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Separation Theorems for Variational Inequalities and a $Sinc$-Collection Method for Second Order Linear Voltera Integro-Variational Differential Equation Via Differential Dominated Complementarities

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Abstract. The purpose of this paper is to introduce the concept T-η-equiinvex function of order λ and $(GDDVCP;\lambda)$. The concept of second order linear Voltera Integro-Variational Differential Equations $(LVIVDE_2)$ with boundary conditions via $(GDDVCP;\lambda)$. The iteration for the solution of $(LVIVDE_2)$ is studied using $Sinc$-collection Method.

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1. Introduction

Variational inequalities are used in the study of calculus of variations and generally in the optimization problems. It is well known that variational inequality theory is a result of research works of Chipot [10], Browder [8], Lions and Stampacchia [24], Mosco [29], Kinderlehrer and Stampacchia [22], etc., and it has interesting applications in the study of obstacles problems, confined plasmas, filtration phenomena, free boundary problems, plasticity and viscoplasticity phenomena, elasticity problem and stochastic optimal control problems.

The complementarity problem has also been considered by many mathematicians, as a large independent division of mathematical programming theory but in Isac's opinion [21] is quite different. The complementarity problem represents a very deep, very interesting and very difficult mathematical problem. This problem is a very nice research domain because it has many interesting applications and deep connections with important area of nonlinear analysis.

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Several authors have proved many fascinating results on various types of variational inequality problems. The existence of the solution to the problem is studied by many authors such as, J. L. Lions and G. Stampacchia [24], R. W. Cottle, F. Giannessi and J. L. Lions [11] to name only a few. The most general and popular forms of inequality with very reasonable conditions are due to F. E. Browder [8], D. Kinderlehrer and G. Stampacchia [22], Behera and Panda [5, 6].

Let X be a Banach space and its dual X^*. Let K be any nonempty subset of X. Let $T : K \to X^*$ be any nonlinear map.

The *Variational Inequality Problem* (VIP) is to:

(VIP) find $x_0 \in K$ such that

$$\langle T(x_0), x - x_0\rangle \geq 0 \quad \text{for all } x \in K,$$

and the *Complementarity Problem* (CP) is to:

(CP) find $x_0 \in K$ such that

$$\langle T(x_0), x_0\rangle = 0.$$

In [10], Chipot has defined the nonlinear variational inequality problems as follows. Let X be a real Banach space and $K \subset X$. Let $\xi \in X^*$ and $T : K \to X^*$. Then the nonlinear variational inequality problem $(NVIP)$ is to:

$(NVIP)$ find $x_0 \in K$ such that

$$\langle T(x_0), x - x_0\rangle \geq \langle \xi, x - x_0\rangle \quad \text{for all } x \in K,$$

and studied various results on it.

In 1981, Hanson [20] has introduced the concept of invex (i.e., invariant convex) function as a generalization of convex function. For the significance of the function $\eta(\text{-},\text{-})$, we refer to the papers of Hanson [20] and Ben-Israel and Mond [7].

The function $\eta(\text{-},\text{-})$ plays quite general role in nature and is applicable to many cases of general interest by enhancing the concept the set of convexity to invexity.

Initially η-invex function was used to study the generalization of optimization problems. With the concept of set of invexity, Behera and Panda [5, 6] studied various generalized variational inequality problems, generalized variational-type inequality problems in Housdorff spaces and reflexive real Banach spaces.

In 2006, Behera and Das [3] introduced the concept of T-η-invex function in ordered topological vector spaces to generalize η-invex function introduced by Hanson [20]. For the significance of the T-η-invex function, we refer to the papers of Behera and Das [3, 4], Das and Satpathy [17, 18], Das and Mohanta [15], Das [12], Das and Behera [14]. In 2011, Das and Sahu [16] has introduced the concept of generalized differential dominated variational inequality problems in Banach spaces and topological vector spaces. In [13], Das introduced the concept of generalized dominated differential variational inequality problems $(GDDVIP_n)$ and complementarity problems in Riemannian n-manifold.

2. Generalized Vector Variational Inequalities and Separation Theorems

For simplicity, we recall the following results studied by Chen [9], and Behera and Das [3].

Lemma 1. *[9, Lemma 2.1] Let (Y,P) be an ordered topological vector space (TVS) with a closed, convex and pointed cone P with $intP \neq \emptyset$. Then, for all $y,z \in Y$, we have*

(i) $y - z \in intP$ *and* $y \notin intP$ *imply* $z \notin intP$*;*

(ii) $y - z \in P$ *and* $y \notin intP$ *imply* $z \notin intP$*;*

(iii) $y - z \in -intP$ *and* $y \notin -intP$ *imply* $z \notin -intP$*;*

(iv) $y - z \in -P$ *and* $y \notin -intP$ *imply* $z \notin -intP$*.*

Remark 1. *[3] For simplicity, we use the following terminologies:*

(a) $y \notin -intP$ if and only if $y \in P$ if and only if $y \geq_P 0$;

(b) $y \in intP$ if and only if $y >_P 0$;

(c) $y \notin intP$ if and only if $y \in -P$ if and only if $y \leq_P 0$;

(d) $y \in -intP$ if and only if $y <_P 0$;

(e) $y - z \notin -intP$ if and only if $y - z \geq_P 0$ (i.e., $y \geq_P z$);

(f) $y - z \notin intP$ if and only if $y - z \leq_P 0$ (i.e., $y \leq_P z$);

(g) $y - z \notin (-intP \bigcup intP)$ if and only if $y - z \in (-P \bigcap P)$

if and only if $y - z =_P 0$, (i.e., $y =_P z$).

For our need, we recall some definitions on the concept of set of invexity and T-η-invex function.

Definition 1. *[20] Let K be any subset of the topological vector space X. Let $\eta : K \times K \to X$ be continuous vector valued mapping. K is said to be η-invex if for all $x,u \in K$ and for all $t \in (0,1)$, we have*

$$u + t\eta(x,u) \in K.$$

Definition 2. *[18] Let K be any η-invex subset of the topological vector space X where $\eta : K \times K \to X$ be continuous vector valued mapping. K is said to be η-invex cone if for all $x,u \in K$ and for all $t \in (0,1)$, we have*

$$\eta(u + t\eta(x,u),u) = t\eta(x,u) \in K.$$

Definition 3. *[3] The mapping $T : K \to L(X,Y)$ is said to be η-monotone if there exists a vector function $\eta : K \times K \to X$ such that*

$$\langle T(u), \eta(x,u)\rangle + \langle T(x), \eta(u,x)\rangle \notin intP \quad \text{for all } x,u \in K,$$

and for strict case; equality holds for $x = u$.

Definition 4. *[3] Let X be topological vector space and K be a nonempty subset of X. Let (Y,P) be an ordered topological vector space equipped with the closed convex pointed cone such that $intP \neq \emptyset$ where $intP$ denotes the interior of P. Let $L(X,Y)$ be the set of all continuous linear functionals from X to Y. Let the pair $\langle f,x\rangle$ denote the value of $f \in L(X,Y)$ at $x \in X$. Let $F : K \to Y$, and $T : K \to L(X,Y)$ be any two maps. Let $\eta : K \times K \to X$ be any vector valued function. Then*

(a) F is T-η-invex on K if for all $x,u \in K$, we have

$$F(x) - F(u) - \langle T(u), \eta(x,u)\rangle \notin -intP,$$

(b) F is not T-η-invex on K if for all $x,u \in K$, we have

$$-F(x) + F(u) + \langle T(u), \eta(x,u)\rangle \notin -intP,$$

alternatively;

$$F(x) - F(u) - \langle T(u), \eta(x,u)\rangle \notin intP.$$

Furthermore,

(i) In strict cases of T-η-invexity of F, the terms "$\notin -intP$", and "$\notin intP$" will be replaced by "$\notin -P$", and "$\notin P$" respectively.

(ii) In particular, if $Y = \mathbb{R}^n$ and $P = \mathbb{R}^n_+$, the terms "$\notin -intP$", "$\notin intP$", "$\notin -P$", and "$\notin P$" will be replaced by "≥ 0", "≤ 0", "> 0" and "< 0" respectively.

Behera and Das [3] explored various equivalent generalized vector variational inequality problems in ordered topological vector spaces using T-η-invex function and η-monotone operator in ordered topological vector spaces. In this note, the separation theorems are explored with the help of generalized vector variational inequality problems and η-monotone operators.

2.1. $(GVVIP)$ and Separation Theorems

Let X be a topological vector space. Let K be any nonempty subset of X. Let $\eta : K \times K \to X$ be a vector valued function. Let (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$ where $intP$ denotes the "interior of P". Let $L(X,Y)$ be the set of all continuous linear functionals from X to Y. Let $T : K \to L(X,Y)$ be any function. For our study, the generalized vector variational inequalities are classified into primal generalized vector variational inequality problems and its equivalent dual generalized vector variational inequality problems [3].

The primal generalized vector variational inequality problem ($PGVVIP$) is to:

($PGVVIP$) find $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \geq_P 0 \quad \text{for all } x \in K.$$

and the dual generalized vector variational inequality problem ($GVVIP$) is to:

($DGVVIP$) find $x_0 \in K$ such that

$$\langle T(x), \eta(x_0, x)\rangle \leq_P 0 \quad \text{for all } x \in K.$$

The generalized vector complementarity problem ($GVCP$) is to:

($GVCP$) find $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle =_P 0 \quad \text{for all } x \in K.$$

The concepts of hypersurfaces and separations associated with T and η are defined as follows. Let X be a topological vector space. Let K be any nonempty subset of X. Let (Y, P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. Let $T : K \to L(X, Y)$ be any map. Let $\eta : K \times K \to X$ be a vector valued map.

