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Certain Classes of Harmonic Functions Associated with Dual Convolution

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Abstract. In this paper, we investigate several properties for the harmonic classes $\mathcal{M}_{\mathcal{H}}(\varphi,\psi;t,\beta,\sigma)$ and $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$. We obtain coefficient bounds, distortion theorem, extreme points, convolution condition, convex combinations and integral operator for these classes.

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1. Introduction

A continuous complex valued functions $f=u+iv$ which is defined in a simply connected complex domain $\mathcal{D}$ is said to be harmonic in $\mathcal{D}$ if both u and v are real harmonic in $\mathcal{D}$. In any simply connected domain we can write

$$f(z)=h(z)+\overline{g(z)}, \tag{1}$$

where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h^{'}(z)|>|g^{'}(z)|$ in $\mathcal{D}$ (see [7]).

Let $\mathcal{A}$ denote the class of the functions f of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k,$$

which are analytic in the open unit disc $U=\{z:z\in C \text{ and } |z|<1\}$ and satisfy the normalization condition $f(0)=f'(0)-1=0$.

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For $1 < \beta \leq \frac{4}{3}$ and $z \in U$, let

$$\mathcal{M}(\beta) = \left\{ f \in A : Re\left\{ \frac{zf'(z)}{f(z)} \right\} < \beta \right\},$$

and

$$\mathcal{N}(\beta) = \left\{ f \in A : Re\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \beta \right\}.$$

These classes $\mathcal{M}(\beta), \mathcal{N}(\beta)$ were extensively studied by Uralegaddi *et al.* [19], see also Owa and Srivastava [13], Porwal and Dixit [16] and Breaz [6].

Denote by $\mathcal{S}_{\mathcal{H}}$, the class of functions f of the form (1) that are harmonic univalent and sense preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, we may express

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}, \; |b_1| < 1, \tag{2}$$

where the analytic functions h and g are of the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g(z) = \sum_{k=1}^{\infty} b_k z^k, \; |b_1| < 1. \tag{3}$$

In 1984 Clunie and Sheil-Small [7] investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $\mathcal{S}_{\mathcal{H}}$ and its subclasses. For more basic results one may refer to the following standard introductory, Porwal [15, Chapter 5] defined the subclass $\mathcal{M}_{\mathcal{H}}(\beta) \subset \mathcal{S}_{\mathcal{H}}$ consisting of harmonic univalent functions $f(z)$ satisfying the following condition:

$$\mathcal{M}_{\mathcal{H}}(\beta) = \left\{ f(z) \in \mathcal{S}_{\mathcal{H}} : Re\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right) < \beta \right\} \left(1 < \beta \leq \frac{4}{3}; z \in U \right).$$

He proved that if $f = h + \overline{g}$, where h and g are given by (3) and if

$$\sum_{k=2}^{\infty} \frac{(k-\beta)}{\beta - 1} \left| a_k \right| + \sum_{k=1}^{\infty} \frac{(k+\beta)}{\beta - 1} \left| b_k \right| \leq 1 \left(1 < \beta \leq \frac{4}{3} \right), \tag{4}$$

then $f(z) \in \mathcal{M}_{\mathcal{H}}(\beta)$.

For $g \equiv 0$ the class of $\mathcal{M}_{\mathcal{H}}(\beta)$ is reduced to the class $\mathcal{M}(\beta)$ studied by Uralegaddi *et al.* [19].

The convolution of two functions of the from

$$\varphi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k \; (\lambda_k \geq 0) \text{ and } \psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k \; (\mu_k \geq 0), \tag{5}$$

is defined as

$$(\varphi * \psi)(z) = z + \sum_{k=2}^{\infty} \lambda_k \mu_k z^k = (\psi * \varphi)(z), \tag{6}$$

while the integral convolution is defined by

$$(\varphi \Diamond \psi)(z) = z + \sum_{k=2}^{\infty} \frac{\lambda_k \mu_k}{k} z^k = (\psi \Diamond \varphi)(z). \tag{7}$$

Motivated by the work of Ahuja [1], we consider the class $\mathcal{M}_{\mathcal{H}}(\varphi, \psi; t, \beta, \sigma)$ of functions of the form (1) satisfying the condition

$$Re\left\{\frac{h(z) * \varphi(z) - \sigma \overline{g(z) * \psi(z)}}{(1-t)z + t\left[h(z)\Diamond\varphi(z) + \sigma\overline{g(z)\Diamond\psi(z)}\right]}\right\} < \beta, \tag{8}$$

where $0 \le t \le 1$, $|\sigma| = 1$, $1 < \beta \le \frac{4}{3}$, $\varphi(z)$ and $\psi(z)$ are given by (5).

