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Dual of a Wilson Frame

Shiv Kumar Kaushik[1,*], Suman Panwar[2]

[1] *Department of Mathematics, Kirori Mal College, University of Delhi, Delhi-110007, India*
[2] *Department of Mathematics, University of Delhi, Delhi-110007, India*

Abstract. In this paper Wilson frame as a generalization of Wilson bases has been defined. A sufficent condition for a Wilson system to be a Wilson Bessel sequence in terms of a Gabor Bessel sequence has been given. It is shown that the canonical dual frame of a Wilson frame may not have a Wilson structure.Also,a sufficient condition for two Wilson Bessel sequences to be dual frames has been given in terms of dual Gabor frames.

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1. Introduction

Gabor systems are time and frequency shifted images of a signal function f, called an atom. Gabor systems have become a popular tool, both in theory and applications. However, one drawback in view of Balian-Low Theorem is that it is impossible to construct Gabor bases for $L^2(\mathbb{R})$ having good time-frequency localization [6]. Replacing the frequency-shift (modulation) with multiplication by suitably chosen trigonometric functions, we get a system called Wilson system which under certain conditions is an orthonormal basis.

Using ideas of Wilson [8, 9], Daubechies, Jaffard and Journe [3] constructed orthonormal Wilson bases which have good localization properties in time and frequency simultaneously. In [4], it has been proved that Wilson bases of exponential decay are unconditional bases for all modulation spaces on $\mathbb{R}$ including the classical Bessel potential space and the Schwartz spaces. Also, Wilson bases are no unconditional bases for the ordinary L^p-spaces for $p \neq 2$ [4]. Approximation properties of Wilson bases are studied in [1].

Generalizations of Wilson bases to non-rectangular lattices are discussed in [7] with motivation from wireless communication and cosines modulated filter banks. Modified Wilson orthonormal bases are studied in [10].

This paper starts with the definition of a Wilson system [5] followed by the definition of a

*Corresponding author.

Email addresses: shikk2003@yahoo.co.in (S. Kaushik), spanwar87@gmail.com (S. Panwar)

Wilson frame.

In this paper, a sufficient condition for a Wilson system to be a Wilson Bessel sequence in terms of a Gabor Bessel sequence is given.It is shown that the canonical dual frame of a Wilson frame may not have a Wilson structure.Finally, a sufficient condition for two Wilson Bessel sequences to be dual frames is given.

2. Preliminaries

Definition 1. *A sequence $\{x_n\}$ in a Hilbert space H is said to be a frame for H if there exist constants A and B with $0 < A \leq B < \infty$ such that*

$$A\|x\|^2 \leq \sum_n |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \ x \in H \tag{1}$$

The positive constants A and B, respectively, are called lower and upper frame bounds for the frame $\{x_n\}$. The inequality (1) in Definition 1, is called the *frame inequality* for the frame $\{x_n\}$.A sequence $\{x_n\} \in H$ is called a *Bessel sequence* if it satisfies upper frame inequality in (1) of Definition 1.

Definition 2. *For a Bessel sequence $\{x_n\}$ in a Hilbert space H, the frame operator S is defined as $S : H \to H$ such that $Sx = \sum\langle x, x_n\rangle x_n$,for all x in H.*

Daubechies, Grossmann and Meyer [3] were credited for combining Gabor analysis with frame theory. They were the first to construct tight frames for $L^2(\mathbb{R})$ having the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, where

$$E_{mb} : L^2(\mathbb{R}) \to Ł^2(\mathbb{R}), (E_{mb}g)(x) = e^{2\pi imbx}g(x - na)$$

and

$$T_{na} : L^2(\mathbb{R}) \to Ł^2(\mathbb{R}), (T_{na}g)(x) = g(x - na), \ a > 0, b > 0.$$

Definition 3. *Let $g \in L^2(\mathbb{R})$ and a, b be positive constants. Then, the sequence $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is called a Gabor system for $L^2(\mathbb{R})$. Further, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is called a Gabor frame for $L^2(\mathbb{R})$,if there exist constants A and B with $0 < A \leq B < \infty$ such that*

$$A\|f\|^2 \leq \sum_n |\langle f, E_{mb}T_{na}g\rangle|^2 \leq B\|f\|^2, \ f \in L^2(\mathbb{R}) \tag{2}$$

The sequence $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is called a Gabor Bessel sequence for $L^2(\mathbb{R})$ if it satisfies the upper frame inequality in (2) in Definition 3.

