EUROPEAN JOURNAL OF MATHEMATICAL SCIENCES
Vol. 2, No. 1, 2013, 85-90
ISSN 2147-5512 – www.ejmathsci.com

On Certain Classes of $p-$Valent Meromorphic Functions Associated with a Family of Integral Operators

Rabha M. El-Ashwah[1], M. K. Aouf[2], Ahmed M. Abd-Eltawab [3,*]

[1] *Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt*
[2] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*
[3] *Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt*

Abstract. This paper gives some inclusion relationships of certain class of $p-$valent meromorphic functions which are defined by using the linear operator $Q^{p,\mu}_{\alpha,\beta,\gamma}$. Further, a property preserving integrals is considered.
2010 Mathematics Subject Classifications: 30C45
Key Words and Phrases: Differential subordination, Hadamard product, meromorphic function, Hypergeometric function.

1. Introduction

For any integer $m > -p$, let $\Sigma_{p,m}$ denote the class of all meromorphic functions f of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1}$$

which are analytic and $p-$valent in the punctured disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \backslash \{0\}$. For convenience, we write $\Sigma_{p,-p+1} = \Sigma_p$. If f and g are analytic in U, we say that f is subordinate to g, written symbolically as, $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function g is univalent in U, we have the equivalence [see for example 5]:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f \in \Sigma_{p,m}$ given by (1), and $g \in \Sigma_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \quad (m > -p; p \in \mathbb{N}), \tag{2}$$

*Corresponding author.

Email addresses: mkaouf127@yahoo.com (M. Aouf), ams03@fayoum.edu.eg (A. Abd-Eltawab)

then the Hadamard product (or convolution) of f and g is given by

$$(f*g)=z^{-p}+\sum_{k=m}^{\infty}a_kb_kz^k=(g*f)(z)\quad (m>-p;p\in\mathbb{N}). \tag{3}$$

For $f\in\Sigma_{p,m}$ we introduce the integral operator $Q^p_{\alpha,\beta,\gamma}:\Sigma_{p,m}\to\Sigma_{p,m}$ as follows:

$$\begin{aligned}Q^p_{\alpha,\beta,\gamma}f(z)=&\frac{\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\beta)\Gamma(\alpha-\gamma+1)}\frac{1}{z^{\beta+p}}\int_0^z\left(1-\frac{t}{z}\right)^{\alpha-\gamma}t^{\beta+p-1}f(t)\,dt\\ =&\frac{1}{z^p}+\frac{\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+p+k)}{\Gamma(\alpha+\beta+p+k-\gamma+1)}a_kz^k\\ &\left(\beta>0;\ \alpha>\gamma-1;\gamma>0;\ p\in\mathbb{N};z\in U^*\right),\end{aligned} \tag{4}$$

and $Q^p_{\gamma-1,\beta,\gamma}f(z)=f(z)\ \left(\beta>0;\gamma>0;\ p\in\mathbb{N};z\in U^*\right)$.

By setting

$$\begin{aligned}f^p_{\alpha,\beta,\gamma}(z)=&z^{-p}+\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta-\gamma+1)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+\beta+p+k-\gamma+1)}{\Gamma(\beta+p+k)}z^k\\ &\left(\beta>0;\ \alpha>\gamma-1;\gamma>0;\ p\in\mathbb{N};z\in U^*\right),\end{aligned} \tag{5}$$

we define a new function $f^{p,\mu}_{\alpha,\beta,\gamma}$ in terms of the Hadamard product as follows

$$\begin{aligned}f^p_{\alpha,\beta,\gamma}(z)*f^{p,\mu}_{\alpha,\beta,\gamma}(z)=&\frac{1}{z^p(1-z)^\mu}\\ &\left(\mu>0;\beta>0;\ \alpha>\gamma-1;\gamma>0;\ p\in\mathbb{N};z\in U^*\right).\end{aligned} \tag{6}$$

Now we introduce the operator $Q^{p,\mu}_{\alpha,\beta,\gamma}:\Sigma_{p,m}\to\Sigma_{p,m}$ as follows:

$$Q^{p,\mu}_{\alpha,\beta,\gamma}f(z)=f^{p,\mu}_{\alpha,\beta,\gamma}(z)*f(z)\quad\left(z\in U^*;f\in\Sigma_{p,m}\right). \tag{7}$$