Definition 5. *For each $x' \in K$, the set $P(x') \subset K$ is said to be $(T, \eta; x')$-hypersurface if*

$$P(x') = \{x \in K : \langle T(x'), \eta(x, x')\rangle =_P 0\} \neq \emptyset.$$

Definition 6. *For each $x' \in K$, the $(T, \eta; x')$-hypersurface $P(x')$ separates the point $x^* \in K$ if*

$$\langle T(x'), \eta(x^*, x')\rangle <_P 0.$$

Definition 7. *For each $x' \in K$, the $(T, \eta; x')$-hypersurface $P(x')$ separates the set $U \subset K$ if*

$$\langle T(x'), \eta(u, x')\rangle <_P 0 \quad \textit{for all } u \in U.$$

Separation of convex and compact subset of K is studied in following theorem.

Theorem 1 (Separation Theorem-I)**.** *Let X be a topological vector space. Let K be any nonempty subset of X. Let (Y, P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. Let $T : K \to L(X, Y)$ be any map. Let $\eta : K \times K \to X$ be a vector valued map. Let K_1 and K_2 be two convex and compact subsets of K such that $K_2 \subset K_1$. Let T be strictly η-monotone on K. Let there exist solution of the problem*

$$x_i \in K_i : \ \langle T(x_i), \eta(x, x_i)\rangle \geq_P 0 \quad \textit{for all } x \in K_i, \ i = 1, 2.$$

Then

(a) $x_1 = x_2$ *if* $T(x_2) = 0$,

(b) the $(T,\eta;x_2)$*-hypersurface* $P(x_2)$ *separates* x_1 *from* K_2 *if* $x_1 \neq x_2$ *and* $T(x_2) \neq 0$.

Proof. K_1 and K_2 are two subsets of X such that $K_2 \subset K_1$. Let $x_1, x_2 \in K_2 \subset K_1$, then for $x_i \in K_i$, $i = 1,2$;

$$\langle T(x_i), \eta(x, x_i)\rangle \geq_P 0$$

for all $x \in K_i$, i.e.,

$$\langle T(x_1), \eta(x, x_1)\rangle \geq_P 0 \tag{1}$$

for all $x \in K_1$ and

$$\langle T(x_2), \eta(x, x_2)\rangle \geq_P 0 \tag{2}$$

for all $x \in K_2$. Taking $x = x_2$ in (1) and $x = x_1$ in (2), we get

$$\langle T(x_1), \eta(x_2, x_1)\rangle \geq_P 0$$

and

$$\langle T(x_2), \eta(x_1, x_2)\rangle \geq_P 0$$

respectively. Adding the above equations, we get

$$\langle T(x_1), \eta(x_2, x_1)\rangle + \langle T(x_2), \eta(x_1, x_2)\rangle \geq_P 0.$$

By the property of strictly η-monotoneness of T on K, we have

$$\langle T(x_1), \eta(x_2, x_1)\rangle + \langle T(x_2), \eta(x_1, x_2)\rangle =_P 0$$

for $x_1 = x_2 \in K_2 \subset K_1 \subset K$ and

$$\langle T(x_1), \eta(x_2, x_1)\rangle + \langle T(x_2), \eta(x_1, x_2)\rangle <_P 0$$

and for $x_1 \neq x_2 \in K_2 \subset K_1 \subset K$. Hence

$$\begin{aligned} \langle T(x_2), \eta(x_1, x_2)\rangle &\leq_P -\langle T(x_1), \eta(x_2, x_1)\rangle \\ &\leq_P 0. \end{aligned} \tag{3}$$

Thus

(a) if $T(x_2) = 0$, we get

$$\langle T(x_2), \eta(x_1, x_2)\rangle =_P 0 \Rightarrow x_1 = x_2;$$

(b) if $T(x_2) \neq 0$ and $x_1 \neq x_2$, then from (2) and (3), we get

$$\langle T(x_2), \eta(x_1, x_2)\rangle <_P 0.$$

Hence the $(T,\eta;x_2)$-hypersurface

$$P(x_2)=\{x\in K:\langle T(x_2),\eta(x,x_2)\rangle =_P 0\}$$

separates x_1 from K_2. This completes the proof of the theorem.

To study the separation theorems on invex sets, the concept of $(\eta;i)$-invex set and contractible $(\eta;i)$-invex set are defined as follows.

Definition 8. *Let X be a topological vector space. Let K be any nonempty subset of X. Let $\eta:K\times K\to X$ be a vector valued map. Let K_1 and K_2 be two subsets of K. For $i=1,2$; K_i is said to be $(\eta;i)$-invex set if $x+t\eta(x_i,x)\in K_i$ for all $x\in K$ and $x_i\in K_i$.*

Definition 9. *Let X be a topological vector space. Let K be any nonempty subset of X. Let $\eta:K\times K\to X$ be a vector valued map. For $i=1,2$; K_i be two $(\eta;i)$-invex subsets of K such that $K_1\cap K_2\neq\emptyset$. K_i is said to be contractible $(\eta;i)$-invex set if for each $i=1,2$; K_i is contractible.*

The following theorem is the extension of Theorem 1 to contractible $(\eta;i)$-invex sets.

Theorem 2 (Separation Theorem-II)**.** *Let X be a topological vector space. Let K be any nonempty subset of X. Let (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP\neq\emptyset$. Let $T:K\to L(X,Y)$ be any map. Let $\eta:K\times K\to X$ be a vector valued map. For $i=1,2$; K_i be contractible $(\eta;i)$-invex, compact, and connected subsets of K. Let T be strictly η-monotone on K. Let there exist solution of the problem*

$$x_i\in K_i:\ \langle T(x_i),\eta(x,x_i)\rangle \geq_P 0\quad \textit{for all } x\in K_i.$$

If η satisfies condition C_0 on K, then

(a) $x_1=x_2$ *if* $T(x_i)=0, i=1,2$;

(b) the $(T,\eta;x_i)$-hypersurface $P(x_i)$ separates x_j from K_i if $x_j\neq x_i$ and $T(x_i)\neq 0$ for $i\neq j$; $i,j=1,2$.

Proof. For $i=1,2$; K_i are two contractible $(\eta;i)$-invex, compact, and connected subsets of K. Let there exist $x_i\in K_i$, $i=1,2$ such that

$$\langle T(x_i),\eta(x,x_i)\rangle \geq_P 0$$

for all $x\in K_i$.
Case-I: When $i=1$, we get

$$\langle T(x_1),\eta(x,x_1)\rangle \geq_P 0 \tag{4}$$

for all $x\in K_1$. Taking $x=x_2$ in (4), we get

$$\langle T(x_1),\eta(x_2,x_1)\rangle \geq_P 0. \tag{5}$$

Since K_2 is $(\eta;2)$-invex,

$$x+t\eta(x_2,x)\in K_2$$

for all $x \in K$, $t \in (0,1)$ and fixed $x_2 \in K_2$. Replacing x by $x + t\eta(x_2, x) \in K_2$ in (4), we get

$$\langle T(x_1), \eta(x + t\eta(x_2, x), x_1)\rangle \geq_P 0 \tag{6}$$

for all $x \in K$. Taking $x = x_1$ in (6), we get

$$\langle T(x_1), \eta(x_1 + t\eta(x_2, x_1), x_1)\rangle \geq_P 0.$$

Applying condition C_0 in the above inequality, we get

$$\langle T(x_1), -t\eta(x_2, x_1)\rangle \geq_P 0,$$

i.e.,

$$-t\langle T(x_1), \eta(x_2, x_1)\rangle \geq_P 0.$$

Thus

$$\langle T(x_1), \eta(x_2, x_1)\rangle \leq_P 0. \tag{7}$$

Thus

(a) if $T(x_1) = 0$, we get

$$\langle T(x_1), \eta(x_2, x_1)\rangle =_P 0 \Rightarrow x_1 = x_2;$$

(b) if $T(x_1) \neq 0$ and $x_1 \neq x_2$, then from (5) and (7), we get

$$\langle T(x_1), \eta(x_2, x_1)\rangle <_P 0.$$

Hence the $(T, \eta; x_1)$-hypersurface

$$P(x_1) = \{x \in K : \langle T(x_1), \eta(x, x_1)\rangle =_P 0\}$$

separates x_2 from K_1. This proves Case-I.

Similarly for $i = 2$, Case-II is proved as follows.

