EUROPEAN JOURNAL OF MATHEMATICAL SCIENCES
Vol. 2, No. 1, 2013, 41-50
ISSN 2147-5512 – www.ejmathsci.com

Generalizations of Hadamard Product of Certain Meromorphic Multivalent Functions with Positive Coefficients

R. M. El-Ashwah[1], M. K. Aouf[2], A. A. M. Hassan[3] and A. H. Hassan[3,*]

[1] *Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt*
[2] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 33516, Egypt*
[3] *Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt*

Abstract. In this paper, we used the generalization of the modified-Hadamard products for p-valent meromorphic functions to obtain some results for the classes $\sum_p S_n^*(\alpha,\beta)$ and $\sum_p K_n(\alpha,\beta)$, which represent the classes of meromorphically p-valent starlike of order α and type β and meromorphically p-valent convex of order α and type β respectively.

2010 Mathematics Subject Classifications: 30C45

Key Words and Phrases: Analytic, p-valent meromorphic, Hadamard product.

1. Introduction

Let $\sum_{p,n}$ denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} a_k z^k \quad (a_k \geq 0; n \geq p; p \in \mathbb{N} = \{1,2,3,\dots\}) \tag{1}$$

that are analytic and p-valent in the punctured disk
$U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}$ $(U = \{z \in \mathbb{C} : |z| < 1\})$.
A function $f(z) \in \sum_{p,n}$ is said to be meromorphically p-valent starlike of order α if it is satisfying the following (see Aouf and Hossen [3] and Kumar et al. [9]):

$$\text{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < p; z \in U^*), \tag{2}$$

*Corresponding author.

Email addresses: `r_elashwah@yahoo.com` (R. El-Ashwah), `mkaouf127@yahoo.com` (M. Aouf), `aam_hassan@yahoo.com` (A. Hassan), `alaahassan1986@yahoo.com` (A. Hassan)

also a function $f(z) \in \sum_{p,n}$ is said to be meromorphically p-valent convex of order α if it is satisfying the following (see Nunokawa and Ahuja [14]):

$$\operatorname{Re}\left\{-\left(1+\frac{zf^{''}(z)}{f^{'}(z)}\right)\right\} > \alpha \quad (0 \le \alpha < p; z \in U^{*}). \tag{3}$$

We denote by $\sum_p S_n^*(\alpha)$ and $\sum_p K_n(\alpha)$ the classes of meromorphically p-valent starlike of order α and meromorphically p-valent convex of order α respectively, we note that

$$f(z) \in \sum_p K_n(\alpha) \Longleftrightarrow -\frac{zf^{'}(z)}{p} \in \sum_p S_n^*(\alpha). \tag{4}$$

We note that the classes $\sum_1 S_1^*(\alpha) = \sum S^*(\alpha)$ and $\sum_1 K_1(\alpha) = \sum K(\alpha)$ are the classes of meromorphically univalent starlike functions of order α and meromorphically univalent convex functions of order α respectively, which have been extensively studied by Pommerenke [15], Clunie [6], Royster [16], Miller [10], Juneja and Reddy [8] and Mogra [12] and others.

Moreover a function $f(z) \in \sum_{p,n}$ is said to be meromorphically p-valent starlike of order α and type β if it is satisfying the following inequality (see Aouf [1] and Mogra [11]):

$$\left|\frac{\frac{zf^{'}(z)}{f(z)}+p}{\frac{zf^{'}(z)}{f(z)}+2\alpha-p}\right| < \beta \quad (0 \le \alpha < p; 0 < \beta \le 1; z \in U^{*}), \tag{5}$$

also a function $f(z) \in \sum_{p,n}$ is said to be meromorphically p-valent convex of order α and type β if it is satisfying the following inequality:

$$\left|\frac{1+\frac{zf^{''}(z)}{f^{'}(z)}+p}{1+\frac{zf^{''}(z)}{f^{'}(z)}+2\alpha-p}\right| < \beta \quad (0 \le \alpha < p; 0 < \beta \le 1; z \in U^{*}). \tag{6}$$

We denote by $\sum_p S_n^*(\alpha,\beta)$ and $\sum_p K_n(\alpha,\beta)$ the classes of meromorphically p-valent starlike of order α and type β and meromorphically p-valent convex of order α and type β respectively, we note that

$$f(z) \in \sum_p K_n(\alpha,\beta) \Longleftrightarrow -\frac{zf^{'}(z)}{p} \in \sum_p S_n^*(\alpha,\beta). \tag{7}$$

We note that the class $\sum_1 S_1^*(\alpha,\beta) = \sum S^*(\alpha,\beta)$ was introduced and studied by Mogra et al. [13].

