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(σ, τ) −Generalized Power Series Over Zip and Weak Zip Rings

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Abstract. In this paper we give a new class of extension rings called the (σ, τ)-*generalized power series ring* $[[R^{S,\leq}; \sigma, \tau]]$ with coefficients in a ring R and exponents in a strictly ordered monoid S which extends Ribenboim's and Ziembowski's constructions of generalized and skew generalized power series rings, respectively. The weak annihilator property of $[[R^{S,\leq}; \sigma, \tau]]$ is investigated in this paper. We also show, under certain conditions, that the (σ, τ)-*generalized power series ring* $[[R^{S,\leq}; \sigma, \tau]]$ is a right zip (weak zip) ring if and only if R is a right zip (weak zip) ring.

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1. On Zip and Weak Zip rings

Throughout this article R denotes an associative ring with identity 1_R, and $\text{nil}(R)$ is the set of all nilpotent elements of R. Recall that a ring R is reduced if it has no nonzero nilpotent elements, a ring R is reversible if $ab = 0$ implies $ba = 0$ for each $a, b \in R$, a ring R is semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$, and a ring R is called NI if $\text{nil}(R)$ forms an ideal of R. Clearly, reduced rings, reversible rings and semicommutative rings are NI rings.

The first time that the concept of zip ring appeared as it is known nowadays was in 1989, by Faith in [5]. Previously, Beachy and Blair in (1975) and Zelmanowitz in (1976) [24], introduced a more general property. Beachy and Blair defined rings whose faithful right ideals are cofaithful in the sense that $r(I_1) = 0$ for a finite subset $I_1 \subseteq I$ (= rings with

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the right Beachy-Blair condition), where $\mathrm{r}_R(I_1) = (0 : I_1) = \{a \in R \mid xa = 0 \text{ for all } x \in I_1\}$ is the right annihilator of I_1.(= rings with the right Beachy-Blair condition) and, Zelmanowitz worked with rings with the "finite intersection property" on annihilator right ideals. Both properties are equivalent, but they were introduced independently and parallelly, obtaining quite different results.

Faith [5] gives the following equivalent definitions of zip rings:

Definition 1. *(i) A ring R is called right zip if the right annihilator $r_R(X)$ of a subset X of R is zero, $r_R(Y) = 0$ for a finite subset $Y \subseteq X$.*

in the above definition, one can equivalentely require that X is a left ideal, so

(ii) A ring R is called right zip if L is a left ideal and $r_R(L) = 0$, then there exists a finitely generated left ideal $L_0 \subseteq L$ such that $r_R(L_0) = 0$. similarly for left zip ring. A ring R is zip if it is right and left zip.

Zelmanowitz [24] stated that any ring satisfying the descending chain condition on right annihilators is a right zip ring, but the converse does not hold, this observation thoght is trivial but important in what follows.

Recall the following definitions; A ring R is left kasch if every maximal left ideal has a nonzero right annihilator, equivalently, every simple left module embeds in R. A ring R is said to be a (left) Goldie ring if: R satisfies the ascending chain condition on left annihilators and R has no infinite direct sum of nonzero left ideals. It should be noted that the ascending chain condition on left annihilators is equivalent to the descending chain condition on right annihilators [7]. The class of right zip rings is enough wide as it can be deduced from the next proposition.

In the following proposition we collect, without proofs, a few basic properties of right zip rings.

(i) Any finite ring is right (and left) zip;

(ii) All domains are zip rings, since the annihilators of a subset X and heir subset $Y \subseteq X$ are zero.

(iii) Any left kasch ring is right zip and conversely if finitely generated left ideals are annihilators[5].

(iv) Left (right) self-injective ring R is right (left) zip if and only if R is psedo forbenius (=PF) [], and a semiprime commutative ring R is zip if and only if R is Goldie [6].

(v) If $R \subseteq S$ are rings such that S_R is an essential extension of R, and if S is a right zip, then so is R.

(vi) Cedò[3]: If S_R is a free left R-module, and if S is a right zip, then so is R.

Since Quasi Forbenius rings are right (left) artinian and right artinian rings are right Goldie, it follows that,

(vii) Quasi Forbenius rings are right (left) zip rings.

(viii) Semiprimary rings and $J^2 = 0$ is right and left zip ([7], Lemma 3-39).

(ix) Every right zip ring satisfies the right Beachy-Blair condition, but there are examples of rings with the left Beachy-Blair condition that are not left zip. See [9]. However, for commutative or reduced rings, it is not difficult to see that both properties are equivalent.

As a generalization of annihilators, Ouyang in [14] introduced the concept of *weak annihilator* (*nilpotent annihilator*). For a nonempty subset X of a ring R, the *weak annihilator* of X in R is defined as follows:

$$\mathrm{N}_R(X) = (\mathrm{nil}(R) : X) = \{a \in R | xa \in \mathrm{nil}(R) \text{ for all } x \in X\}.$$

In general, $\mathrm{r}_R(X) \subseteq \mathrm{N}_R(X)$ and $\mathrm{l}_R(X) \subseteq \mathrm{N}_R(X)$ for any subset X of a ring R. If R is reduced, then $\mathrm{r}_R(X) = \mathrm{l}_R(X) = \mathrm{N}_R(X)$ for any subset X of a ring R.

We can easily prove that

$$ab \in \mathrm{nil}(R) \Leftrightarrow ba \in \mathrm{nil}(R) \text{ for all } a, b \in R.$$

Therefore, there is no distinguish between the right weak annihilator and the left weak annihilator.

It is easy to see that for any subset $X \subseteq R$, $\mathrm{N}_R(X)$ is an ideal of R in case that $\mathrm{nil}(R)$ is an ideal. For more details on the weak annihilator property see [14], [16], [17] and [15].

Definition 2. *A ring R is called a weak zip ring if $\mathrm{N}_R(X) \subseteq \mathrm{nil}(R)$ for a subset X of R, then there exists a finite subset $X_0 \subseteq X$ such that $\mathrm{N}_R(X_0) \subseteq \mathrm{nil}(R)$.*

The next results establish a connection of right, left zip rings with weak zip rings.

Theorem 1. *Suppose that R is an NI ring. Then R is weak zip if and only if $\overline{R} = R/\mathrm{nil}(R)$ is right zip.*

Proof. Let R be weak zip and $\overline{X} = X + \mathrm{nil}(R) \subseteq \overline{R} = R/\mathrm{nil}(R)$ for some set $X \subseteq R$ such that $\mathrm{r}_{\overline{R}}(\overline{X}) = \overline{0} = \mathrm{nil}(R)$. For any element $a \in \mathrm{N}_R(X)$, we deduce that $a \in \mathrm{nil}(R)$. Hence $\mathrm{N}_R(X) \subseteq \mathrm{nil}(R)$. Since R is weak zip, there exists a finite subset $X_0 \subseteq X$ such that $\mathrm{N}_R(X_0) \subseteq \mathrm{nil}(R)$. We thus get a finite subset $X_0 + \mathrm{nil}(R) = \overline{X_0} \subseteq \overline{X} = X + \mathrm{nil}(R)$. Let $\overline{a} \in \mathrm{r}_{\overline{R}}(\overline{X_0})$. Then $\overline{X_0 a} = \overline{0} = \mathrm{nil}(R)$, which implies that $X_0 a \subseteq \mathrm{nil}(R)$. Hence $a \in \mathrm{N}_R(X_0) \subseteq \mathrm{nil}(R)$ and so $\mathrm{r}_{\overline{R}}(\overline{X_0}) = \overline{0} = \mathrm{nil}(R)$. Therefore $\overline{R} = R/\mathrm{nil}(R)$ is right zip. Conversely, let $\overline{R} = R/\mathrm{nil}(R)$ be right zip and $X \subseteq R$ such that $\mathrm{N}_R(X) \subseteq \mathrm{nil}(R)$. It is easy to see that $\mathrm{r}_{\overline{R}}(\overline{X}) = \overline{0} = \mathrm{nil}(R)$. Since $\overline{R} = R/\mathrm{nil}(R)$ is right zip, there exists a finite subset $X_0 + \mathrm{nil}(R) = \overline{X_0} \subseteq \overline{X} = X + \mathrm{nil}(R)$ such that $\mathrm{r}_{\overline{R}}(\overline{X_0}) = \overline{0} = \mathrm{nil}(R)$. We have X_0 is a finite subset of X and for any element $a \in \mathrm{N}_R(X_0)$ we get $\overline{a} \in \mathrm{r}_{\overline{R}}(\overline{X_0}) = \overline{0} = \mathrm{nil}(R)$. Hence $a \in \mathrm{nil}(R)$ and so $\mathrm{N}_R(X_0) \subseteq \mathrm{nil}(R)$. Therefore R is weak zip.