We can easily find from (5), (6), and (7) that

$$\begin{aligned}Q^{p,\mu}_{\alpha,\beta,\gamma}f(z)=&z^{-p}+\frac{\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\beta)}\\ &\cdot\left[\sum_{k=0}^{\infty}\left(\frac{\Gamma(\beta+p+k)}{\Gamma(\alpha+\beta+p+k-\gamma+1)}\right)\frac{(\mu)_{k+p}}{(k+p)!}a_kz^k\right]\\ &\left(\mu>0;\beta>0;\ \alpha>\gamma-1;\gamma>0;\ p\in\mathbb{N};z\in U^*\right),\end{aligned} \tag{8}$$

and $Q^{p,1}_{\gamma-1,\beta,\gamma}f(z)=f(z)\ \left(\beta>0;\ \mu=1;\gamma>0;\ p\in\mathbb{N};z\in U^*\right)$, where $(\mu)_k$ is the Pochhammer symbol defined by

$$(\mu)_k=\begin{cases}1 & (k=0),\\ \mu(\mu+1)\ldots(\mu+k-1) & (k\in\mathbb{N})\end{cases}$$

From (8), it is easy to verify that

$$z\left(Q^{p,\mu}_{\alpha,\beta,\gamma}f(z)\right)' = \mu Q^{p,\mu+1}_{\alpha,\beta,\gamma}f(z) - (\mu+p)\, Q^{p,\mu}_{\alpha,\beta,\gamma}f(z), \tag{9}$$

Remark 1. *(i) For $\mu = 1$, we have $Q^{p,1}_{\alpha,\beta,\gamma} = Q^{p}_{\alpha,\beta,\gamma}$;*

(ii) For $\mu = \gamma = 1$, $Q^{p,1}_{\alpha,\beta,1} = Q^{p}_{\alpha,\beta}$, where the operator $Q^{p}_{\alpha,\beta}$ was introduced and studied by Aqlan et al. [2] (see also [1]);

(iii) For $p = 1$, $Q^{1,\mu}_{\alpha,\beta} = Q^{\mu}_{\alpha,\beta}$, where the operator $Q^{\mu}_{\alpha,\beta}$ was introduced and studied by Wang et al. [7];

(iv) For $p = \mu = \gamma = 1$, $Q^{1,1}_{\alpha,\beta,1} = Q_{\alpha,\beta}$, where the operator $Q_{\alpha,\beta}$ was introduced and studied by Lashin [4].

Definition 1. *We say that a function $f \in \Sigma_{p,m}$ is in the class $\Sigma^{\mu}_{p,m}(\alpha,\beta,\gamma,\lambda)$, if it satisfies the following condition:*

$$\operatorname{Re}\left\{ -\frac{z^{p+1}\left(Q^{p,\mu}_{\alpha,\beta,\gamma}f(z)\right)'}{p} \right\} > \lambda, \quad z \in U^*. \tag{10}$$

where $\beta > 0$, $\alpha > \gamma - 1$, $\gamma > 0$, $\mu > 0$, $0 \le \lambda < 1$, and $p \in \mathbb{N}$.

Using (9) condition (10) can be re-written in the form

$$\operatorname{Re}\left\{ -\mu\frac{z^{p}Q^{p,\mu+1}_{\alpha,\beta,\gamma}f(z)}{p} + (\mu+p)\frac{z^{p}Q^{p,\mu}_{\alpha,\beta,\gamma}f(z)}{p} \right\} > \lambda, \quad 0 \le \lambda < 1, z \in U^*. \tag{11}$$

2. Basic Properties of the Class $\Sigma^{\mu}_{p,m}(\alpha,\beta,\gamma,\lambda)$

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\beta > 0$, $\alpha > \gamma - 1$, $\gamma > 0$, $\mu > 0$, $0 \le \lambda < 1$, and $p \in \mathbb{N}$.

We begin by recalling the following result (Jack's lemma), which we shall apply in proving our inclusion theorems below.