For the functions

$$f_j(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0; j = 1,2; n \geq p; p \in \mathbb{N}), \tag{8}$$

we denote by $\left(f_1 * f_2\right)(z)$ the Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$\left(f_1 * f_2\right)(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^k. \tag{9}$$

For any real numbers r and s, the generalized Hadamard product $\left(f_1 \Delta f_2\right)(r,s;z)$ is given by

$$\left(f_1 \Delta f_2\right)(r,s;z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} \left(a_{k,1}\right)^r \left(a_{k,2}\right)^s z^k. \tag{10}$$

If we take $r = s = 1$, then we have

$$\left(f_1 \Delta f_2\right)(1,1;z) = \left(f_1 * f_2\right)(z) \quad (z \in U^*). \tag{11}$$

In the present paper, applying methods used by Choi et al. [5], Aouf and Silverman [4] and Darwish and Aouf [7], we will obtain several results for the generalized Hadamard product of functions in the classes $\sum_p S_n^*(\alpha,\beta)$ and $\sum_p K_n(\alpha,\beta)$.

2. Main Results

Unless otherwise mentioned, we assume in the reminder of this paper that $0 \leq \alpha < p$, $0 < \beta \leq 1$, $n \geq p$, $p \in \mathbb{N}$ and $z \in U^*$.
In order to prove our results for functions belonging to the classes $\sum_p S_n^*(\alpha,\beta)$ and $\sum_p K_n(\alpha,\beta)$, we shall need the following lemmas given by Aouf [1, 2] see also Mogra [11].

Lemma 1. *Let the function $f(z)$ be defined by (1). Then $f(z)$ is in the class $\sum_p S_n^*(\alpha,\beta)$ if and only if*

$$\sum_{k=n}^{\infty} \left[(k+p) + \beta(k+2\alpha-p)\right] a_k \leq 2\beta\left(p-\alpha\right). \tag{12}$$

Lemma 2. *Let the function $f(z)$ be defined by (1). Then $f(z)$ is in the class $\sum_p K_n(\alpha,\beta)$ if and only if*

$$\sum_{k=n}^{\infty} \frac{k}{p} \left[(k+p) + \beta(k+2\alpha-p)\right] a_k \leq 2\beta\left(p-\alpha\right). \tag{13}$$

Applying Lemma 1 and Lemma 2, we derive:

Theorem 1. *If the functions $f_j(z)$ $\left(j = 1,2\right)$ defined by (8) are in the classes $\sum_p S_n^*(\alpha_j,\beta)$ for each j, then*

$$\left(f_1 \Delta f_2\right)\left(\frac{1}{r}, \frac{r-1}{r}; z\right) \in \sum\nolimits_p S_n^*(\gamma,\beta), \tag{14}$$

where $r > 1$ *and*

$$\gamma = \min_{k\geq n}\left\{ p - \frac{(k+p)\,(1+\beta)}{2\beta\left[1+\left(\frac{k+p+\beta(k+2\alpha_1-p)}{2\beta(p-\alpha_1)}\right)^{\frac{1}{r}}\left(\frac{k+p+\beta(k+2\alpha_2-p)}{2\beta(p-\alpha_2)}\right)^{\frac{r-1}{r}}\right]}\right\}. \tag{15}$$

Proof. Since $f_j(z)\in\sum_p S_n^*(\alpha_j,\beta)\ (j=1,2)$, by using Lemma 1, we have

$$\sum_{k=n}^{\infty}\left(\frac{(k+p)+\beta(k+2\alpha_j-p)}{2\beta\left(p-\alpha_j\right)}\right)a_{k,j}\leq 1 \quad (j=1,2)\,. \tag{16}$$

Moreover,

$$\left\{\sum_{k=n}^{\infty}\left(\frac{(k+p)+\beta(k+2\alpha_1-p)}{2\beta\,(p-\alpha_1)}\right)a_{k,1}\right\}^{\frac{1}{r}}\leq 1, \tag{17}$$

and

$$\left\{\sum_{k=n}^{\infty}\left(\frac{(k+p)+\beta(k+2\alpha_2-p)}{2\beta\,(p-\alpha_2)}\right)a_{k,2}\right\}^{\frac{r-1}{r}}\leq 1. \tag{18}$$