Case-II: When $i = 2$, we get

$$\langle T(x_2), \eta(x, x_2)\rangle \geq_P 0 \tag{8}$$

for all $x \in K_2$. Taking $x = x_2$ in (8), we get

$$\langle T(x_2), \eta(x_1, x_2)\rangle \geq_P 0. \tag{9}$$

Since K_1 is $(\eta; 1)$-invex,

$$x + t\eta(x_1, x) \in K_2$$

for all $x \in K$, $t \in (0,1)$ and fixed $x_2 \in K_2$. Replacing x by $x + t\eta(x_1, x) \in K_2$ in (8), we get

$$\langle T(x_2), \eta(x + t\eta(x_2, x), x_1)\rangle \geq_P 0 \tag{10}$$

for all $x \in K$. Taking $x = x_2$ in (10), we get

$$\langle T(x_2), \eta(x_1 + t\eta(x_2, x_1), x_1)\rangle \geq_P 0.$$

Applying condition C_0 in the above inequality, we get

$$\langle T(x_2), \eta(x_2, x_1)\rangle \leq_P 0. \tag{11}$$

for all $t \in (0,1)$. Thus

(a) if $T(x_2) = 0$, we get

$$\langle T(x_2), \eta(x_2, x_1)\rangle =_P 0 \Rightarrow x_1 = x_2;$$

(b) if $T(x_2) \neq 0$ and $x_1 \neq x_2$, then from (9) and (11), we get

$$\langle T(x_2), \eta(x_2, x_1)\rangle <_P 0.$$

Hence the (T, η)-hypersurface

$$P(x_1) = \{x \in K : \langle T(x_1), \eta(x, x_1)\rangle =_P 0\}$$

separates x_2 from K_1. This completes the proof of the theorem.

3. Cyclically T-η-invex Function, Cyclically η-Monotone Theorems and Cyclically Separation Theorems

In this section, the concept of cyclically compact set of order n is defined to study the cyclically separation theorems of the newly defined problem, called the generalized vector cyclically variational inequality problems. We study the separation theorems in cyclically invex sets.

3.1. $(n+1)$-cyclically Compactness

To study the results on cyclically separation theorems and cyclically η-monotone theorems, the concept of $(n+1)$-cyclically compact set is defined as follows.

Definition 10. *Let X be a topological vector space. Let K be any nonempty subset of X. For any integer $n \geq 1$, K is said to be cyclically compact of order n (in short; $(n+1)$-cyclically compact) if there exist a finite subset K° containing $(n+1)$ cyclic points*

$$x_0, x_1, x_2, \ldots, x_n \in K$$

satisfying

$$x_{i+1} = \begin{cases} x_{i+1} & \text{if } i \leq n-1; \\ x_0 & \text{if } i = n, \end{cases}$$

such that

$$\bigcup_{i=0}^{n} \overline{K}_i = K \quad \text{and} \quad \bigcap_{i=0}^{n} K_i \neq \emptyset;$$

where $\overline{K}_i$ is the closure of the open set $K_i \subset K$ that contains $x_i \in K^\circ$ only but $x_j \notin K_i$ for $i \neq j$.

Example 1. *Let $X = \mathbb{R}$, $K = [a, b] \subset X$. It is obvious that K is not 1-cyclically compact subset of X because there is only one open set $K_0 = (a, b)$ such that $\overline{K}_0 = K$. Now assuming $K_0 = K_1$, we get $x_0 \in K_0 = K_1$ which contradicts the condition that $x_j \notin K_i$ for $i \neq j$. Since for $c \in (a, b)$, $\epsilon > 0$, there exists at least two open sets $K_0 = (a, c + \epsilon)$ and $K_1 = (c - \epsilon, b)$ such that $\overline{K}_0 \bigcup \overline{K}_1 = K$, so K is at least 2-cyclically compact subset of X with respect to the finite subset $K^o = \{a, b\}$ which contains the end points of K only. K can be expressed as 3-cyclically compact subset of X with respect to the finite subset $K^o = \{x_0, x_1, x_2\}$ where $x_0 \in (a, c) \subset K_0 = (a, d)$, $x_1 \in (d, e) \subset K_1 = (d, f)$ and $x_2 \in (f, b) \subset K_2 = (e, b)$ for $a < c < d < e < f < b$ satisfies $\overline{K}_0 \bigcup \overline{K}_1 \bigcup \overline{K}_2 = K$. Similarly for a $(n + 1)$-cyclically compact subset K of X, there are at most $(n + 1)$ no. of such open sets K_i contains the cyclic points x_i, $i = 0, 1, 2, \ldots, n$; satisfying $\bigcap_{i=0}^n K_i \neq \emptyset$ and $\bigcup_{i=0}^n \overline{K}_i = K$ but $x_j \notin K_i$ for $i \neq j$. In this case,*

$$K^o = \{x_i : x_i \in K_i - \bigcup_{i \neq j=0}^{n} \overline{K}_j,\ i = 0, 1, 2\ \ldots, n\}$$

is cyclic finite subset corresponds to K.

3.2. Cyclically Separation Theorem

For our need, we define the concept of cyclically η-monotoneness of T.

Definition 11. *Let X be a topological vector space. Let K be any $(n+1)$-cyclically compact subset of X with respect to the finite subset K^o. Let $\eta : K \times K \to X$ be a vector valued map. Let (Y, P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. The mapping $T : K \to L(X, Y)$ is said to be cyclically η-monotone on K^o if for all $0 \leq k \leq n$,*

$$\sum_{k=0}^{n} \langle T(x_k), \eta(x_{k+1}, x_k) \rangle \leq_P 0,$$

and for strictly cyclically η-monotone on K^o if equality holds for $x_n = x_0$.

Remark 2. *For $n = 1$, let $K = [x_0, x_1]$ where $\{x_0, x_1\}$ are the cyclic numbers, then $x_2 = x_0$. So the definition of cyclically η-monotoness given in Definition 11 coincides with Definition 3. In fact, the condition of cyclically η-monotoness is weaker than the condition of η-monotoness.*

The *generalized cyclically vector variational inequality problem* is of finding a set

$$K^o = \{x_0, x_1, x_2, \ldots, x_n\}$$

such that

$$\sum_{k=0}^{n} \langle T(x_i), \eta(x_{i+1}, x_i) \rangle \geq_P 0.$$

The following theorem shows the existence of generalized cyclically vector variational inequality problem and cyclically separation theorem.

Theorem 3 (Cyclically Separation Theorem-I)**.** *Let X be a topological vector space. Let K be any nonempty subset of X. Let (Y, P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. Let $T : K \to L(X, Y)$ be any map. Let $\eta : K \times K \to X$ be a vector valued map. Let K be a $(n+1)$-cyclically compact with respect to the finite subset $K^o = \{x_0, x_1, x_2, \ldots, x_n\}$ of K. Let $K_i \subset K$ be the open set that contains $x_i \in K^o$ only but $x_j \notin K_i$ for $i \neq j$. Let K_i satisfies $\bigcup_{i=0}^{n} \overline{K}_i = K$ where $\overline{K}_i$ is the closure of K_i. Let T be strictly cyclically η-monotone on K. Let there exist solution of the problem*

$$x_i \in K_i : \langle T(x_i), \eta(x, x_i)\rangle \geq_P 0 \quad \textit{for all } x \in K_i$$

for all $i = 0, 1, 2, \ldots, n$. Then the set K^o solves the generalized vector cyclically variational inequality problems, i.e. for all $x_i \in K^o$, $i = 0, 1, \ldots, n$, we have

$$\sum_{k=0}^{n} \langle T(x_i), \eta(x_{i+1}, x_i)\rangle \geq_P 0$$

and the followings hold:

(a) $x_n = x_0$ if $T(x_n) = 0$,

(b) the $(T, \eta; x_n)$-hypersurface $P(x_n)$ separates x_0 from K_n.

Proof. There exist solution of the problem

$$x_i \in K_i : \langle T(x_i), \eta(x, x_i)\rangle \geq_P 0 \quad \text{for all } x \in K_i$$

for all $i = 0, 1, 2, \ldots, n$; implies

$$\langle T(x_0), \eta(x, x_0)\rangle \geq_P 0 \tag{12}$$

for all $x \in K_0$;

$$\langle T(x_1), \eta(x, x_1)\rangle \geq_P 0 \tag{13}$$

for all $x \in K_1$; and so on; we get

$$\langle T(x_{n-1}), \eta(x, x_{n-1})\rangle \geq_P 0 \tag{14}$$

and

$$\langle T(x_n), \eta(x, x_n)\rangle \geq_P 0 \tag{15}$$

for all $x \in K_n$. Taking $x = x_1$ in (12), $x = x_2$ in (13) and so on; $x = x_n$ in (14) and $x = x_{n+1}$ in (15), we get

$$\langle T(x_0), \eta(x_1, x_0)\rangle \geq_P 0, \tag{16}$$

$$\langle T(x_1), \eta(x_2, x_1)\rangle \geq_P 0, \tag{17}$$

and so on;

$$\langle T(x_{n-1}), \eta(x_n, x_{n-1})\rangle \geq_P 0 \tag{18}$$

and

$$\langle T(x_n), \eta(x_{n+1}, x_n)\rangle \geq_P 0 \tag{19}$$

respectively. Adding the above inequalities, we get

$$\sum_{k=0}^{n}\langle T(x_i), \eta(x_{i+1}, x_i)\rangle \geq_P 0.$$

Thus the set $K^o = \{x_0, x_1, x_2, \ldots, x_n\}$ solves the generalized vector cyclically variational inequality problems. Furthermore, the above equation can be written as

$$\langle T(x_n), \eta(x_{n+1}, x_n)\rangle + \sum_{k=0}^{n-1}\langle T(x_i), \eta(x_{i+1}, x_i)\rangle \geq_P 0.$$