Lemma 1. *[3] Let the (nonconstant) function $w(z)$ be analytic in U, with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then $z_0 w'(z_0) = \xi w(z_0)$, where ξ is a real number and $\xi \ge 1$.*

Theorem 1. *The following inclusion property holds true for the class $\Sigma^{\mu}_{p,m}(\alpha,\beta,\gamma,\lambda)$:*

$$\Sigma^{\mu+1}_{p,m}(\alpha,\beta,\gamma,\lambda) \subset \Sigma^{\mu}_{p,m}(\alpha,\beta,\gamma,\lambda). \tag{12}$$

Proof. Let $f(z) \in \Sigma_{p,m}^{\mu+1}(\alpha,\beta,\gamma,\lambda)$ and define a regular function $w(z)$ in U such that $w(0)=0$, $w(z)\neq -1$ by

$$-\mu Q_{\alpha,\beta,\gamma}^{p,\mu+1}f(z)+(\mu+p)Q_{\alpha,\beta,\gamma}^{p,\mu}f(z)=pz^{-p}\frac{1+(2\lambda-1)w(z)}{1+w(z)}. \tag{13}$$

Differentiating (13) with respect to z, we obtain

$$-\frac{z^{p+1}\left(Q_{\alpha,\beta,\gamma}^{p,\mu+1}f(z)\right)'}{p}=\frac{1+(2\lambda-1)w(z)}{1+w(z)}-\frac{2(1-\lambda)}{\mu}\frac{zw'(z)}{(1+w(z))^2}. \tag{14}$$

We claim that $|w(z)|<1$ for $z\in U$. Otherwise there exists a point $z_0\in U$ such that $\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1$. Applying Jack's lemma, we have

$$z_0w'(z_0)=\xi w(z_0) \quad (\xi\geq 1). \tag{15}$$

From (14) and (15) we have

$$-\frac{z_0^{p+1}\left(Q_{\alpha,\beta,\gamma}^{p,\mu+1}f(z_0)\right)'}{p}=\frac{1+(2\lambda-1)w(z_0)}{1+w(z_0)}-\frac{2(1-\lambda)}{\mu}\frac{\xi w(z_0)}{\left(1+w(z_0)\right)^2}. \tag{16}$$

Since $\mathrm{Re}\left\{\frac{1+(2\lambda-1)w(z_0)}{1+w(z_0)}\right\}=\lambda$, $\xi\geq 1$, and $\frac{\xi w(z_0)}{\left(1+w(z_0)\right)^2}$ is real and positive, we see that $\mathrm{Re}\left\{-\frac{z_0^{p+1}\left(Q_{\alpha,\beta,\gamma}^{p,\mu+1}f(z_0)\right)'}{p}\right\}<\lambda$, which obviously contradicts $f(z)\in\Sigma_{p,m}^{\mu+1}(\alpha,\beta,\gamma,\lambda)$. Hence $|w(z)|<1$ for $z\in U$, and it follows from (13) that $f(z)\in\Sigma_{p,m}^{\mu}(\alpha,\beta,\gamma,\lambda)$. This completes the proof of Theorem 1.

Theorem 2. *Let* **c** *be any real number and $c>0$. If $f(z)\in\Sigma_{p,m}^{\mu}(\alpha,\beta,\gamma,\lambda)$, then*

$$J_{c,p}(z)=\frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)\,dt\ \in\Sigma_{p,m}^{\mu}(\alpha,\beta,\gamma,\lambda) \quad (c>0). \tag{17}$$

Proof. From (17), we have

$$z\left(Q_{\alpha,\beta,\gamma}^{p,\mu}J_{c,p}(z)\right)'=cQ_{\alpha,\beta,\gamma}^{p,\mu}f(z)-(c+p)\,Q_{\alpha,\beta,\gamma}^{p,\mu}J_{c,p}(z). \tag{18}$$

Define a regular function $w(z)$ in U such that $w(0)=0$, $w(z)\neq -1$ by

$$-\frac{z^{p+1}\left(Q_{\alpha,\beta,\gamma}^{p,\mu}J_{c,p}(z)\right)'}{p}=\frac{1+(2\lambda-1)w(z)}{1+w(z)}. \tag{19}$$