By using Holder inequality, we get

$$\sum_{k=n}^{\infty}\left(\frac{(k+p)+\beta(k+2\alpha_1-p)}{2\beta(p-\alpha_1)}\right)^{\frac{1}{r}}\left(\frac{(k+p)+\beta(k+2\alpha_2-p)}{2\beta(p-\alpha_2)}\right)^{\frac{r-1}{r}}\left(a_{k,1}\right)^{\frac{1}{r}}\left(a_{k,2}\right)^{\frac{r-1}{r}}\leq 1 \tag{19}$$

Since

$$(f_1\Delta f_2)\left(\frac{1}{r},\frac{r-1}{r};z\right)=\frac{1}{z^p}+\sum_{k=n}^{\infty}\left(a_{k,1}\right)^{\frac{1}{r}}\left(a_{k,2}\right)^{\frac{r-1}{r}}z^k, \tag{20}$$

we see that

$$\sum_{k=n}^{\infty}\left(\frac{(k+p)+\beta(k+2\gamma-p)}{2\beta\,(p-\gamma)}\right)\left(a_{k,1}\right)^{\frac{1}{r}}\left(a_{k,2}\right)^{\frac{r-1}{r}}\leq 1 \tag{21}$$

with

$$\gamma \leq \min_{k\geq n}\left\{ p - \frac{(k+p)\,(1+\beta)}{2\beta\left[1+\left(\frac{k+p+\beta(k+2\alpha_1-p)}{2\beta(p-\alpha_1)}\right)^{\frac{1}{r}}\left(\frac{k+p+\beta(k+2\alpha_2-p)}{2\beta(p-\alpha_2)}\right)^{\frac{r-1}{r}}\right]}\right\}.$$

Thus, by using Lemma 1, the proof of Theorem 1 is completed.

Corollary 1. *If the functions $f_j(z)$ $(j=1,2)$ defined by (8) are in the class $\sum_p S_n^*(\alpha,\beta)$ for each j, then*

$$(f_1 \Delta f_2)\left(\frac{1}{r}, \frac{r-1}{r}; z\right) \in \sum\nolimits_p S_n^*(\alpha,\beta) \quad (r>1). \tag{22}$$

Proof. In view of Lemma 1, Corollary 1 follows immediately from Theorem 1 by taking $\alpha_j = \alpha$ $(j=1,2)$.

Theorem 2. *Let the functions $f_j(z)$ $(j=1,2)$ defined by (8) are in the classes $\sum_p K_n(\alpha_j,\beta)$ for each j, then*

$$(f_1 \Delta f_2)\left(\frac{1}{r}, \frac{r-1}{r}; z\right) \in \sum\nolimits_p K_n(\gamma,\beta), \tag{23}$$

where $r>1$ and γ is defined by (15).

Proof. Since $f_j(z) \in \sum_p K_n(\alpha_j,\beta)$ $(j=1,2)$, by using Lemma 2, we have

$$\sum_{k=n}^{\infty} \left(\frac{k}{p}\right) \left(\frac{(k+p)+\beta(k+2\alpha_j-p)}{2\beta\left(p-\alpha_j\right)}\right) a_{k,j} \le 1 \quad (j=1,2). \tag{24}$$

Thus the proof of Theorem 2 is similar to that of Theorem 1 where Lemma 2 is used instead of Lemma 1.

Corollary 2. *If the functions $f_j(z)$ $(j=1,2)$ defined by (8) are in the class $\sum_p K_n(\alpha,\beta)$ for each j, then*

$$(f_1 \Delta f_2)\left(\frac{1}{r}, \frac{r-1}{r}; z\right) \in \sum\nolimits_p K_n(\alpha,\beta) \quad (r>1). \tag{25}$$

Proof. In view of Lemma 2, Corollary 2 follows immediately from Theorem 2 by taking $\alpha_j = \alpha$ $(j=1,2)$.