Since $x_0, x_1, x_2, \ldots, x_n \in K^o$, we get $x_{n+1} = x_0$. From the above inequality, we get

$$\langle T(x_n), \eta(x_0, x_n)\rangle + \sum_{k=0}^{n-1}\langle T(x_i), \eta(x_{i+1}, x_i)\rangle \geq_P 0. \tag{20}$$

Since T is strictly cyclically η-monotone on K, we have

$$\sum_{k=0}^{n}\langle T(x_i), \eta(x_{i+1}, x_i)\rangle \leq_P 0$$

for $x_i \in K^o$ which can be written as

$$\langle T(x_n), \eta(x_0, x_n)\rangle + \sum_{k=0}^{n-1}\langle T(x_i), \eta(x_{i+1}, x_i)\rangle \leq_P 0. \tag{21}$$

By the strictly cyclically η-monotoness property of T, and the equations (20), (21) we get

$$\langle T(x_n), \eta(x_0, x_n)\rangle + \sum_{k=0}^{n-1}\langle T(x_i), \eta(x_{i+1}, x_i)\rangle =_P 0,$$

implies $x_n = x_0$. Thus

$$\langle T(x_0), \eta(x_n, x_0)\rangle =_P -\sum_{k=0}^{n-2}\langle T(x_i), \eta(x_{i+1}, x_i)\rangle.$$

Using the equations (16) to (18) in (21), we get

$$\begin{aligned}\langle T(x_n), \eta(x_0, x_n)\rangle &\leq_P -\sum_{k=0}^{n-1}\langle T(x_i), \eta(x_{i+1}, x_i)\rangle \\ &\leq_P 0.\end{aligned} \tag{22}$$

Since $x_{n+1} = x_0$, from (19) we get

$$\langle T(x_n), \eta(x_0, x_n)\rangle \geq_P 0. \tag{23}$$

Thus

(a) if $T(x_n) = 0$, we get

$$\langle T(x_n), \eta(x_0, x_n)\rangle =_P 0 \Rightarrow x_n = x_0;$$

(b) if $T(x_n) \neq 0$ and $x_n \neq x_0$, then from (23), we get

$$\langle T(x_n), \eta(x_0, x_n)\rangle <_P 0.$$

Hence the $(T, \eta; x_n)$-hypersurface

$$P(x_n) = \{x \in K : \langle T(x_n), \eta(x, x_n)\rangle =_P 0\}$$

separates x_0 from K_n. This completes the proof of the theorem.

3.3. Cyclically T-η-invex Function

For our need, we introduced the concept of $(n+1)$-cyclically η-invexed compact set, $(n+1)$-cyclically η-invexed compact cone, cyclically T-η-invex function as follows.

Definition 12. *Let X be a topological vector space. Let K be any nonempty subset of X. K is said to be $(n+1)$-cyclically η-invexed compact if K is $(n+1)$-cyclically compact with respect to the finite subset $K^o = \{x_0, x_1, x_2, \ldots, x_n\}$ of K and there exists a vector function $\eta : K^o \times K^o \to X$ such that*

$$x_k + t\eta(x_{k+1}, x_k) \in K^o$$

for all $0 \leq k \leq n$ and $t \in (0,1)$.

Definition 13. *Let X be a topological vector space. Let K be any nonempty subset of X. Let K be a $(n+1)$-cyclically compact with respect to the finite subset $K^o = \{x_0, x_1, x_2, \ldots, x_n\}$ of K. Let $\eta : K^o \times K^o \to X$ be vector function. K is said to be $(n+1)$-cyclically η-invexed compact cone if*

(a) K^o is η-invex,

(b) for all $0 \leq k \leq n$ and $t \in (0,1)$,

$$\eta(x_k + t\eta(x_{k+1}, x_k), x_k) = t\eta(x_{k+1}, x_k).$$

Definition 14. *Let X be a topological vector space. Let K be any $(n+1)$-cyclically compact subset of X with respect to the finite subset K^o. Let $\eta : K^o \times K^o \to X$ be a vector valued function. Let (Y, P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. The mapping $F : K \to Y$ is said to be cyclically T-η-invex on K^o if*

$$F(x_{k+1}) - F(x_k) - \langle T(x_k), \eta(x_{k+1}, x_k)\rangle \notin -intP,$$

i.e.,

$$F(x_{k+1}) - F(x_k) \geq_P \langle T(x_k), \eta(x_{k+1}, x_k)\rangle$$

for all $0 \leq k \leq n$ and $t \in (0,1)$.

Theorem 4 (Cyclically η-monotone Theorem)**.** *Let X be a topological vector space. Let K be any cyclically compact subset of X with respect to the finite subset K^o. Let $\eta : K^o \times K^o \to X$ be a vector valued function. Let (Y, P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. Let $T : K \to L(X, Y)$ be any map. Let $F : K \to Y$ be any map. Then T is cyclically η-monotone on K^o if F is cyclically T-η-invex on K^o.*

Proof. Since K is cyclically compact subset of X with respect to the finite subset K^o containing $n+1$ no. of cyclic points

$$x_0, x_1, x_2, \ldots, x_n \in K$$

satisfying

$$x_{i+1} = \begin{cases} x_{i+1} & \text{if } i \leq n-1; \\ x_0 & \text{if } i = n, \end{cases}$$

such that

$$\bigcup_{i=0}^{n} \overline{K}_i = K \text{ and } \bigcap_{i=0}^{n} K_i \neq \emptyset$$

where $\overline{K}_i$ is the closure of the open set $K_i \subset K$ that contains $x_i \in K^o$ only but $x_i \notin K_j$ for $i \neq j$. Since F is cyclically T-η-invex on K^o, we have

$$F(x_{k+1}) - F(x_k) \geq_P \langle T(x_k), \eta(x_{k+1}, x_k) \rangle$$

for all $0 \leq k \leq n$ and $t \in (0, 1)$. For $k = 0$,

$$F(x_1) - F(x_0) \geq_P \langle T(x_0), \eta(x_1, x_0) \rangle. \tag{24}$$

For $k = 1$,

$$F(x_2) - F(x_1) \geq_P \langle T(x_1), \eta(x_2, x_1) \rangle, \tag{25}$$

and so on; for $k = n-1$,

$$F(x_n) - F(x_{n-1}) \geq_P \langle T(x_{n-1}), \eta(x_n, x_{n-1}) \rangle, \tag{26}$$

and for $k = n$

$$F(x_{n+1}) - F(x_n) \geq_P \langle T(x_n), \eta(x_{n+1}, x_n) \rangle.$$

By the $(n+1)$-cyclicallyness of K^o, $x_{n+1} = x_0$. Thus

$$F(x_0) - F(x_n) \geq_P \langle T(x_n), \eta(x_0, x_n) \rangle. \tag{27}$$

Adding all the equations from (24) to (27), we get

$$\sum_{k=0}^{n} \langle T(x_k), \eta(x_{k+1}, x_k) \rangle \leq_P 0$$

where $x_{n+1} = x_0$. Hence T is cyclically η-monotone on K^o. This completes the proof of the theorem.

Theorem 5 (Cyclically Separation Theorem-II)**.** *Let X be a topological vector space. Let K be any cyclically compact subset of X with respect to the finite subset K^o. Let $\eta : K^o \times K^o \to X$ be a vector valued function. Let (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. Let $T : K \to L(X,Y)$ be any map. Let $F : K \to Y$ be cyclically T-η-invex on K^o.*

Let $K_i \subset K$ be the open set that contains $x_i \in K^o$ only but $x_j \notin K_i$ for $i \neq j$. Let K_i satisfies

$$\bigcup_{i=0}^{n} \overline{K}_i = K \quad \textit{and} \quad \bigcap_{i=0}^{n} K_i \neq \emptyset;$$

where $\overline{K}_i$ is the closure of K_i. Let there exist solution of the problem

$$x_i \in K_i : \langle T(x_i), \eta(x, x_i)\rangle \geq_P 0 \quad \textit{for all } x \in K_i$$

for all $i = 0, 1, 2, \dots, n$. If η satisfies condition C_0 on K, then

(a) $x_n = x_0$ if $T(x_n) = 0$,

(b) the $(T, \eta; x_n)$-hypersurface $P(x_n)$ separates x_0 from K_n.

Proof. By Theorem 4, F is cyclically T-η-invex on K^o implies T is cyclically η-monotone on K^o, i.e.,

$$\sum_{k=0}^{n} \langle T(x_k), \eta(x_{k+1}, x_k)\rangle \leq_P 0$$

where $x_{n+1} = x_0$. Using Theorem 3, we have

(a) if $T(x_n) = 0$, then

$$\langle T(x_n), \eta(x_0, x_n)\rangle =_P 0 \Rightarrow x_n = x_0;$$

(b) if $T(x_n) \neq 0$ and $x_n \neq x_0$, then (23) concludes

$$\langle T(x_n), \eta(x_0, x_n)\rangle <_P 0.$$

Hence the $(T, \eta; n)$-hypersurface

$$P(x_n) = \{x \in K : \langle T(x_n), \eta(x, x_n)\rangle =_P 0\}$$

separates x_0 from K_n. This completes the proof of the theorem.