Corollary 1. *Suppose that R is an NI ring. Then R is weak zip if and only if $\overline{R} = R/\mathrm{nil}(R)$ is weak zip.*

Given a ring R and an R-bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R,M) = R \oplus M$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2), \text{ where } r_1, r_2 \in R \text{ and } m_1, m_2 \in M.$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ with the usual matrix operations.

Corollary 2. *Let R be a reduced ring. Then $T(R,M)$ is weak zip if and only if R is right zip.*

Proof. Set $A = T(R,M)$. Since R is a reduced ring, we can easily conclude that $\mathrm{nil}(A) \cong \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} | m \in M \right\}$ and hence $\overline{A} = A/\mathrm{nil}(A) \cong R$, which completes the proof.

2. (σ, τ)-generalized power series ring.

Let σ be an endomorphism of R, a ring R is said to be σ-compatible if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\sigma(b) = 0$.

Let $(S, +, \leqslant)$ be a strictly ordered commutative monoid (that is, $(S, \leqslant)$ is an ordered additive monoid satisfying the condition that, if $s < s'$, then $s + t < s' + t$ for $s, s', t \in S$). Recall that a subset X of $(S, \leqslant)$ is said to be *artinian* if every strictly decreasing sequence of elements of X is finite and that X is *narrow* if every subset of pairwise order-incomparable elements of X is finite.

Now, let R be a ring, $(S, \leq)$ a strictly ordered monoid, suppose the two maps $\sigma : S \longrightarrow \mathrm{End}(R)$ and $\tau : S \times S \longrightarrow U(R)$ (the group of invertible elements of R). Let $A = [[R^{S,\leq}; \sigma, \tau]]$ denotes the set of all maps $f : S \longrightarrow R$ such that the support of f ($\mathrm{supp}(f) = \{s \in S | \, f(s) \neq 0\}$) is artinian and narrow. For every $s \in S$ and $f, g \in [[R^{S,\leq}; \sigma, \tau]]$, let

$$X_s(f,g) = \{(u,v) \in S \times S | \, u + v = s, \; f(u) \neq 0, g(v) \neq 0\}.$$

It follows from ([19], Section 2.1) that $X_s(f,g)$ is finite. T he operation of multiplication on A is defined by the following way:

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)\,\sigma_u(g(v))\,\tau(u,v),$$

and $(fg)(s) = 0$ if $X_s(f,g) = \phi$. In order to ensure the associativity, it is necessary to impose two additional conditions on σ and τ, namely, for all $u, v, w \in S$:

(i) $\tau(u,v)\tau(u+v,w) = \sigma_u(\tau(v,w))\tau(u,v+w)$,

(ii) $\sigma_u\sigma_v = \eta(u,v)\sigma_{u+v}$, where $\eta(u,v)$ denotes the automorphism of R induced by the unit $\tau(u,v)$, namely $\eta(u,v)(r) = \tau(u,v)r\tau^{-1}(u,v)$, for all $r \in R$.

It is now routine to check that $A = [[R^{S,\leq};\sigma,\tau]]$, with componentwise addition and the above multiplication rule, is a ring which we call the (σ,τ)-*generalized power series ring.*

It is easy to verify that if $\tau(u,v) = 1$ for all $u, v \in S$, $\sigma : S \longrightarrow \mathrm{End}(R)$ be a monoid homomorphism, then $A = [[R^{S,\leq};\sigma,\tau]] \cong [[R^{S,\leq};\sigma]]$ is the ring of skew generalized power series in the sense of Mazurk and Ziembowski [13].

In the same time, if we set also $\sigma(s) = \mathrm{Id}_R \in \mathrm{End}(R)$ for all $s \in S$, the identity map of R, then $A \cong [[R^{S,\leq}]]$, the ring of generalized power series in the sense of Ribenboim (See [15],[20], and [21]).

It is easy to check that polynomial rings, Laurent polynomial rings, monoid rings, formal power series rings, Laurent power series rings and their skew versions are special cases of A.

For any $r \in R$ and any $s \in S$, we define the map $c_r \in A = [[R^{S,\leq};\sigma,\tau]]$ as follows:

$$c_r(s) = \begin{cases} r \text{ if } s = 0, \\ 0 \text{ if } s \neq 0. \end{cases}$$

Clearly, $\mathrm{supp}(c_r) = 0$ for each $r \in R$. Moreover the identity element of A is $c_1 : S \to R$ given by $c_1(0) = 1$, and $c_1(s) = 0$ for all $s \in S \setminus \{0\}$. It is easy to see that $r \to c_r(0)$ is a ring embedding of R into A via $\sigma_1 = \mathrm{Id}_R$, i.e. $\sigma_s(1) = 1$ for any $s \in S$ and $\tau(s,0) = \tau(0,s) = 1$ for all $s \in S$. For any $r \in R$, $f \in A$, we have $rf = c_r(0)f$.

Let C_f be the content of f which is defined as follows:

$$C_f = \{f(s) \,|\, s \in \mathrm{supp}(f)\}.$$

$\Longrightarrow$ In this paper we extend the results of some recent papers on the rings of skew generalized power series (see for example [13], [16], and [22]) to the class of (σ,τ)-*generalized power series ring* A. Namely we show, under some Armendariz conditions, that the zip and weak zip properties on the base ring R can be transferred to the (σ,τ)-*generalized power series ring* A. The results of Sections 3 and 4 generalize those in [16], [17] and [22].

3. (σ,τ)-Generalized Power Series Over Zip Rings

Extensions of zip rings were studied by several authors. Beachy and Blair [2] showed that if R is a commutative zip ring, then the polynomial ring $R[x]$ is a zip ring. In [8] Hong et al. proved that an Armendariz ring R is a right zip ring if and only if $R[x]$ is a right zip ring. Cortes [4] studied the relationship between right zip property of R and the skew polynomial extensions over R using a skew version of Armendariz condition.

Definition 3. *A ring R is an S-Armendariz if whenever $f, g \in [[R^{S,\leq}]]$ such that $fg = 0$, then $f(u)g(v) = 0$ for all $u, v \in S$.*

Definition 4 ([12])**.** *If σ is a monoid homomorphism, then a ring R is an (S,σ)-Armendariz if whenever $fg=0$, where $f,g\in\left[\left[R^{S,\leq},\sigma\right]\right]$, then $f(u)\,\sigma_u(g(v))=0$ for all $u,v\in S$.*

Definition 5. *A ring R is an (S,σ,τ)-Armendariz if whenever $fg=0$, where $f,g\in A=\left[\left[R^{S,\leq};\sigma,\tau\right]\right]$, then $f(u)\,\sigma_u(g(v))\,\tau(u,v)=0$.*

It is obvious that R is an (S,σ,τ)-Armendariz if and only if it is an (S,σ)-Armendariz ring. For more details on Armendariz ring see [1], [11], [12] and [18].