For literature related to Gabor frames, one may refer to [2]

Definition 4 ([5])**.** *For $g \in L^2(\mathbb{R})$, the associated Wilson system $W(g) = \{\psi_{k,n}g\}_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}$ is given by the functions*

$$\psi_{k,\ n}g = c_n T_{\frac{k}{2}}(E_n + (-1)^{k+n}E_{-n})g \ , \ k \in \mathbb{Z} \text{ and } n \in \mathbb{N}_0,$$

where $c_0 = \frac{1}{2}$, and $c_n = \frac{1}{\sqrt{2}}$ for $n \geq 1$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

3. Main Results

We begin this section with the definition of a Wilson frame.

Definition 5. *The Wilson system* $W(g) = \{\psi_{k,\,n} g\}_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}$ *for* $L^2(\mathbb{R})$ *is called a* Wilson frame*, if there exist constants A and B with* $0 < A \le B < \infty$ *such that*

$$A\|f\|^2 \le \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} |\langle f, \psi_{k,n} g\rangle|^2 \le B\|f\|^2, \text{ for all } f \in L^2(\mathbb{R}). \tag{3}$$

The constants A and B are called lower frame bound and upper frame bound respectively for the Wilson frame $W(g)$.The Wilson system $W(g) = \{\psi_{k,n} g\}_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}$ is called a *Wilson Bessel sequence* if it satisfies upper frame inequality in (3) in Definition 5.

In the following result, we give a sufficient condition for a Wilson system to be a Wilson Bessel sequence in terms of a Gabor Bessel sequence.

Theorem 1. *Let* $g \in L^2(\mathbb{R})$*.Let* $\{E_n T_{\frac{k}{2}} g\}_{n,\,k\in\mathbb{Z}}$ *be a Gabor Bessel sequence with Bessel bound B.Then the Wilson system* $\{\psi_{k,\,n} g\}_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}$ *which can be expressed as*

$$\{(-1)^{kn} c_n (E_n T_{\frac{k}{2}} + (-1)^{k+n} E_{-n} T_{\frac{k}{2}}) g\}_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}$$

is a Wilson Bessel sequence with Bessel bound B.

Proof. For $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, we have

$$\begin{aligned}
\psi_{k,\,n} g =& c_n T_{\frac{k}{2}} E_n g + c_n (-1)^{k+n} T_{\frac{k}{2}} E_{-n} g \\
=& e^{-2\pi i \frac{k}{2} n} c_n E_n T_{\frac{k}{2}} g + e^{-2\pi i \frac{k}{2}(-n)} c_n (-1)^{k+n} E_{-n} T_{\frac{k}{2}} g \\
=& (-1)^{kn} c_n E_n T_{\frac{k}{2}} g + (-1)^{kn+k+n} c_n E_{-n} T_{\frac{k}{2}} g \\
=& (-1)^{kn} c_n (E_n T_{\frac{k}{2}} + (-1)^{k+n} E_{-n} T_{\frac{k}{2}}) g.
\end{aligned}$$

Also, since $\{E_n T_{\frac{k}{2}} g\}_{n,\,k\in\mathbb{Z}}$ is a Gabor Bessel sequence with Bessel bound B, we have

$$\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{Z}}} |\langle E_n T_{\frac{k}{2}} g, f\rangle|^2 \le B\|f\|^2, \text{ for all } f \in L^2(\mathbb{R})$$

Note that

$$\begin{aligned}
\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} |\langle \psi_{k,n} g, f\rangle|^2 &= \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} |\langle (-1)^{kn} c_n (E_n T_{\frac{k}{2}} g + (-1)^{k+n} E_{-n} T_{\frac{k}{2}} g), f\rangle|^2 \\
&= \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} |(-1)^{kn} c_n \langle E_n T_{\frac{k}{2}} g, f\rangle + (-1)^{kn+k+n} c_n \langle E_{-n} T_{\frac{k}{2}} g, f\rangle|^2
\end{aligned}$$