Theorem 6 (Cyclically Separation Theorem-III)**.** *Let X be a topological vector space. Let K be any nonempty subset of X. Let (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. Let $T : K \to L(X,Y)$ be any map. Let $\eta : K \times K \to X$ be a vector valued map. Let K be a $(n+1)$-cyclically compact with respect to the finite subset $K^o = \{x_0, x_1, x_2, \dots, x_n\}$ of K. Let $K_i \subset K$ be the open set that contains $x_i \in K^o$ only but $x_j \notin K_i$ for $i \neq j$. Let K_i satisfies $\bigcup_{i=0}^{n} \overline{K}_i = K$ where $\overline{K}_i$ is the closure of K_i. Assume that* *(i) for $i = 0, 1, 2, \ldots, n$; K_i is contractible $(\eta; i)$-invex, compact, and connected subsets of K,*

(ii) T is strictly η-monotone on K^o,

(iii) there exist solution of the problem

$$x_i \in K_i : \langle T(x_i), \eta(x, x_i)\rangle \geq_P 0$$

for all $x \in K_i$, $i = 0, 1, 2, \ldots, n$,

(iv) η satisfies condition C_0 on K.

Then

(a) $x_n = x_0$ if $T(x_n) = 0$,

(b) the $(T, \eta; x_n)$-hypersurface $P(x_n)$ separates x_0 from K_n.

Proof. There exist solution of the problem

$$x_i \in K_i : \langle T(x_i), \eta(x, x_i)\rangle \geq_P 0 \quad \text{for all } x \in K_i$$

for all $i = 0, 1, 2, \ldots, n$; implies

$$\langle T(x_0), \eta(x, x_0)\rangle \geq_P 0 \tag{28}$$

for all $x \in K_0$;

$$\langle T(x_1), \eta(x, x_1)\rangle \geq_P 0 \tag{29}$$

for all $x \in K_1$; and so on; we get

$$\langle T(x_{n-1}), \eta(x, x_{n-1})\rangle \geq_P 0 \tag{30}$$

and

$$\langle T(x_n), \eta(x, x_n)\rangle \geq_P 0 \tag{31}$$

for all $x \in K_n$. Taking $x = x_1$ in (28), $x = x_2$ in (29) and so on; $x = x_n$ in (30) and $x = x_{n+1}$ in (31), we get

$$\langle T(x_0), \eta(x_1, x_0)\rangle \geq_P 0, \tag{32}$$

$$\langle T(x_1), \eta(x_2, x_1)\rangle \geq_P 0, \tag{33}$$

and so on;

$$\langle T(x_{n-1}), \eta(x_n, x_{n-1})\rangle \geq_P 0 \tag{34}$$

and

$$\langle T(x_n), \eta(x_{n+1}, x_n)\rangle \geq_P 0 \tag{35}$$

respectively. Adding the above inequalities, we get

$$\sum_{k=0}^{n} \langle T(x_i), \eta(x_{i+1}, x_i)\rangle \geq_P 0 \tag{36}$$

Since K_i is $(\eta; i)$-invex,

$$x_i + t\eta(x_{i+1}, x_i) \in K_i$$

for all $i = 0, 1, 2, \ldots, n$, $t \in (0, 1)$. Replacing x_i by $x_i + t\eta(x_{i+1}, x_i) \in K_i$ in (36), we get

$$\sum_{k=0}^{n} \langle T(x_i), \eta(x_i + t\eta(x_{i+1}, x_i), x_i)\rangle \geq_P 0.$$

Applying condition C_0 in the above inequality, we get

$$\sum_{k=0}^{n} \langle T(x_i), -t\eta(x_{i+1}, x_i)\rangle \geq_P 0,$$

i.e.,

$$-t\sum_{k=0}^{n} \langle T(x_i), \eta(x_{i+1}, x_i)\rangle \geq_P 0.$$

Thus

$$\sum_{k=0}^{n} \langle T(x_i), \eta(x_{i+1}, x_i)\rangle \leq_P 0,$$

i.e., T is cyclically η-monotoness. From the above equation and (36), we get

$$\sum_{k=0}^{n} \langle T(x_i), \eta(x_{i+1}, x_i)\rangle =_P 0.$$

which can be written as

$$\langle T(x_n), \eta(x_{n+1}, x_n)\rangle + \sum_{k=0}^{n-1} \langle T(x_i), \eta(x_{i+1}, x_i)\rangle =_P 0.$$

Since $x_0, x_1, x_2, \ldots, x_n \in K^o$, we get $x_{n+1} = x_0$. From the above inequality, we get

$$\langle T(x_n), \eta(x_0, x_n)\rangle + \sum_{k=0}^{n-1} \langle T(x_i), \eta(x_{i+1}, x_i)\rangle =_P 0,$$

i.e.,

$$\begin{aligned} \langle T(x_n), \eta(x_0, x_n)\rangle &=_P -\sum_{k=0}^{n-1} \langle T(x_i), \eta(x_{i+1}, x_i)\rangle \\ &\leq_P 0. \end{aligned} \tag{37}$$

But from (36), we get

$$\langle T(x_n), \eta(x_0, x_n)\rangle \geq_P 0. \tag{38}$$

Thus

$$\langle T(x_n), \eta(x_0, x_n)\rangle =_P 0,$$

but T is cyclically η-monotone, so

(a) if $T(x_n) = 0$,

$$\langle T(x_n), \eta(x_0, x_n)\rangle =_P 0 \Rightarrow x_n = x_0;$$

(b) if $T(x_n) \neq 0$ and $x_n \neq x_0$, then from (37) and (38), we get

$$\langle T(x_n), \eta(x_0, x_n)\rangle <_P 0.$$

Hence the $(T, \eta; n)$-hypersurface

$$P(x_n) = \{x \in K : \langle T(x_n), \eta(x, x_n)\rangle =_P 0\}$$

separates x_0 from K_n. This completes the proof of the theorem.

Theorem 7 (Cyclically Separation Theorem-IV). *Let X be a topological vector space. Let K be any nonempty subset of X. Let (Y, P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. Let $T : K \to L(X, Y)$ be any map. Let $\eta : K \times K \to X$ be a vector valued map. Let K be a $(n+1)$-cyclically compact with respect to the finite subset $K^o = \{x_0, x_1, x_2, \ldots, x_n\}$ of K. Let $K_i \subset K$ be the open set that contains $x_i \in K^o$ only but $x_j \notin K_i$ for $i \neq j$. Let K_i satisfies $\bigcup_{i=0}^{n} \overline{K}_i = K$ where $\overline{K}_i$ is the closure of K_i. Let $F : K \to Y$ be any map. Assume that*

(i) for $i = 0, 1, 2, \ldots, n$; K_i is contractible $(\eta; i)$-invex, compact, and connected subsets of K,

(ii) F is cyclically T-η-invex on K^o,

(iii) $\langle T(x), \eta(x, x)\rangle =_P 0$ for all $x \in K$,

(iv) there exist solution of the problem

$$x_i \in K_i : \langle T(x_i), \eta(x, x_i)\rangle \geq_P 0$$

for all $x \in K_i$, $i = 0, 1, 2, \ldots, n$,

(v) η satisfies condition C_0 on K.

Then

(a) $x_n = x_0$ if $T(x_n) = 0$,

(b) the $(T, \eta; n)$-hypersurface $P(x_n)$ separates x_0 from K_n.

Proof. F is cyclically T-η-invex on K^o, so by Theorem 4, T is η-monotone on K^o. Since all the conditions of Theorem 6 are satisfied, we have

(a) $x_n = x_0$ if $T(x_n) = 0$,

(b) the $(T, \eta; n)$-hypersurface $P(x_n)$ separates x_0 from K_n.

This completes the proof of the theorem.

4. $(GDDVCP;\lambda)$ and Linear Voltera Integro-Variational Differential Inequality Problems

In this section, we study the concept of Linear Voltera-integro variational inequality problems via generalized dominated differential vector variational inequality problems using $Sinc$ collection method.

4.1. T-η-equiinvex Function and $(GDDVCP;\lambda)$

In [16], the concept of generalized dominated differential vector variational inequality problems of order λ $(GDDVVIP;\lambda)$, generalized dominated differential complementarily problems of order λ are introduced in ordered topological vector spaces as follows.

Problem 1. *Let K be a subset of a topological vector space X. Let (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \emptyset$. Let $L(X,Y)$ be the set of all continuous linear functionals from X to Y. Let $\eta : K \times K \to X$ be a vector valued function. Let $T : K \to L(X,Y)$ be any function. Let $F : K \to Y$ be any function.*

$(GDDVVIP;\lambda)$ *Find $x_0 \in K$ such that*

$$\langle(\nabla F - \lambda T)(x_0), \eta(x,x_0)\rangle \notin -intP \quad \text{for all } x \in K,$$

and

$(GDDVCP;\lambda)$ *find $x_0 \in K$ such that*

$$\langle(\nabla F - \lambda T)(x_0), \eta(x,x_0)\rangle \notin -intP \bigcup intP \quad \text{for all } x \in K.$$

If $Y = \mathbb{R}^n$ and $P = \mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y \geq 0\}$, then the terms "$\notin -intP$" and "$-intP \bigcup intP$" will be replaced by "$\geq 0$" and "$= 0$" respectively.

For our need, the concept of T-η-invex of order λ is concentrated in finite dimensional vector spaces.