For a nonempty subset $X\subseteq R$, we define the following subsets of $A=\left[\left[R^{S,\leq};\sigma,\tau\right]\right]$

$$\left[\left[X^{S,\leq};\sigma,\tau\right]\right]=\left\{f\in A|f(s)\in X\bigcup\{0\}\text{ for each }s\in S\right\},$$

$$\mathrm{r}_A(X)=\{f\in A\,|c_xf=0\text{ for all }x\in X\},$$

$$\left[\left[\mathrm{r}_R(X)^{S,\leq};\sigma,\tau\right]\right]=\{f\in A\,|f(s)\in\mathrm{r}_R(X)\text{ for all }s\in S\},\text{ and}$$

$$\left[\left[(\mathrm{nil}(R))^{S,\leq};\sigma,\tau\right]\right]=\{f\in A\,|f(s)\in\mathrm{nil}(R)\text{ for all }s\in S\}.$$

We begin with the following Lemma and use it without further mention.

Lemma 1. *Suppose that R is a ring. Then*

$$\mathrm{r}_A(X)=\left[\left[\mathrm{r}_R(X)^{S,\leq};\sigma,\tau\right]\right]$$

for any $X\subseteq R$.

Proof. Let $f\in\mathrm{r}_A(X)$, then $0=c_xf$ for each $x\in X$. So for each $s\in\mathrm{supp}(f)$, we have

$$0=(c_xf)(s)=x\sigma_0(f(s))\,\tau(0,s)$$

Since $\sigma_0=\mathrm{Id}_R$ and $\tau(0,s)=1$, we conclude that $xf(s)=0$ thus $f(s)\in\mathrm{r}_R(X)$ for each $s\in S$. Therefore, $f\in\left[\left[\mathrm{r}_R(X)^{S,\leq};\sigma,\tau\right]\right]$ and we have

$$\mathrm{r}_A(X)\subseteq\left[\left[\mathrm{r}_R(X)^{S,\leq};\sigma,\tau\right]\right].$$

On the other hand, suppose $f\in\left[\left[\mathrm{r}_R(X)^{S,\leq};\sigma,\tau\right]\right]$, then $xf(s)=0$ for each $s\in\mathrm{supp}(f)$.

$$0=xf(s)=x\sigma_0(f(s))\,\tau(0,s)=(c_xf)(s).$$

Hence, $f\in\mathrm{r}_A(X)$ and it follows that

$$\left[\left[\mathrm{r}_R(X)^{S,\leq};\sigma,\tau\right]\right]\subseteq\mathrm{r}_A(X).$$

Consequently, $\mathrm{r}_A(X)=\left[\left[\mathrm{r}_R(X)^{S,\leq};\sigma,\tau\right]\right]$.

Definition 6. *A ring R is said to be S-compatible if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\sigma_s(b) = 0$ for all $s \in S$.*

Using Lemma 1, we have the following map, $\varphi : \mathrm{r}_R\left(2^R\right) \longrightarrow \mathrm{r}_A\left(2^A\right)$ defined by

$$\varphi(I) = \left[\left[I^{S,\leq};\sigma,\tau\right]\right] \text{ for every } I \in \mathrm{r}_R\left(2^R\right), \text{where}$$

$$\mathrm{r}_R\left(2^R\right) = \{\mathrm{r}_R(X) \,|X \subseteq R\} \text{ and } \mathrm{r}_A\left(2^A\right) = \{\mathrm{r}_A(Y) \,|Y \subseteq A\}.$$

Obviously φ is an injective map.

The following Lemma shows that φ is a bijective map if and only if R is (S,σ,τ)-Armendariz ring.

Lemma 2. *Suppose that R is an S-compatible ring. Then the following are equivalent:*
(1) R is (S,σ,τ)-Armendariz ring.
(2) φ is a bijective map.

Proof. Suppose that R is an (S,σ,τ)-Armendariz ring. Let $Y \subseteq A$, then $C(Y) = \bigcup_{f\in Y} C_f$. From Lemma 1, it is sufficient to show that

$$\mathrm{r}_A(f) = \left[\left[\mathrm{r}_R(C_f)^{S,\leq};\sigma,\tau\right]\right] = \varphi\left(\mathrm{r}_R(C_f)\right) \text{ for all } f \in Y.$$

Let $g \in \mathrm{r}_A(f)$, then $fg = 0$. Since R is (S,σ,τ)-Armendariz

$$0 = f(u)\,\sigma_u(g(v))\,\tau(u,v)$$

for each $u \in \mathrm{supp}(f)$ and $v \in \mathrm{supp}(g)$. Since $\tau(u,v) \in U(R)$ and R is S-compatible, we get $0 = f(u)\,g(v)$.

Hence $g(v) \in \mathrm{r}_R(C_f)$, consequently $g \in \left[\left[\mathrm{r}_R(C_f)^{S,\leq};\sigma,\tau\right]\right]$ and we have

$$\mathrm{r}_A(f) \subseteq \left[\left[\mathrm{r}_R(C_f)^{S,\leq};\sigma,\tau\right]\right].$$

Conversely, let $g \in \left[\left[\mathrm{r}_R(C_f)^{S,\leq};\sigma,\tau\right]\right]$, this implies that $g(v) \in r_R(C_f)$ for each $v \in \mathrm{supp}(g)$. Then $0 = f(u)\,g(v)$ for each $v \in \mathrm{supp}(g)$ and $u \in \mathrm{supp}(f)$. Since R is S-compatible and $\tau(u,v) \in U(R)$, it follows that

$$0 = f(u)\,\sigma_u(g(v))\,\tau(u,v).$$

Thus, for each $s \in S$, we have

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)\,\sigma_u(g(v))\,\tau(u,v) = 0.$$

Therefore, $g \in \mathrm{r}_A(f)$ and it follows that

$$\left[\left[\mathrm{r}_R(C_f)^{S,\leq};\sigma,\tau\right]\right] \subseteq \mathrm{r}_A(f).$$

Now, we can conclude the following:

$$\mathrm{r}_A(Y)=\bigcap_{f\in Y}\mathrm{r}_A(f)=\bigcap_{f\in Y}\left[\left[\mathrm{r}_R(C_f)^{S,\leq};\sigma,\tau\right]\right]=\left[\left[\bigcap_{f\in Y}\mathrm{r}_R(C_f)^{S,\leq};\sigma,\tau\right]\right]$$
$$\left[\left[\mathrm{r}_R\left(\bigcup_{f\in Y}C_f\right)^{S,\leq};\sigma,\tau\right]\right]=\left[\left[\mathrm{r}_R(C(Y))^{S,\leq};\sigma,\tau\right]\right]=\varphi(\mathrm{r}_R(C(Y))).$$

Thus φ is surjective.

(2) $\Longrightarrow$ (1)

Let $f,g\in A$ such that $fg=0$, then $g\in\mathrm{r}_A(f)$. By the assumption, we have

$$\mathrm{r}_A(f)=\varphi(r_R(C_f))=\left[\left[\mathrm{r}_R(C_f)^{S,\leq};\sigma,\tau\right]\right].$$

Hence $g\in\left[\left[\mathrm{r}_R(C_f)^{S,\leq};\sigma,\tau\right]\right]$ which implies that $g(v)\in\mathrm{r}_R(C_f)$ for each $v\in S$. So, $fc_{g(v)}=0$. Thus,

$$0=\left(fc_{g(v)}\right)(u)=f(u)\,\sigma_u(g(v))\,\tau(u,v)$$

for each $u\in$supp(f) and $v\in$supp(g). Hence R is (S,σ,τ)-Armendariz ring.