$$=\sum_{k\in\mathbb{Z}}|\frac{1}{2}\langle E_0T_{\frac{k}{2}}g,f\rangle+(-1)^k\frac{1}{2}\langle E_0T_{\frac{k}{2}}g,f\rangle|^2$$
$$+\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}}}|\frac{1}{\sqrt{2}}(-1)^{kn}\langle E_nT_{\frac{k}{2}}g,f\rangle+\frac{(-1)^{kn+k+n}}{\sqrt{2}}\langle E_{-n}T_{\frac{k}{2}}g,f\rangle|^2$$
$$\leq\frac{1}{4}\sum_{k\in\mathbb{Z}}(|\langle E_0T_{\frac{k}{2}}g,f\rangle|^2+|\langle E_0T_{\frac{k}{2}}g,f\rangle|^2)$$
$$+\frac{1}{2}\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}}}(|\langle E_nT_{\frac{k}{2}}g,f\rangle|^2+|\langle E_{-n}T_{\frac{k}{2}}g,f\rangle|^2)$$

Hence,we have

$$\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}|\langle\psi_{k,n}g,f\rangle|^2\leq\sum_{k,n\in\mathbb{Z}}|\langle E_nT_{\frac{k}{2}}g,f\rangle|^2\leq B\|f\|^2,\text{ for all} f\in\ L^2(\mathbb{R}).$$

Remark 1. *By frame decomposition, we know that if $\{x_n\}$ is a frame in a Hilbert space H with frame operator S, then $x=\sum\langle x,S^{-1}x_n\rangle x_n$, for all x in H. In case of a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, we know that*

$$\begin{aligned} f &= \sum_{m,n\in\mathbb{Z}}\langle f,S^{-1}(E_{mb}T_{na}g)\rangle E_{mb}T_{na}g\\ &= \sum_{m,n\in\mathbb{Z}}\langle f,E_{mb}T_{na}S^{-1}g\rangle E_{mb}T_{na}g. \end{aligned}\tag{4}$$

In the following Theorem we prove that (4) of Remark 1 is partially satisfied by a Wilson frame.

Theorem 2. *Let $g\in L^2(\mathbb{R})$ and assume that $\{\psi_{k,\ n}g\}_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}$ is a Wilson Bessel sequence with frame operator S. Let $k'\in\mathbb{Z},n'\in\mathbb{N}_0$. If $k'+n'$ is even, then $S\psi_{k',\ n'}g=\psi_{k',\ n'}Sg$. Further, if $\{\psi_{k,\ n}g\}_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}$ is a Wilson frame, then $S^{-1}\psi_{k',\ n'}g=\psi_{k',\ n'}S^{-1}g$.*

Proof. Let $f\in L^2(\mathbb{R})$,and assume that $\{\psi_{k,\ n}g\}_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}$ is a Wilson Bessel sequence. We have

$$S\psi_{k',n'}f=\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}\langle\psi_{k',n'}f,\psi_{k,n}g\rangle\psi_{k,n}g$$
$$=\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+kn}c_nc_n'\langle E_{n'}T_{\frac{k'}{2}}f,E_nT_{\frac{k}{2}}g\rangle\psi_{k,n}g$$
$$+\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+kn+k+n}c_nc_n'\langle E_{n'}T_{\frac{k'}{2}}f,E_{-n}T_{\frac{k}{2}}g\rangle\psi_{k,n}g$$

$$+\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+kn+k'+n'}c_n c_n'\langle E_{-n'}T_{\frac{k'}{2}}f, E_n T_{\frac{k}{2}}g\rangle\psi_{k,n}g$$
$$+\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+k'+n'+kn+k+n}c_n c_n'\langle E_{-n'}T_{\frac{k'}{2}}f, E_{-n} T_{\frac{k}{2}}g\rangle\psi_{k,n}g$$

Also, we have

$$\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+kn}c_n c_n'\langle E_{n'}T_{\frac{k'}{2}}f, E_n T_{\frac{k}{2}}g\rangle\psi_{k,n}g$$
$$=\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+kn}c_n c_n'\langle E_{n'}T_{\frac{k'}{2}}f, E_n T_{\frac{k}{2}}g\rangle\{(-1)^{kn}c_n(E_n T_{\frac{k}{2}}+(-1)^{k+n}E_{-n}T_{\frac{k}{2}})g\}$$