Definition 15. *Let X be a reflexive real Banach Space. Let $K \subset X$. Let $\eta : K \times K \to X$ be any vector valued map. Let $T : K \to L(X,\mathbb{R}^n) \equiv \mathbb{R}^n$ be any nonlinear map. The mapping $F : K \to \mathbb{R}^n$ is said to be T-η-invex of order $\lambda \geq 0$ on K if for all $x, x' \in K$, we have*

$$F(x) - F(x') - \lambda\langle T(x'), \eta(x,x')\rangle \geq 0.$$

Remark 3. *If $\lambda = 1$, Definition 15 coincides with Definition 4 of T-η-invex function.*

Remark 4. *Solving of F-generalized variational inequality problem (GVIP)$_F$, i.e., to :*

***(GVIP)**$_F$: find $x' \in K$ such that*

$$F(x) - F(x') - \lambda\langle T(x'), \eta(x,x')\rangle \geq 0 \quad \text{for all } x \in K,$$

is equivalent to solve the generalized variational conditional minimization problem (GVCMP):

(GVCMP)*: find* $x' \in K$ *such that*

$$F(x) - F(x') \geq 0$$

subject to

$$\langle T(x'), \eta(x, x') \rangle \geq 0 \quad \textit{for all } x \in K,$$

and the generalized variational conditional differential inequality problem (GVCDIP):

(GVCDIP)*: find* $x' \in K$ *such that*

$$\langle \nabla F(x'), \eta(x, x') \rangle \geq 0$$

subject to

$$\langle T(x'), \eta(x, x') \rangle \geq 0 \quad \textit{for all } x \in K.$$

To study the existence theorem of the problem $(GDDCP; \lambda)$, the concept of T-η-equiinvex of order λ is defined as follows.

Definition 16. *Let* X *be a reflexive real Banach Space. Let* $K \subset X$*. Let* $\eta : K \times K \to X$ *be any vector valued map. Let* $T : K \to L(X, \mathbb{R}^n) \equiv \mathbb{R}^n$ *be any nonlinear map. The* $F : K \to \mathbb{R}^n$ *is said to be* T*-*η*-equiinvex of order* λ *on* K *if there exist a* $\lambda \geq 0$ *such that for all* $x, x' \in K$*, we have*

$$F(x) - F(x') - \lambda \langle T(x'), \eta(x, x') \rangle = 0$$

where the parameter λ*, called the constant of proportionality.*

Theorem 8. *Let* X *be a reflexive real Banach Space. Let* $K \subset X$ *be a compact subset of* X*. Let* $\eta : K \times K \to X$ *be any vector valued map. Let* $T : K \to L(X, \mathbb{R}^n) \equiv \mathbb{R}^n$ *be any nonlinear map. Let* $F : K \to \mathbb{R}^n$ *be* T*-*η*-equiinvex of order* $\lambda > 0$ *at* $x_0 \in K$*. If* K *is* η*-invex cone, then* x_0 *solves* $(GDDCP; \lambda)$*.*

Proof. F is T-η-equiinvex of order $\lambda > 0$ at $x_0 \in K$, i.e., have

$$F(x) - F(x_0) - \lambda \langle T(x_0), \eta(x, x_0) \rangle = 0,$$

i.e.,

$$F(x) - F(x_0) = \lambda \langle T(x_0), \eta(x, x_0) \rangle$$

for all $x \in K$. Since K is η-invex cone, we have $x_0 + t\eta(x, x_0) \in K$ and

$$\eta(x_0 + t\eta(x, x_0), x_0) = t\eta(x, x_0)$$

for all $x \in K$ and $t \in (0, 1)$. Replacing x by $x_0 + t\eta(x, x_0)$ in the above equation, we get

$$F(x_0 + t\eta(x, x_0)) - F(x_0) = \lambda \langle T(x_0), \eta(x_0 + t\eta(x, x_0), x_0) \rangle,$$

i.e.,

$$F(x_0 + t\eta(x, x_0)) - F(x_0) = \lambda\langle T(x_0), t\eta(x, x_0)\rangle$$

for all $x \in K$ and $t \in (0, 1)$. Thus

$$F(x_0 + t\eta(x, x_0)) - F(x_0) = \lambda t\langle T(x_0), \eta(x, x_0)\rangle,$$

i.e.,

$$\frac{F(x_0 + t\eta(x, x_0)) - F(x_0)}{t} = \lambda\langle T(x_0), \eta(x, x_0)\rangle,$$

for all $x \in K$ and $t \in (0, 1)$. Taking limit $t \to 0$, we get

$$\langle \nabla F(x_0), \eta(x, x_0)\rangle = \lambda\langle T(x_0), \eta(x, x_0)\rangle$$

for all $x \in K$. Hence x_0 solves $(GDDCP; \lambda)$. This completes proof of the theorem.

4.2. Generalized Integro-Differential Dominated Equillibrium Problem

From Theorem 8, it is observed that if a function F is T-η-equiinvex of order λ at $x_0 \in K$, then x_0 can be a solution of the integral equation associated with F under certain condition, which is define a new type of problem, called Generalized Integro-Differential Dominated Variational Equillibrium Problem $(GIDDEP)$.

Problem 2 $(GIDDVEP)$**.** *Let $M \subset X$ be nonempty subset of a reflexive real Banach Space X. Let $F : M \to \mathbb{R}^n$, $\eta : M \times M \to X$ be the maps for which $\langle \nabla F(t), \eta(x, t)\rangle$ exists. Let $T : M \to L(X, \mathbb{R}^n) \equiv \mathbb{R}^n$ be any nonlinear map such that $\langle T(t), \eta(x, t)\rangle$ exists. If the map F is T-η-equiinvex of order λ (or at least T-η-invex of order λ) on M, we can find a analytic function $y : M \to \mathbb{R}^n$ such that*

$$\int_a^x \langle \nabla F(t), \eta(x, t)\rangle \; y(t) \; dt = \lambda \int_a^x \langle T(t), \eta(x, t)\rangle \; y(t) \; dt \tag{39}$$

for all $x \in M$. In this case, the weight function $y(x)$ is said be the function of equilibrium.

4.3. *Sinc* function

Let $M \subset X$ be nonempty subset of a reflexive real Banach Space X. Let $p(x)$, $q(x)$, $F(x)$ and $K(x, t)$ be analytic functions, then the second order linear Voltera Integro-Differential Equation $(LVIDE_2)$-BVP is to find a analytic function $y(x)$ such that

$$y'' + p(x)y' + q(x)y = F(x) + \rho \int_a^x K(x, t)y(t)dt, x \in [a, b] \tag{40}$$

with boundary condition

$$y(a) = y_a \quad \text{and} \quad y(b) = y_b$$

where ρ is a parameter; y_a and y_b are real constants.

Numerical methods for the solution of $(LVIDE_2)$-BVP have been studied by the authors [1, 2, 28, 27, 34]. Theorems which list the conditions for the existence and uniqueness of solutions of such problems are contained in a bood by Agarwal [1]. Two point boundary value problem for integro-differential equation of second order is discussed by Morchalo [28]. Also, Morchalo [27] studied two point boundary value problem for integro-differential equation of higher order. A reliable algorithm for solving the boundary value problems for higher order integro-differential equation has been studied by Wazwaz [34]. Babolian et al [2], applied operational matrices of piecewise constant orthogonal functions for solving Voltera integral and integro-differential equations. They first obtained Laplace transform of the problem and found numerical inversion of Laplace transform by operational matrices. $Sinc$ Methods have increasing been recognized as powerful tools for attacking problems in applied physics adn engineering. The books [25, 33] provide excellent overviews of methods based on $Sinc$ functions for solving ordinary, partial differential equations and integro-differential equations. In [19, 26], the $Sinc$ collection procedures for the eigen vaule problems are presented. Ng [30] employed the preconditioned conjugate gradient method with boundary value problem using $Sinc$-Galerkin method. Koonpraset [23] presented a block matrix formulation for the $Sinc$-Galerkin technique applied to Wind-driven current problem form oceanography. In [32, 31] , the $Sinc$ methods are used for the numerical solutions of integral equations. Approximation by $Sinc$ functions are typified by errors of the form

$$\epsilon_k = O\left(e^{-\frac{k}{h}}\right),$$

where $k > 0$ is a constant and h is a step size. To make the paper self contained, we recall the concept of $Sinc$ function and its properties.