Salem in [22] studied the relationship between right zip property of R and right zip property of the ring of skew generalized power series over R using a skew version of Armendariz condition compatible with the structure of the skew generalized power series ring.

Motivated by the above we introduce our main result in this section concerning the transfer of right zip property between R and the ring of (σ,τ)-generalized power series as follows:

Theorem 2. *Let $(S,\leq)$ a strictly ordered monoid, R an (S,σ,τ)-Armendariz and S-compatible ring. Then A is a right zip ring if and only if R is a right zip ring.*

Proof. ($\Longrightarrow$) Suppose A is a right zip ring and $X\subseteq R$ satisfies $\mathrm{r}_R(X)=0$. Let $Y=\{c_x|x\in X\}\subseteq A$. Then, by Lemma 1,

$$\mathrm{r}_A(Y)=\mathrm{r}_A(c_X)=\mathrm{r}_A(X)=\left[\left[\mathrm{r}_R(X)^{S,\leq};\sigma,\tau\right]\right]=0.$$

Since A is a right zip ring, there exists a finite subset $Y_0=\{c_{x_1},\ldots,c_{x_\mathrm{n}}\}\subseteq Y$ such that $\mathrm{r}_A(Y_0)=\mathrm{r}_A(c_{x_1},\ldots,c_{x_\mathrm{n}})=0$ for some $x_1,\ldots,x_\mathrm{n}\in X_0\subseteq X$. From Lemma 1, we have

$$0=\mathrm{r}_A(Y_0)=\mathrm{r}_A(X_0)=\left[\left[\mathrm{r}_R(X_0)^{S,\leq};\sigma,\tau\right]\right].$$

Hence $\mathrm{r}_R(X_0)=0$. Thus R is a right zip ring.

($\Longleftarrow$) Conversely, suppose that R is a right zip ring and $Y \subseteq A$ be such that $\mathrm{r}_A(Y) = 0$. Let

$$T = C(Y) = \bigcup_{f \in Y} C_f = \bigcup_{f \in Y} \{f(u) |\ u \in \mathrm{supp}(f)\}.$$

Then, by Lemma 1,

$$0 = \mathrm{r}_A(Y) = \left[\left[\mathrm{r}_R(C(Y))^{S,\leq}; \sigma, \tau\right]\right] = \left[\left[\mathrm{r}_R(T)^{S,\leq}; \sigma, \tau\right]\right].$$

Thus, $\mathrm{r}_R(T) = 0$. Since R is a right zip ring, there exists a finite subset $T_0 \subseteq T$ such that $\mathrm{r}_R(T_0) = 0$. For each $f(u) \in T_0$, there exists $f_u \in Y$ such that $f_u(u_i) = f(u)$ for some $u_i \in \mathrm{supp}(f_u)$. Let Y_0 be the minimal subset of Y, with respect to inclusion, such that $f_u \in Y_0$ for each $f(u) \in T_0$. Thus Y_0 is a nonempty finite subset of Y. Now we show that $\mathrm{r}_A(Y_0) = 0$. Suppose that $\mathrm{r}_A(Y_0) \neq 0$ and let $g \in \mathrm{r}_A(Y_0) \setminus \{0\}$, then $f_u g = 0$ for all $f_u \in Y_0$. Since R is (S, σ, τ)-Armendariz,$f_u(w)\ \sigma_w(g(v))\tau(w, v) = 0$. Since $\tau(w, v) \in U(R)$ and R is an σ-compatible ring, $f_u(w) g(v) = 0$ for all $w \in \mathrm{supp}(f_u)$ and $v \in \mathrm{supp}(g)$. Hence $g(v) \in \mathrm{r}_R(C(Y_0)) = \mathrm{r}_R(T_0) = 0$ for all $v \in \mathrm{supp}(g)$, a contradiction. Therefore $\mathrm{r}_A(Y_0) = 0$ and hence A is a right zip ring.

From Theorem 2 if we set $\tau(u, v) = 1$ for every $u, v \in S$ and $\sigma : S \longrightarrow \mathrm{End}(R)$ be a monoid homomorphism, we get immediately the following Corollary:

Corollary 3. *([22], Theorem 2.3) Suppose that S is a strictly ordered monoid, $\sigma : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism, R is an S-compatible and (S, σ)-Armendariz ring. Then R is a right (left) zip ring if and only if $\left[\left[R^{S,\leq}; \sigma\right]\right]$ is a right (left) zip ring.*

As a special case of the last result we get the following Corollary, if we set $\tau(u, v) = 1$ for every $u, v \in S$ and $\sigma_s = \mathrm{Id}_R$, for every $s \in S$.

Corollary 4. *([23], Theorem 2.3) Let S be a strictly ordered monoid and R an S-Armendariz ring. Then R is a right zip ring if and only if $\left[\left[R^{S,\leq}\right]\right]$ is a right zip ring.*

4. (σ, τ) −Generalized Power Series Over Weak Zip Rings

In this section, we mainly discuss the weak annihilator property of the ring $A = \left[\left[R^{S,\leq}; \sigma, \tau\right]\right]$.

The next two Lemmas appear in [16] and [17].

Lemma 3. *Let X, Y be subsets of a ring R. Then the following hold:*

(i) $X \subseteq Y$ *implies* $\mathrm{N}_R(X) \supseteq \mathrm{N}_R(Y)$.

(ii) $X \subseteq \mathrm{N}_R((\mathrm{N}_R(X)))$.

(iii) $\mathrm{N}_R(X) = \mathrm{N}_R((\mathrm{N}_R(\mathrm{N}_R(X))))$.

Lemma 4. *Let $\sigma : S \longrightarrow \mathrm{End}(R)$ be a map, $x \mapsto \sigma_x$. If σ_x is compatible for all $x \in S$, then, for each $a, b \in R$, the following hold:*

(i) $ab \in \operatorname{nil}(R) \Longleftrightarrow a\sigma_x(b) \in \operatorname{nil}(R)$.

(ii) $ab \in \operatorname{nil}(R) \Longleftrightarrow \sigma_x(a)\, b \in \operatorname{nil}(R)$.

Proposition 1. *Let $(S, \leq)$ be a strictly totally ordered monoid, R an S-compatible* NI *ring with* $\operatorname{nil}(R)$ *nilpotent. Then $f \in \operatorname{nil}(A)$ if and only if $f(s) \in \operatorname{nil}(R)$ for every $s \in \operatorname{supp}(f)$.*

Proof. ($\Rightarrow$)Suppose that $f \in \operatorname{nil}(A)$, then there exists some positive integer k such that $f^k = 0$. We will use transfinite induction on the strictly totally ordered monoid $(S, \leq)$ to show that $f(s) \in \operatorname{nil}(R)$ for every $s \in$supp(f). Let x_0 be the minimal element of supp(f) with respect to the order $\leq$. If $v_1, v_2, \ldots, v_k \in$supp$(f)$ are such that $v_1 + v_2 + \ldots + v_k = kx_0$, then $x_0 \leq v_i$ for all $1 \leq i \leq k$. If $x_0 < v_i$ for some $1 \leq i \leq k$, then $kx_0 = \underbrace{x_0 + x_0 + \ldots + x_0}_{k\text{-times}} < v_1 + \cdots + v_k = kx_0$ a contradiction. Thus $x_0 = v_i$ for $1 \leq i \leq k$.
Hence from $f^k = 0$, it follows that:

$$0 = f^k(kx_0) = \left(f^{k-1} f\right)(kx_0)$$

$$\Rightarrow 0 = f^{k-1}((k-1)x_0)\, \sigma_{(k-1)x_0}(f(x_0))\, \tau((k-1)x_0, x_0)$$

$$\begin{aligned}\Rightarrow \quad & 0 = f(x_0)\, \sigma_{x_0}(f(x_0))\, \tau(x_0, x_0)\, \sigma_{2x_0}(f(x_0))\, \tau(2x_0, x_0) \cdots \\ & \sigma_{(k-1)x_0}(f(x_0))\, \tau((k-1)x_0, x_0).\end{aligned}$$