Note that for $f\in L^2(\mathbb{R})$,

$$T_aE_bf(x)=\exp(-2\pi iba)E_bT_af(x). \tag{5}$$

Using commutator relations given in (5) in Theorem 2

$$\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+kn}c_n c_n'\langle E_{n'}T_{\frac{k'}{2}}f, E_n T_{\frac{k}{2}}g\rangle\psi_{k,n}g$$
$$=\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'}c_n^2 c_n'\langle f,\exp(2\pi i\frac{k}{2}(n-n'))E_{n-n'}T_{\frac{k-k'}{2}}g\rangle E_nT_{\frac{k}{2}}g$$
$$+\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+k+n}c_n^2 c_n'\langle f,\exp(2\pi i\frac{k}{2}(n-n'))E_{n-n'}T_{\frac{k-k'}{2}}g\rangle E_{-n}T_{\frac{k}{2}}g$$

performing the change of variables $n\to n+n', k\to k+k'$ and using the commutator relations given in (5) in Theorem 2 again, we obtain

$$\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+kn}c_n c_n'\langle E_{n'}T_{\frac{k'}{2}}f, E_n T_{\frac{k}{2}}g\rangle\psi_{k,n}g$$
$$=(-1)^{k'n'}c_n'(\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}c_n^2\langle f,E_nT_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g$$
$$+\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k+n}c_n^2\langle f,E_nT_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g)$$

Similarly,

$$\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+kn+k+n}c_n c_n'\langle E_{n'}T_{\frac{k'}{2}}f, E_{-n} T_{\frac{k}{2}}g\rangle\psi_{k,n}g$$

$$=(-1)^{k'n'}c'_n(\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g$$

$$+\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k+n}c_n^2\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g)$$

and

$$\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+kn+k'+n'}c_nc'_n\langle E_{-n'}T_{\frac{k'}{2}}f,E_nT_{\frac{k}{2}}g\rangle\psi_{k,n}g$$

$$=(-1)^{k'n'}c'_n((-1)^{k'+n'}\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}c_n^2\langle f,E_nT_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g$$

$$+\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k+n}c_n^2\langle f,E_nT_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g)$$

and

$$\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k'n'+k'+n'+kn+k+n}c_nc'_n\langle E_{-n'}T_{\frac{k'}{2}}f,E_{-n}T_{\frac{k}{2}}g\rangle\psi_{k,n}g$$

$$=(-1)^{k'n'+k'+n'}c'_n(\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}c_n^2\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g$$

$$+\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}(-1)^{k+n}c_n^2\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g)$$

Finally,we obtain

$$S\psi_{k',n'}f=(-1)^{k'n'}c'_n\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}c_n^2\{\langle f,E_nT_{\frac{k}{2}}g\rangle$$

$$E_{n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g+(-1)^{k+n}\langle f,E_nT_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g\}$$

$$+(-1)^{k'n'}c'_n\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}c_n^2\{\langle f,E_{-n}T_{\frac{k}{2}}g\rangle$$

$$E_{n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g+(-1)^{k+n}\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g\}$$

$$+(-1)^{k'n'}c'_n\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}c_n^2\{(-1)^{k+n}\langle f,E_nT_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g\}$$

$$+(-1)^{k'n'+k'+n'}c'_n\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}c_n^2\{\langle f,E_nT_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g$$

$$+(-1)^{k+n}\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g\}$$

$$+(-1)^{k'n'+k'+n'}c'_n\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g\}$$

Also, we have

$$\begin{aligned}Sf=&\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f,E_nT_{\frac{k}{2}}g\rangle E_nT_{\frac{k}{2}}g+(-1)^{k+n}\langle f,E_nT_{\frac{k}{2}}g\rangle E_{-n}T_{\frac{k}{2}}g\}\\ &+\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{(-1)^{k+n}\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_nT_{\frac{k}{2}}g+\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{-n}T_{\frac{k}{2}}g\}\end{aligned}$$

Therefore

$$\begin{aligned}\psi_{k',n'}Sf=&(-1)^{k'n'}c'_n\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f,E_nT_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g+(-1)^{k+n}\langle f,E_nT_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g\}\\ &+(-1)^{k'n'}c'_n\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g\\ &\quad+(-1)^{k+n}\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g\}\\ &+(-1)^{k'n'+k'+n'}c'_n\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f,E_nT_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g\\ &\quad+(-1)^{k+n}\langle f,E_nT_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g\}\\ &+(-1)^{k'n'+k'+n'}c'_n\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g\\ &\quad+(-1)^{k+n}\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g\}\end{aligned}$$