Theorem 3. *Let the functions $f_j(z)$ $(j=1,2,\ldots,m)$ defined by (8) are in the classes $\sum_p S_n^*(\alpha_j,\beta)$ for each j, and let the function $F_m(z)$ defined by*

$$F_m(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} \left(\sum_{j=1}^{m} \left(a_{k,j}\right)^r\right) z^k \quad (z \in U^*; r \ge 2). \tag{26}$$

Then $F_m(z) \in \sum_p S_n^*(\gamma_m,\beta)$ $(z \in U)$, where

$$\gamma_m \le p - \frac{m(1+\beta)(n+p)\left[2\beta(p-\alpha)\right]^r}{2m\beta\left[2\beta(p-\alpha)\right]^r + 2\beta\left[n+p+\beta(n+2\alpha-p)\right]^r}, \tag{27}$$

where

$$\alpha = \min_{1\le j\le m} \left\{\alpha_j\right\} \tag{28}$$

and

$$\left[m(1+\beta)(n+p) - 2\beta pm\right]\left[2\beta(p-\alpha)\right]^r \le 2\beta p\left[n+p+\beta(n+2\alpha-p)\right]^r \tag{29}$$

Proof. Since $f_j(z) \in \sum_p S_n^*(\alpha_j, \beta)$, by using Lemma 1, we obtain

$$\sum_{k=n}^{\infty} \left(\frac{(k+p) + \beta(k + 2\alpha_j - p)}{2\beta\left(p - \alpha_j\right)} \right) a_{k,j} \leq 1 \quad (j = 1, 2, \ldots, m), \tag{30}$$

and

$$\sum_{k=n}^{\infty} \left(\frac{(k+p)+\beta(k+2\alpha_j-p)}{2\beta(p-\alpha_j)} \right)^r \left(a_{k,j}\right)^r \leq \left\{ \sum_{k=n}^{\infty} \left(\frac{(k+p)+\beta(k+2\alpha_j-p)}{2\beta(p-\alpha_j)} \right) a_{k,j} \right\}^r \leq 1. \tag{31}$$

It follows from (31) that

$$\sum_{k=n}^{\infty} \left\{ \frac{1}{m} \sum_{j=1}^{m} \left(\frac{(k+p) + \beta(k + 2\alpha_j - p)}{2\beta\left(p - \alpha_j\right)} \right)^r \left(a_{k,j}\right)^r \right\} \leq 1. \tag{32}$$

Putting

$$\alpha = \min_{1 \leq j \leq m} \left\{\alpha_j\right\},$$

and by virtue of (32), we find that

$$\sum_{k=n}^{\infty} \left(\frac{(k+p)+\beta(k+2\gamma_m-p)}{2\beta(p-\gamma_m)} \right) \sum_{j=1}^{m} \left(a_{k,j}\right)^r \leq \sum_{k=n}^{\infty} \left\{ \frac{1}{m} \left(\frac{(k+p)+\beta(k+2\alpha-p)}{2\beta(p-\alpha)} \right)^r \sum_{j=1}^{m} \left(a_{k,j}\right)^r \right\}$$

$$\leq \sum_{k=n}^{\infty} \left\{ \frac{1}{m} \sum_{j=1}^{m} \left(\frac{(k+p)+\beta(k+2\alpha_j-p)}{2\beta(p-\alpha_j)} \right)^r \left(a_{k,j}\right)^r \right\} \leq 1 \tag{33}$$

if

$$\gamma_m \leq p - \frac{m\left(1+\beta\right)\left(k+p\right)\left[2\beta\left(p-\alpha\right)\right]^r}{2m\beta\left[2\beta\left(p-\alpha\right)\right]^r + 2\beta\left[k+p+\beta\left(k+2\alpha-p\right)\right]^r} \quad (k \geq n). \tag{34}$$

Now let

$$g(k) = p - \frac{m\left(1+\beta\right)\left(k+p\right)\left[2\beta\left(p-\alpha\right)\right]^r}{2m\beta\left[2\beta\left(p-\alpha\right)\right]^r + 2\beta\left[k+p+\beta\left(k+2\alpha-p\right)\right]^r} \quad (k \geq n).$$