Since $\tau(x, y)$ is invertible for all $x, y \in S$, we have

$$f(x_0)\, \sigma_{x_0}(f(x_0))\, \tau(x_0, x_0)\, \sigma_{2x_0}(f(x_0))\, \tau(2x_0, x_0) \cdots \sigma_{(k-1)x_0}(f(x_0)) = 0$$

Since R is S-compatible, we have

$$\begin{aligned}& f(x_0)\, \sigma_{x_0}(f(x_0))\, \tau(x_0, x_0)\, \sigma_{2x_0}(f(x_0))\, \tau(2x_0, x_0) \cdots \\ & \sigma_{(k-2)x_0}(f(x_0))\, \tau((k-2)x_0, x_0)\, f(x_0) = 0.\end{aligned}$$

$$\begin{aligned}\Rightarrow\ & f(x_0)\, \sigma_{x_0}(f(x_0))\, \tau(x_0, x_0)\, \sigma_{2x_0}(f(x_0))\, \tau(2x_0, x_0) \cdots \\ & \sigma_{(k-2)x_0}(f(x_0))\, \tau((k-2)x_0, x_0)\, f(x_0) \in \operatorname{nil}(R).\end{aligned}$$

$$\begin{aligned}\Rightarrow\ & f(x_0)\, f(x_0)\, \sigma_{x_0}(f(x_0))\, \tau(x_0, x_0)\, \sigma_{2x_0}(f(x_0))\, \tau(2x_0, x_0) \cdots \\ & \sigma_{(k-2)x_0}(f(x_0))\, \tau((k-2)x_0, x_0) \in \operatorname{nil}(R).\end{aligned}$$

Since R is NI and $\tau(x, y) \in U(R)$, we get

$$\begin{aligned}& (f(x_0))^2\, \sigma_{x_0}(f(x_0))\, \tau(x_0, x_0)\, \sigma_{2x_0}(f(x_0))\, \tau(2x_0, x_0) \cdots \\ & \sigma_{(k-2)x_0}(f(x_0)) \in \operatorname{nil}(R).\end{aligned}$$

Since R is S-compatible NI ring, we have

$$(f(x_0))^2 \sigma_{x_0}(f(x_0))\tau(x_0,x_0)\sigma_{2x_0}(f(x_0))\tau(2x_0,x_0)\cdots \sigma_{(k-3)x_0}(f(x_0))\tau((k-3)x_0,x_0)f(x_0)\in \mathrm{nil}(R).$$

$$\Rightarrow (f(x_0))^3 \sigma_{x_0}(f(x_0))\tau(x_0,x_0)\sigma_{2x_0}(f(x_0))\tau(2x_0,x_0)\cdots \sigma_{(k-3)x_0}(f(x_0))\tau((k-3)x_0,x_0)\in \mathrm{nil}(R).$$

$$\Longrightarrow \cdots \Longrightarrow f(x_0)\in \mathrm{nil}(R).$$

Now suppose that $p \in$supp(f) is such that for any $u \in$supp(f) with $u<p$, $f(u)\in \mathrm{nil}(R)$ for $u \in$supp(f). We will show that $f(p)\in \mathrm{nil}(R)$. For convenience, we write

$$X_{kp}(f,\ldots,f)=\{(u_1,\ldots,u_k)\in S\times\cdots\times S|\ u_1+\ldots+u_k=kp,\ f(u_i)\neq 0, 1\le i\le k\},$$

as

$$\{(p,p,\ldots,p)\}\cup\{(u_{i_1},u_{i_2},\ldots,u_{i_k})\,|i=2,3,\ldots,\mathrm{n}\}$$

and for each

$$(u_{i_1},u_{i_2},\ldots,u_{i_k})\in\{(u_{i_1},u_{i_2},\ldots,u_{i_k})\,|i=2,3,\ldots,\mathrm{n}\},$$

there exists some $1\le l\le k$ such that $u_{i_l}\neq p$. We show that for each

$$(u_{i_1},u_{i_2},\ldots,u_{i_k})\in\{(u_{i_1},u_{i_2},\ldots,u_{i_k})\,|i=2,3,\ldots,\mathrm{n}\}$$

there exists some $1\le t\le k$ such that $u_{i_t}<p$. If $u_{i_l}<p$, then we are done. So we may assume that $u_{i_l}>p$. If for all $1\le j\le k$, and $j\neq l$, $u_{i_j}\ge p$, then $kp<u_{i_1}+u_{i_2}+\ldots+u_{i_k}=kp$, a contradiction. Thus for each

$$(u_{i_1},u_{i_2},\ldots,u_{i_k})\in\{(u_{i_1},u_{i_2},\ldots,u_{i_k})\,|i=2,3,\ldots,\mathrm{n}\}$$

there exists some $1\le t\le k$ such that $u_{i_t}<p$. Then, by induction hypothesis, we obtain $f(u_{i_t})\in \mathrm{nil}(R)$. By Lemma 4, $1.f(u_{i_t})\in \mathrm{nil}(R)$ implies $1.\sigma_s(f(u_{i_t}))=\sigma_s(f(u_{i_t}))\in \mathrm{nil}(R)$ for each $s\in S$.

Set $w=u_{i_1}+\ldots+u_{i_{(k-1)}}$. From $f^k=0$, we have

$$\begin{aligned}0&=f^k(kp)\\&=f(p)\sigma_p(f(p))\tau(p,p)\sigma_{2p}(f(p))\tau(2p,p)\cdots\sigma_{(k-1)p}(f(p))\tau((k-1)p,p)\\&+\left(\sum_{i=2}^{\mathrm{n}} f(u_{i_1})\sigma_{u_{i_1}}(f(u_{i_2}))\tau(u_{i_1},u_{i_2})\cdots\sigma_w(f(u_{i_k}))\tau(w,u_{i_k})\right).\end{aligned}$$

Since R is NI, we get

$$f(u_{i_1})\sigma_{u_{i_1}}(f(u_{i_2}))\tau(u_{i_1},u_{i_2})\cdots\sigma_w(f(u_{i_k}))\tau(w,u_{i_k})\in \mathrm{nil}(R)$$

for all $2\le i\le \mathrm{n}$.