We note that:

i) $\mathcal{M}_{\mathcal{H}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 1, \beta, 1) = \mathcal{M}_{\mathcal{H}}(\beta)$ (see [15]);

ii)

$$\mathcal{M}_{\mathcal{H}}(z + \sum_{k=2}^{\infty} k\Gamma_k(\alpha_1) z^k, z + \sum_{k=2}^{\infty} k\Gamma_k(\alpha_1) z^k, 1, \beta, 1) = \mathcal{M}_{\mathcal{H}}(\alpha_1, \beta),$$
$$(\alpha_i > 0, i = 1, \ldots, q; \beta_j > 0, j = 1, \ldots, s; q \le s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\})$$

where (see [14])

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1}(\alpha_2)_{k-1}\ldots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}(\beta_2)_{k-1}\ldots(\beta_s)_{k-1}} \cdot \frac{1}{(k-1)!}(k \ge 2).$$

Also we note that:

i)

$$\mathcal{M}_{\mathcal{H}}(\frac{z+z^2}{(1-z)^3}, \frac{z+z^2}{(1-z)^3}; 1, \beta, -1)$$
$$= \mathcal{N}_{\mathcal{H}}(\beta) = Re\left\{\frac{z^2 h''(z) + z h'(z) + \overline{z^2 g''(z)} + \overline{z g'(z)}}{z h'(z) - \overline{z g'(z)}}\right\} < \beta;$$

ii)

$$\mathcal{M}_{\mathcal{H}}(z + \sum_{k=2}^{\infty} k^{n+1} z^k, z + \sum_{k=2}^{\infty} k^{n+1} z^k; 1, \beta, (-1)^n)$$

$$=\mathscr{M}_{\mathscr{H}}(\beta,n)=Re\left\{\frac{D^{n+1}h(z)+(-1)^{n+1}\overline{D^{n+1}g(z)}}{D^{n}h(z)+(-1)^{n}\overline{D^{n}g(z)}}\right\}<\beta,$$

for $n\in\mathbb{N}_0$ and where D^n is the modified Salagean differential operator (see [11, 18, 20]);

iii)

$$\mathscr{M}_{\mathscr{H}}(z+\sum_{k=2}^{\infty}k^{-n}z^k,z+\sum_{k=2}^{\infty}k^{-n}z^k;1,\beta,(-1)^{n+1})$$
$$=\mathscr{L}_{\mathscr{H}}(\beta,n)=Re\left\{\frac{I^{n}h(z)+(-1)^{n}\overline{I^{n}g(z)}}{I^{n+1}h(z)+(-1)^{n+1}\overline{I^{n+1}g(z)}}\right\}<\beta,$$

for $n\in\mathbb{N}_0$ and where I^n is the modified Salagean integral operator (see [8], with $p=1$, also see [18]);

iv)

$$\mathscr{M}_{\mathscr{H}}(z+\sum_{k=2}^{\infty}k\left[1+\lambda(k-1)\right]^{n}z^k,z+\sum_{k=2}^{\infty}k\left[1+\lambda(k-1)\right]^{n}z^k;1,\beta,(-1)^{n})$$
$$=\mathscr{M}_{\mathscr{H}}(\beta,n,\lambda)=Re\left\{\frac{z\left(D_{\lambda}^{n}h(z)\right)'-(-1)^{n}\overline{z\left(D_{\lambda}^{n}g(z)\right)'}}{D_{\lambda}^{n}h(z)+(-1)^{n}\overline{D_{\lambda}^{n}g(z)}}\right\}<\beta,$$

where $\lambda\geq 0$, $n\in\mathbb{N}_0$, and D_λ^n is the modified Al-Oboudi operator (see [2, 21], also see [3], with $p=1$);

v)

$$\mathscr{M}_{\mathscr{H}}(z+\sum_{k=2}^{\infty}k\left[1+\lambda(k-1)\right]^{-n}z^k,z+\sum_{k=2}^{\infty}k\left[1+\lambda(k-1)\right]^{-n}z^k;1,\beta,(-1)^{n})$$
$$=\mathscr{L}_{\mathscr{H}}(\beta,n,\lambda)=Re\left\{\frac{z\left(I_{\lambda}^{n}h(z)\right)'-(-1)^{n}\overline{z\left(I_{\lambda}^{n}g(z)\right)'}}{I_{\lambda}^{n}h(z)+(-1)^{n}\overline{I_{\lambda}^{n}g(z)}}\right\}<\beta,$$