Then

$$g'(k) = \frac{\begin{array}{c} 2\beta m\left(1+\beta\right)\left[2\beta\left(p-\alpha\right)\right]^r \left\{\left[k+p+\beta\left(k+2\alpha-p\right)\right]^{r-1}\left[\left(1+\beta\right)\left(k+p\right)\cdot\right.\right. \\ \left.\left.\cdot\left(r-1\right)+2\beta\left(p-\alpha\right)\right]-m\left[2\beta\left(p-\alpha\right)\right]^r\right\} \end{array}}{\left(2m\beta\left[2\beta\left(p-\alpha\right)\right]^r + 2\beta\left[k+p+\beta\left(k+2\alpha-p\right)\right]^r\right)^2}$$

$$= \frac{\begin{array}{c} 2\beta m\left(1+\beta\right)\left[2\beta\left(p-\alpha\right)\right]^r \left\{2\beta p\left[n+p+\beta\left(n+2\alpha-p\right)\right]\left[k+p+\beta(k+2\alpha\right.\right. \\ \left.-p)\right]^{r-1}\left[\left(1+\beta\right)\left(k+p\right)\left(r-1\right)+2\beta\left(p-\alpha\right)\right] \\ \left.-2\beta mp\left[n+p+\beta\left(n+2\alpha-p\right)\right]\left[2\beta\left(p-\alpha\right)\right]^r\right\} \end{array}}{2\beta p\left[n+p+\beta\left(n+2\alpha-p\right)\right]\left(2m\beta\left[2\beta\left(p-\alpha\right)\right]^r + 2\beta\left[k+p+\beta\left(k+2\alpha-p\right)\right]^r\right)^2}$$

$$=\frac{A(k)}{B(k)},$$

where

$$A(k)=2\beta m\left(1+\beta\right)\left[2\beta\left(p-\alpha\right)\right]^{r}\left\{2\beta p\left[n+p+\beta\left(n+2\alpha-p\right)\right]\left[k+p+\beta\cdot\right.\right.$$
$$\cdot\left(k+2\alpha-p\right)\big]^{r-1}\left[\left(1+\beta\right)\left(k+p\right)\left(r-1\right)+2\beta\left(p-\alpha\right)\right]-2\beta mp\left[n+p+\beta\left(n+\right.\right.$$
$$\left.\left.2\alpha-p\right)\right]\left[2\beta\left(p-\alpha\right)\right]^{r}\Big\}\quad\left(k\geq n\right).$$

Then

$$A(k)\geq 2\beta m\left(1+\beta\right)\left[2\beta\left(p-\alpha\right)\right]^{r}\Big\{2\beta p\left[n+p+\beta\left(n+2\alpha-p\right)\right]^{r}\left[\left(1+\beta\right)\left(k\right.\right.$$
$$\left.+p\right)\left(r-1\right)+2\beta\left(p-\alpha\right)\big]-2\beta mp\left[n+p+\beta\left(n+2\alpha-p\right)\right]\left[2\beta\left(p-\alpha\right)\right]^{r}\Big\}$$

Using (29), we have

$$A(k)\geq 2\beta\left(n+p\right)m^{2}\left(1+\beta\right)^{2}\left[2\beta\left(p-\alpha\right)\right]^{2r}\left\{\left(r-1\right)\left[\beta\left(n-p\right)+\left(n+p\right)\right]-2\alpha\beta\right\}\geq 0,$$

for all $0\leq\alpha<p$, $0<\beta\leq 1$, $n\geq p$ and $r\geq 2$. Then we have $g'(k)\geq 0$ for all $0\leq\alpha<p$, $0<\beta\leq 1$, $n\geq p$ and $r\geq 2$. Hence

$$\gamma_{m}\leq p-\frac{m\left(1+\beta\right)\left(n+p\right)\left[2\beta\left(p-\alpha\right)\right]^{r}}{2m\beta\left[2\beta\left(p-\alpha\right)\right]^{r}+2\beta\left[n+p+\beta\left(n+2\alpha-p\right)\right]^{r}}. \tag{35}$$

Using (29), we can see that $0\leq\gamma_{m}<p$. Thus the proof of Theorem 3 is completed.