Note that if $k'+n'$ is an even integer,then $S\psi_{k',n'}f=\psi_{k',n'}Sf$ for all $f\in L^2(\mathbb{R})$. Thus $S\psi_{k',n'}=\psi_{k',n'}S$.
Further, if $\{\psi_{k,\,n}g\}_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}}$ is a Wilson frame, then S is invertible. So, we have
$S^{-1}S\psi_{k',n'}S^{-1}=S^{-1}\psi_{k',n'}SS^{-1}$. Hence,$\psi_{k',n'}(S^{-1}g)=S^{-1}(\psi_{k',n'}g)$

Remark 2. *The result in Theorem* 2 *does not hold if for $k'\in\mathbb{Z}$,and $n'\in\mathbb{N}_0$,$k'+n'$is an odd integer.*
In this case, we have

$$\begin{aligned}S\psi_{k',n'}f=&c'_n\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f,E_nT_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g+(-1)^{k+n}\langle f,E_nT_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g\}\\ &+c'_n\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{n'}T_{\frac{k'}{2}}gE_{-n}T_{\frac{k}{2}}g+(-1)^{k+n}\langle f,E_{-n}T_{\frac{k}{2}}g\rangle E_{-n'}T_{\frac{k'}{2}}gE_nT_{\frac{k}{2}}g\}\end{aligned}$$

$$+ c'_n \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{(-1)^{k+n}\langle f, E_n T_{\frac{k}{2}} g\rangle E_{n'} T_{\frac{k'}{2}} g E_{-n} T_{\frac{k}{2}} g\}$$
$$- c'_n \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f, E_n T_{\frac{k}{2}} g\rangle E_{-n'} T_{\frac{k'}{2}} g E_n T_{\frac{k}{2}} g + (-1)^{k+n}\langle f, E_{-n} T_{\frac{k}{2}} g\rangle E_{n'} T_{\frac{k'}{2}} g E_n T_{\frac{k}{2}} g\}$$
$$- c'_n \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f, E_{-n} T_{\frac{k}{2}} g\rangle E_{-n'} T_{\frac{k'}{2}} g E_{-n} T_{\frac{k}{2}} g\}$$

and

$$\begin{aligned}\psi_{k',n'} S f =& c'_n \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f, E_n T_{\frac{k}{2}} g\rangle E_{n'} T_{\frac{k'}{2}} g E_n T_{\frac{k}{2}} g + (-1)^{k+n}\langle f, E_n T_{\frac{k}{2}} g\rangle E_{n'} T_{\frac{k'}{2}} g E_{-n} T_{\frac{k}{2}} g\}\\
&+ c'_n \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f, E_{-n} T_{\frac{k}{2}} g\rangle E_{n'} T_{\frac{k'}{2}} g E_{-n} T_{\frac{k}{2}} g + (-1)^{k+n}\langle f, E_{-n} T_{\frac{k}{2}} g\rangle E_{n'} T_{\frac{k'}{2}} g E_n T_{\frac{k}{2}} g\}\\
&- c'_n \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f, E_n T_{\frac{k}{2}} g\rangle E_{-n'} T_{\frac{k'}{2}} g E_n T_{\frac{k}{2}} g + (-1)^{k+n}\langle f, E_n T_{\frac{k}{2}} g\rangle E_{-n'} T_{\frac{k'}{2}} g E_{-n} T_{\frac{k}{2}} g\}\\
&- c'_n \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} c_n^2\{\langle f, E_{-n} T_{\frac{k}{2}} g\rangle E_{-n'} T_{\frac{k'}{2}} g E_{-n} T_{\frac{k}{2}} g + (-1)^{k+n}\langle f, E_{-n} T_{\frac{k}{2}} g\rangle E_{-n'} T_{\frac{k'}{2}} g E_n T_{\frac{k}{2}} g\}\end{aligned}$$

Thus, $\psi_{k',n'} S f \neq S\psi_{k',n'} f$.

Remark 3. *The canonical dual frame of a frame* $\{x_n\}$ *with the frame operator* S *is given by* $\{S^{-1}\{x_n\}\}$. *The canonical dual frame of a Gabor frame has a Gabor structure but Theorem* 2 *and Remark* 2 *shows that the canonical dual frame for a Wilson frame does not have a Wilson structure.*

Now we give a sufficient condition for the Wilson systems $W(g)$ and $W(h)$ to be dual Wilson frames. First, we prove a result in the form of a Lemma which will be used in the main result.