Then

$$-\left(\sum_{i=2}^{\mathrm{n}} f\left(u_{i_1}\right)\sigma_{u_{i_1}}\left(f\left(u_{i_2}\right)\right)\tau\left(u_{i_1},u_{i_2}\right)\cdots\sigma_w\left(f\left(u_{i_k}\right)\right)\tau\left(w,u_{i_k}\right)\right)$$
$$=f\left(p\right)\sigma_p\left(f\left(p\right)\right)\tau\left(p,p\right)\sigma_{2p}\left(f\left(p\right)\right)\tau\left(2p,p\right)\cdots\sigma_{(k-1)p}\left(f\left(p\right)\right)\tau\left(\left(k-1\right)p,p\right)\in\mathrm{nil}\left(R\right).$$

Hence

$$f\left(p\right)\sigma_p\left(f\left(p\right)\right)\tau\left(p,p\right)\sigma_{2p}\left(f\left(p\right)\right)\tau\left(2p,p\right)\cdots$$
$$\sigma_{(k-1)p}\left(f\left(p\right)\right)\tau\left(\left(k-1\right)p,p\right)f\left(p\right)\in\mathrm{nil}\left(R\right).$$

Since $\tau\left(u,v\right)$ is invertible,

$$f\left(p\right)\sigma_p\left(f\left(p\right)\right)\tau\left(p,p\right)\sigma_{2p}\left(f\left(p\right)\right)\tau\left(2p,p\right)\cdots$$
$$\sigma_{(k-2)p}\left(f\left(p\right)\right)\tau\left(\left(k-2\right)p,p\right)f\left(p\right)\in\mathrm{nil}\left(R\right).$$

By Lemma 4, since R is an S-compatible, we have

$$\left(f\left(p\right)\right)^2\sigma_p\left(f\left(p\right)\right)\tau\left(p,p\right)\sigma_{2p}\left(f\left(p\right)\right)\tau\left(2p,p\right)\cdots$$
$$\sigma_{(k-2)p}\left(f\left(p\right)\right)\tau\left(\left(k-2\right)p,p\right)\in\mathrm{nil}\left(R\right).$$

By continuing applying the same procedure, we get

$$f\left(p\right)\in\mathrm{nil}\left(R\right),$$

for any $p\in$supp(f)

Therefore, $f\left(s\right)\in\mathrm{nil}\left(R\right)$ for all $s\in S$.

($\Leftarrow$) Assume that $f\left(s\right)\in\mathrm{nil}\left(R\right)$ for all $s\in$supp(f). Then, by Lemma 4, $1.f\left(s\right)\in\mathrm{nil}\left(R\right)$ implies $1.\sigma_x\left(f\left(s\right)\right)\in\mathrm{nil}\left(R\right)$ for all $x\in S$. Since $\mathrm{nil}\left(R\right)$ is nilpotent, there exists some positive integer k such that $\left(\mathrm{nil}\left(R\right)\right)^k=0$. Now we show that $f^k=0$. For every $y\in$supp$\left(f^k\right)$, we write

$$\left\{\left(u_1,\ldots,u_k\right)\in S\times\cdots\times S|u_1+\ldots+u_k=y,\ f\left(u_i\right)\neq0,1\leq i\leq k\right\}$$

as

$$\left\{\left(u_{i_1},u_{i_2},\ldots,u_{i_k}\right)|i=1,2,\ldots,\mathrm{n}\right\}.$$

Then we have

$$f^k\left(y\right)=\sum_{i=1}^{\mathrm{n}} f\left(u_{i_1}\right)\sigma_{u_{i_1}}\left(f\left(u_{i_2}\right)\right)\tau\left(u_{i_1},u_{i_2}\right)\cdots\sigma_w\left(f\left(u_{i_k}\right)\right)\tau\left(w,u_{i_k}\right).$$

Since, by assumption, for any $1\leq i\leq n$, we have

$$f\left(u_{i_1}\right)\sigma_{u_{i_1}}\left(f\left(u_{i_2}\right)\right)\tau\left(u_{i_1},u_{i_2}\right)\cdots\sigma_w\left(f\left(u_{i_k}\right)\right)\tau\left(w,u_{i_k}\right)\in\left(\mathrm{nil}\left(R\right)\right)^k=0.$$

Thus $f^k\left(y\right)=0$ for any $y\in$supp$\left(f^k\right)$. Hence $f^k=0$ for some positive integer k. Therefore, $f\in\mathrm{nil}\left(A\right)$.

Now, according to Proposition 1, we can derive the following Corollary:

Corollary 5. *Let* $(S, \leq)$ *be a strictly totally ordered monoid and* R *an* S*-compatible* NI *ring with* $\operatorname{nil}(R)$ *nilpotent. Then*

(i) $A = \left[\left[R^{S,\leq}; \sigma, \tau\right]\right]$ *is an* NI *ring.*

(ii) $\operatorname{nil}(A) = \left[\left[\operatorname{nil}(R)^{S,\leq}; \sigma, \tau\right]\right]$.

It was proved in [16] that: If R is a right noetherian NI ring, $(S, \leq)$ a strictly totally ordered monoid, $\sigma : S \to \operatorname{End}(R)$ a compatible monoid homomorphism, and $f \in \left[\left[R^{S,\leq}; \sigma\right]\right]$. Then $f \in \operatorname{nil}\left(\left[\left[R^{S,\leq}; \sigma\right]\right]\right)$ if and only if $f(s) \in \operatorname{nil}(R)$ for all $s \in S$. Hence the following Corollary is a generalization of the mentioned Ouyang's result.

Corollary 6. *Let* $(S, \leq)$ *be a strictly totally ordered monoid,* R *an* S*-compatible right noetherian* NI *ring. Then* $f \in \operatorname{nil}\left(\left[\left[R^{S,\leq}; \sigma\right]\right]\right)$ *if and only if* $f(s) \in \operatorname{nil}(R)$ *for every* $s \in S$.

Proof. Since R is a right noetherian NI ring, by Levitzki's Theorem [10], $\operatorname{nil}(R)$ is nilpotent. Then the result follows from Proposition 1.

If we set $\tau(u, v) = 1$ for all $u, v \in S$ and $\sigma : S \to \operatorname{End}(R)$ be a monoid homomorphism, we get the results of [16] as corollaries from Proposition 1 as follows:

Corollary 7. *([16], Proposition 5) Let* $(S, \leq)$ *be a strictly totally ordered monoid and* R *an* S*-compatible* NI *ring with* $\operatorname{nil}(R)$ *nilpotent. Then* $f \in \operatorname{nil}\left(\left[\left[R^{S,\leq}; \sigma\right]\right]\right)$ *if and only if* $f(s) \in \operatorname{nil}(R)$ *for every* $f \in \left[\left[R^{S,\leq}; \sigma\right]\right]$ *and* $s \in S$.

Corollary 8. *([16], Corollary 6) Let* $(S, \leq)$ *be a strictly totally ordered monoid and* R *an* S*-compatible* NI *ring with* $\operatorname{nil}(R)$ *nilpotent. Then we have:*

(i) $\left[\left[R^{S,\leq}; \sigma\right]\right]$ *is an* NI *ring.*

(ii) $\operatorname{nil}\left(\left[\left[R^{S,\leq}; \sigma\right]\right]\right) = \left[\left[\operatorname{nil}(R)^{S,\leq}; \sigma\right]\right]$.

It was proved in [19] that if R is a noetherian commutative ring, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid and $f \in \left[\left[R^{S,\leq}\right]\right]$, then $f \in \operatorname{nil}\left(\left[\left[R^{S,\leq}\right]\right]\right)$ if and only if $f(s) \in \operatorname{nil}(R)$ for all $s \in S$.

Note that if $(S, \leq)$ is a cancellative torsion-free strictly ordered monoid, then by Ribenboim [20], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$ (that is, for all $s, t \in S$, $s \leq t$ implies $s \leq' t$). Thus if we set $\tau(u, v) = 1$ for all $u, v \in S$ and $\sigma_s = \operatorname{Id}_R$ for all $s \in S$, we get directly the results of Ribenboim [19] as corollaries of Proposition 1.

In [15], Ouyang introduced the notion of weak zip rings as follows:

Recall that a ring R is *weak zip ring* if $\mathrm{N}_R(X) \subseteq \operatorname{nil}(R)$ for a subset X of R, then there exists a finite subset $X_0 \subseteq X$ such that $\mathrm{N}_R(X_0) \subseteq \operatorname{nil}(R)$.