Taking $r=2$ and $\alpha_{j}=\alpha$ $\left(j=1,2,\ldots,m\right)$ in Theorem 3, we obtain the following corollary:

Corollary 3. *Let the functions $f_{j}(z)$ $\left(j=1,2,\ldots,m\right)$ defined by (8) are in the class $\sum_{p}S_{n}^{*}(\alpha,\beta)$ for each j, and let the function $F_{m}(z)$ defined by*

$$F_{m}(z)=\frac{1}{z^{p}}+\sum_{k=n}^{\infty}\left(\sum_{j=1}^{m}\left(a_{k,j}\right)^{2}\right)z^{k}\quad\left(z\in U^{*}\right). \tag{36}$$

Then $F_{m}(z)\in\sum_{p}S_{n}^{*}(\delta_{m},\beta)$ $(z\in U)$, where

$$\delta_{m}=p-\frac{m\left(1+\beta\right)\left(n+p\right)\left[2\beta\left(p-\alpha\right)\right]^{2}}{2m\beta\left[2\beta\left(p-\alpha\right)\right]^{2}+2\beta\left[n+p+\beta\left(n+2\alpha-p\right)\right]^{2}}, \tag{37}$$

and

$$\left[m\left(1+\beta\right)\left(n+p\right)-2\beta pm\right]\left[2\beta\left(p-\alpha\right)\right]^{2}\leq 2\beta p\left[n+p+\beta\left(n+2\alpha-p\right)\right]^{2}. \tag{38}$$

The result is sharp, the extremal functions are

$$f_{j}(z)=\frac{1}{z^{p}}+\frac{2\beta\left(p-\alpha\right)}{\left(n+p\right)+\beta(n+2\alpha-p)}z^{n}\quad\left(j=1,2,\ldots,m\right). \tag{39}$$

Taking $\beta = p = 1$, $m = 2$ and $n = 1$ in Corollary 3, we obtain the following corollary:

Corollary 4 ([8, Theorem 10]). *Let the functions $f_j(z)$ $(j = 1,2)$ defined by (8) are in the class $\sum_1 S_1^*(\alpha,1) = \sum S^*(\alpha)$ for each j, and let the function $F_2(z)$ defined by*

$$F_2(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(a_{k,1}^2 + a_{k,2}^2 \right) z^k \quad \left(z \in U^* \right). \tag{40}$$

Then $F_2(z) \in \sum S^*(\delta_2)\,(z \in U^*)$, where

$$\delta_2 = 1 - \frac{4(1-\alpha)^2}{(1+\alpha)^2 + 2(1-\alpha)^2}, \tag{41}$$

and

$$3 - 2\sqrt{2} \le \alpha \le 1. \tag{42}$$

Theorem 4. *Let the functions $f_j(z)$ $(j = 1,2,\ldots,m)$ defined by (8) are in the classes $\sum_p K_n(\alpha_j,\beta)$ for each j, and let the function $F_m(z)$ defined by (26). Then $F_m(z) \in \sum_p K_n(\mu_m,\beta)$ $(z \in U)$, where*

$$\mu_m \le p - \frac{m\,(1+\beta)\,(n+p)\,\left[2\beta\,(p-\alpha)\right]^r p^{r-1}}{2m\beta\left[2\beta\,(p-\alpha)\right]^r p^{r-1} + 2\beta\left[n+p+\beta\,(n+2\alpha-p)\right]^r n^{r-1}} \quad (r \ge 2), \tag{43}$$

where

$$\alpha = \min_{1 \le j \le m} \left\{ \alpha_j \right\}$$

and

$$\left[m\,(1+\beta)\,(n+p)\,p^{r-1} - 2\beta m p^r\right]\left[2\beta\,(p-\alpha)\right]^r \le 2\beta p\left[n+p+\beta\,(n+2\alpha-p)\right]^r n^{r-1}. \tag{44}$$

Proof. Since $f_j(z) \in \sum_p K_n(\alpha_j,\beta)$ $(j = 1,2,\ldots,m)$, using Lemma 2, we obtain

$$\sum_{k=n}^{\infty} \left(\frac{k}{p}\right) \left(\frac{(k+p) + \beta(k+2\alpha_j - p)}{2\beta\left(p - \alpha_j\right)} \right) a_{k,j} \le 1 \quad (j = 1,2,\ldots,m). \tag{45}$$

Thus the proof of Theorem 4 is similar to that of Theorem 3 where Lemma 2 is used instead of Lemma 1, therefore it is omitted.