Lemma 1. *For* f, g *in* $L^2(\mathbb{R})$ *let* $W(g)$ *and* $W(h)$ *be two Wilson Bessel sequences.* $W(g)$ *and* $W(h)$ *are dual Wilson frames if and only if*

$$\langle e, f\rangle = \frac{1}{2}\sum_{k,n\in\mathbb{Z}} \langle (-1)^{k+n} T_{\frac{k}{2}} E_{-n} h, f\rangle\langle e, T_{\frac{k}{2}} E_n g\rangle + \frac{1}{2}\sum_{k,n\in\mathbb{Z}} \langle T_{\frac{k}{2}} E_n h, f\rangle\langle e, T_{\frac{k}{2}} E_n g\rangle,$$

for all e, f *in* $L^2(\mathbb{R})$

Proof. If $W(g)$ and $W(h)$ are two Bessel sequences, then they are dual frames if and only if

$$\langle e, f\rangle = \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} \langle W(h), f\rangle\langle e, W(g)\rangle, \text{ for all } e, f \in L^2(\mathbb{R})$$

$$\Leftrightarrow \langle e,f\rangle = \sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}_0}} \langle e, c_n T_{\frac{k}{2}} E_n g + (-1)^{k+n} c_n T_{\frac{k}{2}} E_{-n} g\rangle \langle c_n T_{\frac{k}{2}} E_n h + (-1)^{k+n} c_n T_{\frac{k}{2}} E_{-n} h, f\rangle$$

$$\Leftrightarrow \langle e,f\rangle = \frac{1}{2}\sum_{k,n\in\mathbb{Z}} \langle T_{\frac{k}{2}} E_n h, f\rangle \langle e, T_{\frac{k}{2}} E_n g\rangle + \frac{1}{2}\sum_{k\in\mathbb{Z}} (-1)^k \langle T_{\frac{k}{2}} E_0 h, f\rangle \langle e, T_{\frac{k}{2}} E_0 g\rangle$$
$$+ \frac{1}{2}\sum_{\substack{k\in\mathbb{Z}\\ n\in\mathbb{N}}} (-1)^{k+n} \langle T_{\frac{k}{2}} E_{-n} h, f\rangle \langle e, T_{\frac{k}{2}} E_n g\rangle + (-1)^{k-n} \langle T_{\frac{k}{2}} E_n h, f\rangle \langle e, T_{\frac{k}{2}} E_{-n} g\rangle$$

$$\Leftrightarrow \langle e,f\rangle = \frac{1}{2}\sum_{k,n\in\mathbb{Z}} \langle T_{\frac{k}{2}} E_n h, f\rangle \langle e, T_{\frac{k}{2}} E_n g\rangle + \frac{1}{2}\sum_{k,n\in\mathbb{Z}} \langle (-1)^{k+n} T_{\frac{k}{2}} E_{-n} h, f\rangle \langle e, T_{\frac{k}{2}} E_n g\rangle$$

Theorem 3. *Let $g, h \in L^2(\mathbb{R})$ and suppose that*

(a) $\{T_{\frac{k}{2}} E_n h\}_{k,n\in\mathbb{Z}}$ *and* $\{T_{\frac{k}{2}} E_n g\}_{k,n\in\mathbb{Z}}$ *be dual frames.*

(b) $\{(-1)^{k+n} T_{\frac{k}{2}} E_{-n} h\}_{k,n\in\mathbb{Z}}$ *and* $\{T_{\frac{k}{2}} E_n g\}_{k,n\in\mathbb{Z}}$ *be dual frames.*

Then the Wilson systems $W(g)$ and $W(h)$ are dual Wilson frames.

Proof. By hypothesis (a) and (b),

$$\langle e,f\rangle = \sum_{k,n\in\mathbb{Z}} \langle T_{\frac{k}{2}} E_n h, f\rangle \langle e, T_{\frac{k}{2}} E_n g\rangle$$

and

$$\langle e,f\rangle = \sum_{k,n\in\mathbb{Z}} \langle (-1)^{k+n} T_{\frac{k}{2}} E_{-n} h, f\rangle \langle e, T_{\frac{k}{2}} E_n g\rangle \text{ for all } e, f \in L^2(\mathbb{R})$$

Now using Lemma 1 the result follows.

Remark 4. *In view of Theorem 3 and commutator relations in (5) in Theorem 2 a sufficient condition for two Wilson Bessel sequences $W(g)$ and $W(h)$ to be dual frames in terms of dual Gabor Bessel sequences is obtained.*

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