In the next Theorem we generalize the result due to Salem [22].

Theorem 3. *Let* $(S, \leq)$ *be a strictly totally ordered monoid and* R *an* S*-compatible* NI *ring with* $\mathrm{nil}(R)$ *nilpotent. Then* $A = \left[\left[R^{S,\leq}; \sigma, \tau\right]\right]$ *is a weak zip ring if and only if* R *is a weak zip ring.*

Proof. Suppose that $A = \left[\left[R^{S,\leq}; \sigma, \tau\right]\right]$ is a weak zip ring and $X \subseteq R$ such that $\mathrm{N}_R(X) \subseteq \mathrm{nil}(R)$. Let $Y = \{c_x \in A | x \in X\} \subseteq A$ and $0 \neq f \in \mathrm{N}_A(Y)$. Then $c_x f \in \mathrm{nil}(A)$ for each $c_x \in Y$ and $x \in X$. Using Proposition 1, we get

$$(c_x f)(u) = x\sigma_0(f(u))\tau(0, u) \in \mathrm{nil}(R)$$

for each $u \in \mathrm{supp}(f)$ and $x \in X$. Hence $xf(u) \in \mathrm{nil}(R)$ for each $u \in \mathrm{supp}(f)$ and $x \in X$. Therefore $f(u) \in \mathrm{N}_R(X) \subseteq \mathrm{nil}(R)$ for each $u \in \mathrm{supp}(f)$. Then, using proposition 1 again, $f \in \mathrm{nil}(A)$. Therefore $\mathrm{N}_A(Y) \subseteq \mathrm{nil}(A)$. Since A is a weak zip ring, there exists a finite subset $Y_0 \subseteq Y$ such that $\mathrm{N}_A(Y_0) \subseteq \mathrm{nil}(A)$, where $Y_0 = \{c_{x_i} | i = 1, \ldots, \mathrm{n}\}$ and $X_0 = \{x_i | i = 1, \ldots, \mathrm{n}\} \subseteq X$. Let $f \in \mathrm{N}_A(Y_0)$, then $c_{x_i} f \in \mathrm{nil}(A)$. From Proposition 1, we have

$$(c_{x_i} f)(u) = x_i\sigma_0(f(u))\tau(0, u) = x_i f(u) \in \mathrm{nil}(R) \text{ for each } u \in \mathrm{supp}(f) \text{ and } x_i \in X_0 .$$

Thus $\mathrm{N}_R(X_0) = \bigcup_{f \in \mathrm{N}_A(Y_0)} \{f(u) | u \in \mathrm{supp}(f)\} \subseteq \mathrm{nil}(R)$. Therefore R is a weak zip ring.

Conversely, assume that R is a weak zip ring and $Y \subseteq A$ such that $\mathrm{N}_A(Y) \subseteq \mathrm{nil}(A)$. Let $T = C(Y) \subseteq R$ and $a \in \mathrm{N}_R(T)$, then $f(u)a \in \mathrm{nil}(R)$ for each $f \in Y$ and $u \in S$. Since R is an S-compatible NI ring, we have

$$\begin{aligned} & f(u)\sigma_u(a)\tau(u, 0) \in \mathrm{nil}(R) \Rightarrow f(u)\sigma_u(c_a(0))\tau(u, 0) \in \mathrm{nil}(R) \\ & \Rightarrow (fc_a)(u) \in \mathrm{nil}(R) \text{ for each } f \in Y \text{ and } u \in S. \end{aligned}$$

Using Proposition 1, we get $fc_a \in \mathrm{nil}(A)$. Hence $c_a \in \mathrm{N}_A(Y) \subseteq \mathrm{nil}(A)$. Therefore, by using Proposition 1 again, it follows that $a \in \mathrm{nil}(R)$ and we have $\mathrm{N}_R(T) \subseteq \mathrm{nil}(R)$ for $T \subseteq R$. Since R is a weak zip ring, there exists a finite subset $T_0 \subseteq T$ such that $\mathrm{N}_R(T_0) \subseteq \mathrm{nil}(R)$. Hence, for each $t \in T_0$, there exists $f_t \in Y$ such that $t \in \{f_t(u) | u \in \mathrm{supp}(f_t)\}$. Let Y_0 be a minimal subset of Y such that $f_t \in Y_0$ for each $t \in T_0$. It is clear that Y_0 is a finite subset of Y. Let $T_1 = C(Y_0)$. Hence $T_0 \subseteq T_1$ and by Lemma 3, we have

$$\mathrm{N}_R(T_1) \subseteq \mathrm{N}_R(T_0) \subseteq \mathrm{nil}(R).$$

Suppose that $g \in \mathrm{N}_A(Y_0)$, then $f_t g \in \mathrm{nil}(A)$ for each $f_t \in Y_0$. Using Proposition 1, we have $(f_t g)(w) \in \mathrm{nil}(R)$ for each $w \in \mathrm{supp}(f_t g) \subseteq \mathrm{supp}(f_t) + \mathrm{supp}(g)$. We use transfinite induction to show that $f_t(u) g(v) \in \mathrm{nil}(R)$. Since $\mathrm{supp}(f_t)$ and $\mathrm{supp}(g)$ are well ordered subsets of a totally ordered sets, let u_0 and v_0 be their minimal elements respectively. Thus

$$(f_t g)(u_0 + v_0) = f_t(u_0)\sigma_{u_0}(g(v_0))\tau(u_0, v_0) +$$

$$\sum_{(u_i,v_i)\in X_{u_0+v_0}(f,g)\setminus\{(u_0,v_0)\}} f_t\left(u_i\right)\sigma_{u_i}\left(g\left(v_i\right)\right)\tau\left(u_i,v_i\right)$$

where, $u_i > u_0$ and $v_i > v_0$ for some i. Therefore $u_0+v_0<u_i+v_0 = u_0+v_0$ a contradiction. Thus $u_0 = u_i$ for each i. Similarly, $v_0 = v_i$ for each i. Therefore by Proposition 1, we have

$$\left(f_t g\right)\left(u_0+v_0\right) = f_t\left(u_0\right)\sigma_{u_0}\left(g\left(v_0\right)\right)\tau\left(u_0,v_0\right) + \sum_{(u_i,v_i)\in X_{u_0+v_0}(f,g)\setminus\{(u_0,v_0)\}} f_t\left(u_i\right)\sigma_{u_i}\left(g\left(v_i\right)\right)\tau\left(u_i,v_i\right) \in \mathrm{nil}\left(R\right)$$

$$\left(f_t g\right)\left(u_0+v_0\right) = f\left(u_0\right)\sigma_{u_0}\left(g\left(v_0\right)\right)\tau\left(u_0,v_0\right)$$

Since R is an S-compatible NI ring, we have $f_t\left(u_0\right)g\left(v_0\right) \in \mathrm{nil}\left(R\right)$. Thus $g\left(v_0\right) \in \mathrm{N}_R\left(f_t\left(u_0\right)\right)$.