From (18) and (19) we have

$$cQ^{p,\mu}_{\alpha,\beta,\gamma}f(z) - (c+p)\, Q^{p,\mu}_{\alpha,\beta,\gamma}J_{c,p}(z) = pz^{-p}\frac{1+(2\lambda-1)w(z)}{1+w(z)}. \tag{20}$$

Differentiating (20) with respect to z, and using (19)) we obtain

$$-\frac{z^{p+1}\left(Q^{p,\mu}_{\alpha,\beta,\gamma}f(z)\right)'}{p} = \frac{1+(2\lambda-1)w(z)}{1+w(z)} - \frac{2(1-\lambda)}{c}\frac{zw'(z)}{(1+w(z))^2}. \tag{21}$$

The remaining part of the proof of Theorem 2 is similar to that of Theorem 1.

Theorem 3. *If $f(z) \in \Sigma_{p,m}$, and satisfy the condition*

$$\operatorname{Re}\left\{-\frac{z^{p+1}\left(Q^{p,\mu}_{\alpha,\beta,\gamma}\ f(z)\right)'}{p}\right\} > \lambda - \frac{(1-\lambda)}{2c} \quad (c>0). \tag{22}$$

Then the function

$$J_{c,p}(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)\,dt \ \in \Sigma^{\mu}_{p,m}(\alpha,\beta,\gamma,\lambda).$$

The proof of Theorem 3 is similar to that of Theorem 2 and so we omit it.

Theorem 4. *Let $f(z)$ be defined by*

$$J_{c,p}(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)\,dt \quad (c>0). \tag{23}$$

If $J_{c,p}(z) \in \Sigma^{\mu}_{p,m}(\alpha,\beta,\gamma,\lambda)$, then $f(z) \in \Sigma^{\mu}_{p,m}(\alpha,\beta,\gamma,\lambda)$ in $|z| < \frac{c}{1+\sqrt{c^2+1}}$.

Proof. Since $F(z) \in \Sigma^{\mu}_{p,m}(\alpha,\beta,\gamma,\lambda)$ we can write

$$-z\left(Q^{p,\mu}_{\alpha,\beta,\gamma}\ J_{c,p}(z)\right)' = pz^{-p}\left[\lambda + (1-\lambda)u(z)\right], \tag{24}$$

where $u(z) \in P$, the class of functions with positive real part in the unit disk U and normalized by $u(0) = 1$. We can re-write (24) as

$$-\mu Q^{p,\mu+1}_{\alpha,\beta,\gamma}J_{c,p}(z) + (\mu+p)Q^{p,\mu}_{\alpha,\beta,\gamma}J_{c,p}(z) = pz^{-p}\left[\lambda + (1-\lambda)u(z)\right]. \tag{25}$$

Differentiating (25) with respect to z, and using (18) we obtain

$$\left(-\frac{z^{p+1}\left(Q^{p,\mu}_{\alpha,\beta,\gamma}\ f(z)\right)'}{p} - \lambda\right)(1-\lambda)^{-1} = u(z) + \frac{1}{c}zu'(z). \tag{26}$$

Using the well-known estimate [see 6] $\left|zu^{'}(z)\right| \leq \frac{2r}{1-r^2} \operatorname{Re} u(z), |z| = r$, (26) yields

$$\operatorname{Re}\left\{\left(-\frac{z^{p+1}\left(Q^{p,\mu}_{\alpha,\beta,\gamma} f(z)\right)^{'}}{p} - \lambda\right)(1-\lambda)^{-1}\right\} \geq \left(1 - \frac{1}{c}\frac{2r}{1-r^2}\right)\operatorname{Re} u(z). \qquad (27)$$

The right-hand side of (27) is positive if $r < \frac{c}{1+\sqrt{c^2+1}}$.

The result is sharp for the function $f(z)$ defined by $f(z) = \frac{1}{cz^{c+p-1}}\left(z^{c+p} J_{c,p}(z)\right)^{'}$ where $J_{c,p}(z)$ is given by $\left(Q^{p,\mu}_{\alpha,\beta,\gamma} J_{c,p}(z)\right)^{'} = -pz^{-p-1}\frac{1+(2\lambda-1)z}{1+z}$.

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