Definition 17. *The $Sinc$ function is defined on the real line by*

$$Sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$$

For any $h > 0$, the translated $Sinc$ functions with evenly space nodes are given as follows.

$$S(j,h)(x) = Sinc\left(\frac{x - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \ldots,$$

which are called the j^{th} $Sinc$ function. The $Sinc$ function form for the interpolating point $x_k = kh$ is given by

$$S(j,h)(kh) = \delta_{jh}^{(0)} = \begin{cases} 1 & \text{if } k = j; \\ 0 & \text{if } k \neq j. \end{cases}$$

Let

$$\sigma_{kj} = \int_{0}^{k-j} \frac{\sin(\pi t)}{\pi t} dt, \quad \text{and} \quad \delta_{kj}^{(-1)} = \frac{1}{2} + \sigma_{kj}$$

then define a matrix whose $(k,j)^{\text{th}}$ entry is given by $\delta_{kj}^{(-1)}$ as

$$I^{(-1)} = \left[\delta_{kj}^{(-1)}\right].$$

If F is defined on real line, then for $h > 0$ the series

$$C(F,h)(x) = \sum_{j=-\infty}^{\infty} F(jh) Sinc\left(\frac{x - jh}{h}\right),$$

is called the Whittaker cardinal expansion of F, whenever the series converges. But in practical, we need to use some specific numbers of terms in the above series such as $j = -N, \ldots, N$, where N is the number of $Sinc$ grid points. They are based in the infinite strip D_d in the complex plane.

$$D_d = \left\{\omega = u + iv : |v| < d \leq \frac{\pi}{2}\right\}.$$

To construct approximation on the interval $\Gamma = [a, b]$, we consider the conformal map

$$\phi(z) = \ln\left(\frac{z - a}{b - z}\right).$$

The map ϕ carries the eye-shaped region

$$D = \left\{z = x + iy : \left|\arg\left(\frac{z - a}{b - z}\right)\right| < d \leq \frac{\pi}{2}\right\}.$$

For the $Sinc$ method, the basis functions on the interval $\Gamma = [a, b]$ for $z \in D$ are derived from the composite translated $Sinc$ functions,

$$S_j(z) = S(j,h) \circ \phi(z) = Sinc\left(\frac{\phi(z) - jh}{h}\right).$$

The function

$$z = \phi^{-1}(\omega) = \frac{a + be^{\omega}}{1 + e^{\omega}},$$

is an inverse map of $\omega = \phi(z)$. Define the range of ϕ^{-1} on the real line as

$$\Gamma = \left\{\psi(y) = \phi^{-1}(y) \in D : -\infty < y < \infty\right\} = [a, b].$$

The $Sinc$ grid points $z_k \in \Gamma$ in D will be denoted by x_k because they are real. For evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \quad k = 0, \pm 1, \pm 2, \ldots.$$

For further explanation of the procedure, the important class of functions is denoted by $L_\alpha(D)$. The properties of the functions in $L_\alpha(D)$ and detailed discussions are given in [25, 33]. We recall the following definition and theorems for our purpose.

Definition 18. *[35] Let $L_\alpha(D)$ be the set of all analytic functions y in D, for which there exist a constant C such that*

$$y(z) \le C\frac{|\rho(z)|^\alpha}{[1+|\rho(z)|]^{2\alpha}}$$

for all $z \in D$, $0 < \alpha \le 1$ where $\rho(z) = e^{\phi(z)}$.

Theorem 9. *[35] Let $y \in L_\alpha(D)$, N be a positive integer, and h be the step length obtained by the formula $h = \sqrt{\frac{\pi d}{\alpha N}}$, then there exist a positive constant C_1, independent of N, such that*

$$\sup_{x\in\Gamma}\left| y(x) - \sum_{j=-N}^{N} y(x_j)S(j,h)\circ\phi(x)\right| \le C_1 e^{-\sqrt{\pi d\alpha N}}.$$

Theorem 10. *[35] Let $\frac{y}{\phi'} \in L_\alpha(D)$, with $0 < \alpha \le 1$ and N be a positive integer, and h be the step length obtained by the formula $h = \sqrt{\frac{\pi d}{\alpha N}}$, then there exist a positive constant C_2, independent of N, such that*

$$\left|\int_a^{x_k} y(t)dt - h\sum_{k=-N}^{N}\delta_{kj}^{-1}\frac{y_k}{\phi'(t_k)}\right| \le C_2 e^{-\sqrt{\pi d\alpha N}}.$$

The n^{th} derivative of $y(x)$ at some point x_k can be approximated by using the finite number of as

$$y^{(n)}(x_k)=h^{-n}\sum_{j=-N}^{N}\delta_{jk}^{n} y_k \tag{41}$$

where

$$\delta_{jk}^{n} = h^n\frac{d^n}{d\phi^n}\{S(j,h)\circ\phi(x)\}_{x=x_k}. \tag{42}$$

In particular,

$$\delta_{jk}^{0} = \{S(j,h)\circ\phi(x)\}_{x=x_k} = \begin{cases} 1 & \text{if } k=j; \\ 0 & \text{if } k\neq j, \end{cases}$$

$$\delta_{jk}^{1} = h\frac{d}{d\phi}\{S(j,h)\circ\phi(x)\}_{x=x_k} = \begin{cases} 0 & \text{if } k=j; \\ \frac{(-1)^{k-j}}{k-j} & \text{if } k\neq j, \end{cases}$$

and

$$\delta_{jk}^{2} = h^2\frac{d^2}{d\phi^2}\{S(j,h)\circ\phi(x)\}_{x=x_k} = \begin{cases} \frac{-\pi^2}{3} & \text{if } k=j; \\ \frac{-2(-1)^{k-j}}{(k-j)^2} & \text{if } k\neq j. \end{cases}$$

4.4. $(LVIVDE_2)$ via $(GDDCP;\lambda)$

Let $X=\mathbb{R}$, $M=[a,b]$. Let $T:M\to X^*\equiv\mathbb{R}$. Let $F:M\to\mathbb{R}$ be T-η-invex of order λ on K. Let $x_0\in K$ solves $(GDDCP;\lambda)$, i.e.,

$$\langle\nabla F(x_0),\eta(x,x_0)\rangle=\lambda\langle T(x_0),\eta(x,x_0)\rangle \tag{43}$$

for all $x\in M$. Let for the given analytic functions $p(x)$, $q(x)$, there exist a analytic function $y(x)$ on M such that

$$y^{(n)}+c_{(n-1)}(x)y^{(n-1)}+\ldots+c_1(x)y'+c_0(x)y=F(x)+\int_a^x\langle\nabla F(t),\eta(x,t)\rangle\;y(t)\,dt, \tag{44}$$

which the general form of the problem given in (40). In particular, the problem given in (44) coincides with $(LVIDE_2)$ given in (40) if $n=2$, $K(x,t)=\frac{1}{\rho}\langle\nabla F(t),\eta(x,t)\rangle$ where ρ is the parameter. We call the problem given in (44) as higher order linear generalized Voltera integro-differential equation $(GLVIDE_n)$. By Remark 2, we define the the variational form of integro-differential equation as

$$y^{(n)}+c_{(n-1)}(x)y^{(n-1)}+\ldots+c_1(x)y'+c_0(x)y=F(x)+\lambda\int_a^x\langle T(t),\eta(x,t)\rangle\;y(t)\,dt \tag{45}$$

for all $x\in M$ which we call, the higher order linear Voltera integro-variational differential equation $(LVIVDE_n)$ and it coincides with $(LVIDE_2)$ given in (40) if $n=2$, and $T(x)=I$, the identity map.

4.5. Second order (*LVIVDE*) Boundary Value Problem and its solution

The main purpose of this section is to study the second order linear Voltera Integro-Variational Differential Equation $(LVIVDE_2)$, i.e., to find a finite collection procedure for the solution of $(LVIVDE_2)$ given with boundary conditions which is developed by using the $Sinc$ functions. Here the properties of the $Sinc$ functions are utilized to evaluate the unknown coefficients.

Let $X=\mathbb{R}$ and its dual be X^*. Let $M=[a,b]\subset X$. Let $F,y,p,q:M\to\mathbb{R}$ be any maps. Let $\eta:M\times M\to X$ be any vector valued map and let $T:M\to X^*$ be any map. Let $F(x)$, $y(x)$, $p(x)$, $q(x)$ and $\langle T(t),\eta(x,t)\rangle$ be analytic functions. We consider the boundary value problem with second order linear Voltera integro-variational differential equation $(LVIVDE_2)$-BVP of the form

$$y''+p(x)y'+q(x)y=F(x)+\lambda\int_a^x\langle T(t),\eta(x,t)\rangle\;y(t)dt,\quad x\in M \tag{46}$$

with boundary condition

$$y(a)=y_a\quad\text{and}\quad y(b)=y_b$$

where λ is a parameter; y_a and y_b are real constants. $y(x)$ is the solution to be determined.

Now $(LVIVDE_2)$-BVP is approximated by the following linear combination of the $Sinc$ functions

$$y(x) \approx y_n(x) = \sum_{j=-N}^{N} y_j S(j,h) \circ \phi(x), \quad n = 2N+1.$$

For the second term on the right hand side (8), we assume that

$$\frac{\langle T(t), \eta(x,t)\rangle}{\phi'(t)} \in L_\alpha(D),$$

then by setting $x = x_k$ and using Theorem 10, we obtain

$$\int_a^{x_k} \langle T(t), \eta(x,t)\rangle y(t)dt \approx h \sum_{j=-N}^{N} \delta_{jk}^{-1} \frac{\langle T(t), \eta(x,t)\rangle}{\phi'(t)} y_j, \tag{47}$$

where y_j denotes an approximate value of $y(x_j)$. Replacing x by x_k, $y(x)$ by $y_n(x_k)$ and using (47) in (46), we get the collocation result:

$$y_n''(x_k)+p(x)y_n'(x_k)+q(x_k)y_n(x_k) = F(x_k)+\lambda h \sum_{j=-N}^{N} \delta_{jk}^{-1} \frac{\langle T(t), \eta(x,t)\rangle}{\phi'(t)} y_j \quad x \in [a,b], \tag{48}$$

where

$$y_n(x) = \sum_{j=-N}^{N} y_j [S(j,h) \circ \phi(x)], \tag{49}$$

$$y_n'(x) = \sum_{j=-N}^{N} y_j [S(j,h) \circ \phi(x)]', \tag{50}$$

and

$$y_n''(x) = \sum_{j=-N}^{N} y_j [S(j,h) \circ \phi(x)]''. \tag{51}$$

At $x = x_k$, we have

$$\begin{aligned} \left[S(j,h)\circ\phi(x)\right]'\Big|_{x=x_k} &= \frac{d}{d\phi}\left[S(j,h)\circ\phi(x)\right]\phi'(x)\Bigg|_{x=x_k} \\ &= \phi'(x_k)\frac{d}{d\phi}\left[S(j,h)\circ\phi(x)\right]\Bigg|_{x=x_k} \\ &= \phi'(x_k)h^{-1}\delta_{jk}^{(1)}, \end{aligned}$$

$$\left[S(j,h)\circ\phi(x)\right]''\Big|_{x=x_k} = \left[\phi'(x)\frac{d}{d\phi}\left[S(j,h)\circ\phi(x)\right]\right]'\Bigg|_{x=x_k}$$

$$= \left[\phi''(x) \frac{d}{d\phi} \left[S(j,h) \circ \phi(x) \right] + \phi'(x) \frac{d^2}{d\phi^2} \left[S(j,h) \circ \phi(x) \right] \right] \Bigg|_{x=x_k}$$
$$= \phi'(x_k) h^{-1} \delta^{(1)}_{jk} + (\phi'(x_k))^2 h^{-2} \delta^{(2)}_{jk}$$