Suppose that $f_t\left(u\right)g\left(v\right) \in \mathrm{nil}\left(R\right)$ for each $u \in \mathrm{supp}\left(f_t\right)$ and $v \in \mathrm{supp}\left(g\right)$ such that $u+v<w \in \mathrm{supp}\left(f_t g\right)$. We will show that $f_t\left(u\right)g\left(v\right)$ is nilpotent for each $u+v = w \in \mathrm{supp}\left(f_t g\right)$. So it follows that $g\left(v_i\right) \in \mathrm{N}_R\left(f_t\left(u_i\right)\right) \subseteq \mathrm{nil}\left(R\right)$. For each $w \in \mathrm{supp}\left(f_t g\right)$ we have

$$X_w\left(f_t,g\right) = \left\{\left(u,v\right)|u+v = w|u \in \mathrm{supp}\left(f_t\right), v \in \mathrm{supp}\left(g\right)\right\}$$

is a finite subset. Let

$$X_w\left(f_t,g\right) = \left\{\left(u_i,v_i\right)|i = 1,\ldots,\mathrm{n}\right\}.$$

Since $(S,\leq)$ is a totally ordered monoid, we can easily conclude that S is cancellative. Assume that $u_1<u_2<\ldots<u_{\mathrm{n}}$ if $u_1 = u_2$ and $u_1+v_1 = u_2+v_2$, then $v_1 = v_2$, and thus $(u_1,v_1) = (u_2,v_2)$. From the above ordering on u_i and v_i we get

$$\left(f_t g\right)\left(w\right) = \sum_{(u,v)\in X_w(f,g)} f_t\left(u\right)\sigma_u\left(g\left(v\right)\right)\tau\left(u,v\right)$$

$$\begin{aligned}\left(f_t g\right)\left(w\right) &= \sum_{i=1}^{n} f_t\left(u_i\right)\sigma_{u_i}\left(g\left(v_i\right)\right)\tau\left(u_i,v_i\right)\\ &= f_t\left(u_1\right)\sigma_{u_1}\left(g\left(v_1\right)\right)\tau\left(u_1,v_1\right)+\cdots+f_t\left(u_{\mathrm{n}}\right)\sigma_{u_{\mathrm{n}}}\left(g\left(v_{\mathrm{n}}\right)\right)\tau\left(u_{\mathrm{n}},v_{\mathrm{n}}\right)\end{aligned} \tag{1}$$

By Proposition 1, $\left(f_t g\right)\left(w\right) = \sum_{i=1}^{n} f_t\left(u_i\right)\sigma_{u_i}\left(g\left(v_i\right)\right)\tau\left(u_i,v_i\right) \in \mathrm{nil}\left(R\right)$. Note that $u_1+v_i<u_i+v_i = w$ for each $2 \leq i \leq n$. By induction hypothesis we have, $f_t\left(u_1\right)g\left(v_i\right) \in \mathrm{nil}\left(R\right)$ for each $2 \leq i \leq n$, and since R is σ-compatible NI ring, $f_t\left(u_1\right)\sigma_{u_1}\left(g\left(v_i\right)\right)\tau\left(u_1,v_i\right)f_t\left(u_1\right) \in \mathrm{nil}\left(R\right)$ for each $2 \leq i \leq n$. From Equation 1, we have

$$\begin{aligned}f_t\left(u_1\right)\sigma_{u_1}\left(g\left(v_1\right)\right)\tau\left(u_1,v_1\right) &= \left(f_t g\right)\left(w\right) - f_t\left(u_2\right)\sigma_{u_2}\left(g\left(v_2\right)\right)\tau\left(u_2,v_2\right) - \cdots\\ &-g\left(v_1\right)f_t\left(u_{\mathrm{n}}\right)\sigma_{u_{\mathrm{n}}}\left(g\left(v_{\mathrm{n}}\right)\right)\tau\left(u_{\mathrm{n}},v_{\mathrm{n}}\right) \in \mathrm{nil}\left(R\right)\end{aligned} \tag{2}$$

Multiply Equation 2 from the left side by $g\left(v_1\right)$ it follows that:

$$g\left(v_1\right)f_t\left(u_1\right)\sigma_{u_1}\left(g\left(v_1\right)\right)\tau\left(u_1,v_1\right) = g\left(v_1\right)\left(f_t g\right)\left(w\right) -$$

$$g(v_1) f_t(u_2) \sigma_{u_2}(g(v_2)) \tau(u_2, v_2) - \cdots - g(v_1) f_t(u_n) \sigma_{u_n}(g(v_n)) \tau(u_n, v_n) \in \mathrm{nil}(R)$$

Since R is an NI ring, it follows that $g(v_1) f_t(u_1) \sigma_{u_1}(g(v_1)) \tau(u_1, v_1) \in \mathrm{nil}(R)$. Since R is an S-compatible NI ring. Then $g(v_1) f_t(u_1) \in \mathrm{nil}(R)$ and it follows that $g(v_1) \in \mathrm{N}_R(f_t(u_1)) \subseteq \mathrm{nil}(R)$.

From Equation 1, we have

$$\begin{aligned} f_t(u_2) \sigma_{u_2}(g(v_2)) \tau(u_2, v_2) &= (f_t g)(w) - f_t(u_1) \sigma_{u_1}(g(v_1)) \tau(u_1, v_1) \\ -f_t(u_3) \sigma_{u3}(g(v_3)) \tau(u_3, v_3) \cdots &- f_t(u_n) \sigma_{u_n}(g(v_n)) \tau(u_n, v_n) \in \mathrm{nil}(R) \end{aligned} \quad (3)$$

Multiply Equation 3 from the left by $g(v_2)$, we obtain $g(v_2) f_t(u_2) \in \mathrm{nil}(R)$ by the same way as above. Continuing this process, we prove that $g(v_i) f_t(u_i) \in \mathrm{nil}(R)$ for each $u_i \in \mathrm{supp}(f_t)$ and $v_i \in \mathrm{supp}(g)$ for all $1 \leq i \leq n$. Consequently, $g(v_i) f_t(u_i) \in \mathrm{nil}(R)$ for all $1 \leq i \leq n$ and thus $g(v) \in \mathrm{N}_R(T_1) \subseteq \mathrm{nil}(R)$ for each $v \in \mathrm{supp}(g)$. Thus $g \in \mathrm{nil}(A)$ by Proposition 1. Hence $\mathrm{N}_A(Y_0) \subseteq \mathrm{nil}(A)$ and it follows that A is a weak zip ring.

Corollary 9. *Let $(S, \leq)$ be a strictly totally ordered monoid and R an S-compatible right noetherian* NI *ring. Then A is a weak zip ring if and only if R is a weak zip ring.*

Proof. Since R is a right noetherian NI ring, by Levitzki's Theorem [10], $\mathrm{nil}(R)$ is nilpotent. Then the result follows from Theorem 3.

As special cases of the construction of the $(\sigma, \tau)-$ generalized power series ring A, we get some results of ([22] and [23]) as corollaries.

If $\tau(u, v) = 1$ for all $u, v \in S$ and $\sigma : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism for all $s \in S$, by Theorem 3 and Corollary 9, we get

Corollary 10. *([22], Theorem 3.10) Let $(S, \leq)$ be a strictly totally ordered monoid and R is an S-compatible* NI *ring with* $\mathrm{nil}(R)$ *nilpotent. Then $\left[\left[R^{S,\leq}; \sigma\right]\right]$ is a weak zip ring if and only if R is a weak zip ring.*

Corollary 11. *([22], Theorem 3.10) Let $(S, \leq)$ be a strictly totally ordered monoid and R is a right Noetherian, S-compatible* NI *ring. Then $\left[\left[R^{S,\leq}, \sigma\right]\right]$ is a weak zip ring if and only if R is a weak zip ring.*

If we set $\tau(u, v) = 1$ for all $u, v \in S$ and $\sigma(s) = \mathrm{Id}_R$ for all $s \in S$, we have the following:

Corollary 12. *([23], Theorem 3.5) Let R be a semicommutative right Noetherian ring, $(S, \leq)$ a strictly totally ordered monoid. Then R is a weak zip ring if and only if $[[R^{S,\leq}]]$ is a weak zip ring.*

Proof. Since R is semicommutative, $\mathrm{nil}(R)$ is an ideal by [11]. Therefore R is an NI ring. Hence the result follows by Corollary 9.

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