Taking $r = 2$ and $\alpha_j = \alpha$ $(j = 1,2,\ldots,m)$ in Theorem 4, we obtain the following corollary:

Corollary 5. *Let the functions $f_j(z)$ $(j = 1,2,\ldots,m)$ defined by (8) are in the class $\sum_p K_n(\alpha,\beta)$ for each j, and let the function $F_m(z)$ defined by (36). Then $F_m(z) \in \sum_p K_n(\lambda_m,\beta)$ $(z \in U^*)$, where*

$$\lambda_m = p - \frac{m\,(1+\beta)\,(n+p)\,\left[2\beta\,(p-\alpha)\right]^2 p}{2m\beta\left[2\beta\,(p-\alpha)\right]^2 p + 2\beta\left[n+p+\beta\,(n+2\alpha-p)\right]^2 n}, \tag{46}$$

and

$$\left[m\,(1+\beta)\,(n+p)\,p - 2\beta m p^2\right]\left[2\beta\,(p-\alpha)\right]^2 \le 2\beta p\left[n+p+\beta\,(n+2\alpha-p)\right]^2 n. \tag{47}$$

Taking $\beta = p = 1$, $m = 2$ and $n = 1$ in Corollary 5, we obtain the following corollary:

Corollary 6. *Let the functions* $f_j(z)$ $\left(j = 1, 2\right)$ *defined by (8) are in the class* $\sum_1 K_1(\alpha, 1) = \sum K(\alpha)$ *for each* j*, where* α *satisfy (42) and let the function* $F_2(z)$ *defined by (40), then* $F_2(z) \in \sum_1 K_1(\delta_2, 1) = \sum K(\delta_2)$*, where* δ_2 *is defined by (41).*

Remark 1. *Putting* $\beta = p = 1$ *in all the above results, we obtain the results obtained by Aouf and Silverman [4].*

References

[1] M K Aouf. Certain classes of meromorphic multivalent functions with positive coefficients. Mathematics and Computer Modelling, 47:328-340, 2008.

[2] M K Aouf. Hadamard products of certain meromorphic $p-$valent starlike and $p-$valent convex functions. Journal of Inequalities in Pure and Apllied Mathematics, 10(2). 43:1-7. 2009.

[3] M K Aouf and H M Hossen. New criteria for meromorphic $p-$valent starlike functions. Tsukuba Journal of Mathematics, 17(2):481-486. 1993.

[4] M K Aouf and H Silverman. Generalization of Hadamard product of certain meromorphic univalent functions with positive coefficients. Demonstratio Mathematica, 10(2):382-288. 2008.

[5] J H Choi, Y C Kim and S Owa. Generalizations of Hadamard products of functions with negative coefficients. Journal of Mathematical Analysis and Applications, 199:495-501. 1996.

[6] J Clunie. On meromorphic schlicht functions. Journal of London Mathematical Society, 34:215-216. 1959.

[7] H E Darwish and M K Aouf. Generalizations of modified-Hadamard products of $p-$valent functions with negative coefficients. Mathematical and Computer Modelling, 49:38-45. 2009.

[8] O P Juneja and T R Reddy. Meromorphic starlike univalent functions with positive coefficients. Annales Universitatis Mariae Curie-Sklodowska. Sectio A, 39:65-76. 1985.

[9] S S Kumar, V Ravichandran and G Murugusundaramoorthy. Classes of meromorphic $p-$valent parabolic starlike functions with positive coefficients. Australian Journal of Mathematical Analysis and Applications, 2(2):1-9. 2005.

[10] J E Miller. Convex meromorphic mappings and related functions. Proceedings of the American Mathematical Society, 25:220-228. 1970.

[11] M L Mogra. Meromorphic multivalent functions with positive coefficients I. Mathematica Japonica, 35(1):1-11. 1990.

[12] M L Mogra. Hadamard products of certain meromorphic univalent functions. Journal of Mathematical Analysis and Applications, 157(1):10-16. 1991.

[13] M L Mogra, T R Reddy and O P Juneja. Meromorphic univalent functions with positive coefficients. Bulletin of the Australian Mathematical Society, 32:161-176. 1985.

[14] M Nunokawa and O P Ahuja. On meromorphic starlike and convex functions. Indian Journal of Pure Applied Mathematics, 32(7): 1027-1032. 2001.

[15] Ch Pommerenke. On meromorphic starlike functions. Pacific Journal of Mathematics, 13:221-235. 1963.

[16] W C Royster. Meromorphic starlike multivalent functions. Transactions of the American Mathematical Society, 107:300-308. 1963.