Thus

$$F(x_k) = \sum_{j=-N}^{N} [\frac{1}{h^2}(\phi'(x_k))^2 \; \delta^{(2)}_{jk} + \frac{1}{h} \left[\phi''(x_k) + p(x_k) \; \phi'(x_k) \right] \; \delta^{(1)}_{jk} + q(x_k) \; \delta^{(0)}_{jk}$$
$$- \lambda h \frac{\langle T(t_j), \eta(x_k, t_j) \rangle}{\phi'(t_j)} \; \delta^{(-1)}_{jk}] y_j$$

Multiplying the above equation by $\frac{h^2}{(\phi'(x_k))^2}$, we get

$$\frac{h^2}{(\phi'(x_k))^2} F(x_k) = \sum_{j=-N}^{N} [\delta^{(2)}_{jk} + h \left[\frac{\phi''(x_k)}{(\phi'(x_k))^2} + \frac{p(x_k)}{\phi'(x_k)} \right] \; \delta^{(1)}_{jk} + h^2 \frac{q(x_k)}{(\phi'(x_k))^2} \delta^{(0)}_{jk}$$
$$- \lambda h^3 \frac{\langle T(t_j), \eta(x_k, t_j) \rangle}{\phi'(t_j)(\phi'(x_k))^2} \delta^{(-1)}_{jk}] y_j$$

Since

$$\delta^{(0)}_{jk} = \delta^{(0)}_{kj}, \quad \delta^{(1)}_{jk} = -\delta^{(1)}_{kj}, \quad \delta^{(2)}_{jk} = \delta^{(2)}_{kj}$$

and

$$\frac{\phi''(x_k)}{(\phi'(x_k))^2} = - \left(\frac{1}{\phi'(x_k)} \right)^2,$$

we get

$$\frac{h^2}{(\phi'(x_k))^2} F(x_k) = \sum_{j=-N}^{N} [\delta^{(2)}_{jk} + h \left[\frac{p(x_k)}{\phi'(x_k)} - \left(\frac{1}{\phi'(x_k)} \right)^2 \right] \delta^{(1)}_{jk} + h^2 \frac{q(x_k)}{(\phi'(x_k))^2} \delta^{(0)}_{jk} \tag{52}$$
$$- \lambda h^3 \frac{\langle T(t_j), \eta(x_k, t_j) \rangle}{\phi'(t_j)(\phi'(x_k))^2} \delta^{(-1)}_{jk}] y_j \tag{53}$$

Set $I^{(m)} = \delta^{(m)}_{jk}$, $m = -1, 0, 1, 2$ where $\delta^{(m)}_{jk}$ is the $(k, j)^{\text{th}}$ element of the matrix $I^{(m)}$,

$$\tau = \frac{\langle T(t_j), \eta(x_k, t_j) \rangle}{\phi'(t_j)(\phi'(x_k))^2},$$

and

$$D \left(\frac{1}{\phi'} \right) = diag \left(\frac{1}{\phi'(x_{-N})}, \frac{1}{\phi'(x_{-(N-1)})}, \ldots, \frac{1}{\phi'(x_N)} \right).$$

τ and $I^{(m)}$, $m = -1, 0, 1, 2$ are the square matrices of order $(2N+1) \times (2N+1)$. The linear system (52) is written in the matrix form

$$AY = P, \tag{54}$$

where

$$A = I^{(2)} + hD\left(\left(\frac{1}{\phi'}\right)' - \frac{P}{\phi'}\right) I^{(1)} + h^2 D\left(\frac{q}{\phi'}^{2}\right) I^{(0)} - \lambda h^3 \left(\tau \circ I^{(-1)}\right),$$

$$P = h^2\left[\frac{F(x_{-N})}{[\phi'(x_{-N})]^2}, \frac{F(x_{-(N-1)})}{[\phi'(x_{-(N-1)})]^2}, \ldots, \frac{F(x_N)}{[\phi'(x_N)]^2}\right],$$

and

$$Y = \left[Y_{-N}, Y_{-(N-1)}), \ldots, Y_N\right]$$

where "$\circ$" denotes the *Hadamard* matrix multiplication. The above linear system containing $(2N+1)$ equations with $(2N+1)$ unknown coefficients $[Y_{-N}, Y_{-(N-1)}), \ldots, Y_N]$. Solving the linear system, we can obtain the approximate solution of $(LVIVDE_2)$-BVP as follows

$$y_n(x) = \sum_{j=-N}^{N} y_j[S(j,h) \circ \phi(x)], \quad n = 2N+1.$$

4.6. BVP with Modified Second Order (*LVIVDE*) and its Solution

Given

$$y(a) = y_a \quad \text{and} \quad y(b) = y_b$$

not all zero. Let

$$u_n(x) = y(x) - \frac{b-x}{b-a} y_a - \frac{x-a}{b-a} y_b,$$

then $(LVIVDE_2)$ reduces to

$$u'' + p(x)u' + q(x)u = G(x) + \lambda \int_a^x \langle T(t), \eta(x,t)\rangle\, u(t)dt, \quad x \in [a,b] \tag{55}$$

where

$$u(a) = 0, \quad u(b) = 0 \tag{56}$$

and

$$\begin{aligned} G(x) =& F(x) - \frac{y_b - y_a}{b-a} p(x) - \frac{(y_b - y_a)x - a y_b + b y_a}{b-a} q(x) \\ &+ \lambda \int_a^x \frac{(y_b - y_a)t - a y_b + b y_a}{2(b-a)} \langle T(t), \eta(x,t)\rangle\, dt. \end{aligned}$$

Using (54) and (56), we get the modified approximate solution

$$u_n(x) = \sum_{j=-N}^{N} y_j [S(j,h) \circ \phi(x)], \quad n = 2N + 1.$$

4.7. Numerical Examples

The *Sinc* collocation method is used to find the approximate solution of $(LVIVDE_2)$-BVP for different value of N, d and $\alpha \in (0,1]$. For our examples, we take $d = \frac{\pi}{2}$, and $\alpha = 1$ which gives $h = \frac{\pi}{\sqrt{2N}}$. The errors are reported on the set of *Sinc* grid points

$$S = [x_{-N}, x_{-(N-1)}, \ldots, x_0 \ldots, x_N],$$

and

$$x_k = \frac{e^{kh}}{1 + e^{kh}}, \quad k = -N, \ldots N.$$

The maximum error on the *Sinc* grid pints

$$\epsilon_S(h) = \max_{-N \le j \le N} |y(x_j) - y_n(x_j)|.$$

In the following example, we illustrate the performance of *Sinc* collocation method to find the approximate solution of $(LVIVDE_2)$-BVP with a given exact solution.

Example 2. *Consider the*

$$y'' - \frac{1}{x} y' + y = F(x) + \lambda \int_a^x \langle T(t), \eta(x,t) \rangle \; y(t) dt, \quad x \in [0,1] \tag{57}$$

with boundary condition

$$y(0) = 0 \quad \textit{and} \quad y(1) = 0$$

with the exact solution $y(x) = x - x^3$, *where*

$$T(x) = x, \quad \eta(x,u) = \frac{e^u \sin t}{x u^2}$$

and

$$F(x) = -x^3 - x \left(2 - \frac{x e^x}{\sqrt{2}} \sin \left(x - \frac{\pi}{4} \right) \right) + \frac{e^x}{x} (x \cos x - \sin x - e^{-x})$$

for all $x, u \in [0,1]$.

This example has been solved for differential values of N and $h = \frac{\pi}{\sqrt{2N}}$. *The maximum of absolute errors on the Sinc grid S are tabulated in Table 1 which indicates that the error decreases more rapidly as N increases.*

Table 1: Results for example (2)

N	h	$\epsilon_S(h)$
5	0.9934588266	0.592003×10^{-4}
10	0.7024814731	0.965024×10^{-5}
15	0.5735737721	0.223322×10^{-5}
20	0.4967294133	0.630761×10^{-5}
25	0.4442882938	0.204361×10^{-5}
30	0.4055778676	0.732063×10^{-7}
35	0.3754921418	0.283447×10^{-7}
40	0.3542407366	0.1166816×10^{-8}

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