for $\lambda\geq 0$ and $n\in\mathbb{N}_0$, and where I_λ^n is modified integral operator see ([4], with $p=1$, also see [10], with $\ell=0$);

vi)

$$\mathscr{M}_{\mathscr{H}}\left(z+\sum_{k=2}^{\infty}k\left(\frac{1+\ell+\lambda(k-1)}{1+\ell}\right)^{m}z^k,z+\sum_{k=2}^{\infty}k\left(\frac{1+\ell+\lambda(k-1)}{1+\ell}\right)^{m}z^k;1,\beta,(-1)^{m}\right)$$

$$=\mathcal{M}_{\mathcal{H}}(\beta,m,\ell,\lambda)=Re\left\{\frac{z(J^m(\lambda,\ell)h(z))'-(-1)^m z\overline{(J^m(\lambda,\ell)g(z))}'}{J^m(\lambda,\ell)h(z)+(-1)^m\overline{J^m(\lambda,\ell)g(z)}}\right\}<\beta,$$

where $\lambda\geq 0$, $\ell>-1$, $m\in\mathbb{Z}=\{0,\pm 1,\ldots\}$, and $J^m(\lambda,\ell)$ is the modified Prajapat operator (see [9, 17], with $p=1$).

Further, let for $\sigma=1,\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$ be the subclass of $\mathcal{M}_{\mathcal{H}}(\varphi,\psi;t,\beta,\sigma)$ consisting of functions of the form:

$$f(z)=z+\sum_{k=2}^{\infty}\left|a_k\right|z^k-\sum_{k=1}^{\infty}\left|b_k\right|\overline{z}^k(|b_1|<1). \tag{9}$$

In this paper, we obtained the coefficient bounds for the classes $\mathcal{M}_{\mathcal{H}}(\varphi,\psi;t,\beta,\sigma)$ and $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$. We also obtain distortion theorem, extreme points, convolution, convex combinations and integral operator for functions in the class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$.

2. Coefficient Bounds and Distortion Theorem

Unless otherwise mentioned, we assume in the reminder of this paper that $0\leq t\leq 1$, $|\sigma|=1$, $1<\beta\leq\frac{4}{3}$ and $z\in U$. We begin with a sufficient condition for functions in the class $\mathcal{M}_{\mathcal{H}}(\varphi,\psi;t,\beta,\sigma)$ and obtain distortion theorem for functions in the class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$.

Theorem 1. *Let $f=h+\overline{g}$, where h and g are given by* (3)*, and satisfy the condition*

$$\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-t\beta\right)\left|a_k\right|+\sum_{k=1}^{\infty}\frac{\mu_k}{k}\left(k+t\beta\right)\left|b_k\right|\leq\beta-1, \tag{10}$$

where

$$k^2(\beta-1)\leq\lambda_k(k-t\beta)\text{ and }k^2(\beta-1)\leq\mu_k(k+t\beta)\text{ for }k\geq 2. \tag{11}$$

Then $f(z)$ is sense-preserving, harmonic univalent in U and $f(z)\in\mathcal{M}_{\mathcal{H}}(\varphi,\psi;t,\beta,\sigma)$.

Proof. If $z_1\neq z_2$, then by using (11), we have

$$\left|\frac{f(z_2)-f(z_1)}{h(z_2)-h(z_1)}\right|\geq 1-\left|\frac{g(z_2)-g(z_1)}{h(z_2)-h(z_1)}\right|=1-\left|\frac{\sum\limits_{k=1}^{\infty}b_k\left(z_2^k-z_1^k\right)}{\left(z_2-z_1\right)+\sum\limits_{k=2}^{\infty}a_k\left(z_2^k-z_1^k\right)}\right|$$

$$\geq 1-\frac{\sum\limits_{k=1}^{\infty}k\left|b_k\right|}{1-\sum\limits_{k=2}^{\infty}k\left|a_k\right|}\geq 1-\frac{\sum\limits_{k=1}^{\infty}\frac{\mu_k}{k}\left(\frac{k+t\beta}{\beta-1}\right)\left|b_k\right|}{1-\sum\limits_{k=2}^{\infty}\frac{\lambda_k}{k}\left(\frac{k-t\beta}{\beta-1}\right)\left|a_k\right|}\geq 0,$$

