*(A))) \subseteq A$. Since every α-$\mathscr{I}$-open set is semi-$\mathscr{I}$-open [4], we have $cl^*(A) = cl^*(int(A))$. Further since A is α-$\mathscr{I}$-open we obtain

$$A \subseteq int(cl^*(A)) = int(cl^*(int(A))) \subseteq int(cl(int(A))) \subseteq int(cl(int^*(A))) \subseteq A.$$

Thus we have $A = int(cl^*(A))$. This shows that A is $\mathscr{I}$-R-open.

Remark 9. *In an ideal space $(X,\tau,\mathscr{I})$, the following hold:*

1. *The notions of open sets and β-$\mathscr{I}$-regular-sets are independent.*
2. *The notions of open sets and β-$\mathscr{I}$-closed sets are independent.*
3. *The notions of open sets and pre-gsp-$\mathscr{I}$-closed sets are independent.*
4. *The notions of α-$\mathscr{I}$-open sets and pre-gsp-$\mathscr{I}$-closed sets are independent.*
5. *The notions of α-$\mathscr{I}$-open sets and α-gsp-$\mathscr{I}$-closed sets are independent.*

Example 12. *Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,\{b\},\{a,d\},\{a,b,d\},X\}$ and $\mathscr{I}=\{\emptyset,\{b\}\}$. Then,*

1. *$A=\{a,b,d\}$ is an open set but neither a β-$\mathscr{I}$-regular set nor a β-$\mathscr{I}$-closed set.*
2. *$A=\{a,b,d\}$ is an open set but not a pre-gsp-$\mathscr{I}$-closed set.*
3. *$A=\{a\}$ is β-$\mathscr{I}$-regular, β-$\mathscr{I}$-closed and pre-gsp-$\mathscr{I}$-closed but not an open set.*
4. *$A=\{a,b,d\}$ is an α-$\mathscr{I}$-open set but neither a pre-gsp-$\mathscr{I}$-closed set nor an α-gsp-$\mathscr{I}$-closed set.*
5. *$A=\{a\}$ is both pre-gsp-$\mathscr{I}$-closed and α-gsp-$\mathscr{I}$-closed but not an α-$\mathscr{I}$-open set.*

Definition 12. *A function $f:(X,\tau,\mathscr{I})\to(Y,\sigma)$ is said to be $\mathscr{I}$-R-continuous (resp. β-$\mathscr{I}$-perfectly continuous) if $f^{-1}(V)$ is an $\mathscr{I}$-R-open set (resp. β-$\mathscr{I}$-regular set) in $(X,\tau,\mathscr{I})$ for each open set V of (Y,σ).*

Definition 13. *A function $f:(X,\tau,\mathscr{I})\to(Y,\sigma)$ is said to be contra pre-gsp-$\mathscr{I}$-continuous (resp. contra α-gsp-$\mathscr{I}$-continuous) if $f^{-1}(V)$ is a pre-gsp-$\mathscr{I}$-closed set (resp. α-gsp-$\mathscr{I}$-closed set) in $(X,\tau,\mathscr{I})$ for each open set V of (Y,σ).*

Theorem 7. *For a function $f:(X,\tau,\mathscr{I})\to(Y,\sigma)$ the following are equivalent:*

1. *f is $\mathscr{I}$-R-continuous.*
2. *f is continuous and β-$\mathscr{I}$-perfectly continuous.*
3. *f is continuous and contra β-$\mathscr{I}$-continuous.*
4. *f is continuous and contra pre-gsp-$\mathscr{I}$-continuous.*
5. *f is α-$\mathscr{I}$-continuous and contra pre-gsp-$\mathscr{I}$-continuous.*
6. *f is α-$\mathscr{I}$-continuous and contra α-gsp-$\mathscr{I}$-continuous.*

Proof. This is an immediate consequence of Theorem 6.

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