which proves the univalent. Also f is sense-preserving in U since

$$\begin{aligned}|h^{'}(z)| &\geq 1-\sum_{k=2}^{\infty}k\left|a_k\right||z|^{k-1} > 1-\sum_{k=2}^{\infty}k\left|a_k\right| \\ &\geq 1-\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(\frac{k-t\beta}{\beta-1}\right)\left|a_k\right| \\ &\geq \sum_{k=1}^{\infty}\frac{\mu_k}{k}\left(\frac{k+t\beta}{\beta-1}\right)\left|b_k\right| \\ &\geq \sum_{k=1}^{\infty}k\left|b_k\right||z|^{k-1} \geq |g^{'}(z)|.\end{aligned}$$

Now we show that $f \in \mathscr{M}_{\mathscr{H}}(\varphi,\psi;t,\beta,\sigma)$. We only need to show that if (10) holds then the condition (8) is satisfied, then we want to prove that

$$\left|\frac{\frac{h(z)*\varphi(z)-\overline{\sigma g(z)*\psi(z)}}{(1-t)z+t\left[h(z)\Diamond\varphi(z)+\sigma\overline{g(z)\Diamond\psi(z)}\right]}-1}{\frac{h(z)*\varphi(z)-\sigma\overline{g(z)*\psi(z)}}{(1-t)z+t\left[h(z)\Diamond\varphi(z)+\sigma\overline{g(z)\Diamond\psi(z)}\right]}-(2\beta-1)}\right| < 1,\ z\in U.$$

We have

$$\left|\frac{\frac{h(z)*\varphi(z)-\sigma\overline{g(z)*\psi(z)}}{(1-t)z+t\left[h(z)\Diamond\varphi(z)+\sigma\overline{g(z)\Diamond\psi(z)}\right]}-1}{\frac{h(z)*\varphi(z)-\overline{\sigma g(z)*\psi(z)}}{(1-t)z+t\left[h(z)\Diamond\varphi(z)+\sigma\overline{g(z)\Diamond\psi(z)}\right]}-(2\beta-1)}\right|$$

$$\leq \frac{\sum\limits_{k=2}^{\infty}\frac{\lambda_k}{k}(k-t)\left|a_k\right|+\sum\limits_{k=1}^{\infty}\frac{\mu_k}{k}(k+t)\left|b_k\right|}{2(\beta-1)-\sum\limits_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-2\beta t+t\right)\left|a_k\right|-\sum\limits_{k=1}^{\infty}\frac{\mu_k}{k}\left(k+2\beta t-t\right)\left|b_k\right|}.$$

The last expression is bounded above by 1, if

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{\lambda_k}{k}(k-t)\left|a_k\right|+\sum_{k=1}^{\infty}\frac{\mu_k}{k}(k+t)\left|b_k\right| \\ &\leq 2(\beta-1)-\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-2\beta t+t\right)\left|a_k\right|-\sum_{k=1}^{\infty}\frac{\mu_k}{k}\left(k+2\beta t-t\right)\left|b_k\right|,\end{aligned}$$

which is equivalent to

$$\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-t\beta\right)\left|a_k\right|+\sum_{k=1}^{\infty}\frac{\mu_k}{k}\left(k+t\beta\right)\left|b_k\right| \leq \beta-1. \tag{12}$$

But (12) is true by hypothesis and the Theorem is proved. □

In the following theorem, it is shown that the condition (10) is also necessary for function $f(z)$ given by (9) and belongs to $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$.

Theorem 2. *Let the function $f(z)$ given by (9). Then $f(z)\in\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$, if and only if the coefficient bound (10) holds.*

Proof. Since $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)\subseteq\mathcal{M}_{\mathcal{H}}(\varphi,\psi;t,\beta,\sigma)$, we only need to prove the only if part of the theorem. To this end for functions $f\in\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$, we notice that the necessary and sufficient condition to be in the class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$ is that

$$Re\left\{\frac{h(z)*\varphi(z)-\overline{g(z)*\psi(z)}}{(1-t)z+t\left[h(z)\Diamond\varphi(z)+\overline{g(z)\Diamond\psi(z)}\right]}\right\}<\beta.$$

This is equivalent to

$$Re\left\{\frac{(\beta-1)z-\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-t\beta\right)\left|a_k\right|z^k-\sum_{k=1}^{\infty}\frac{\mu_k}{k}\left(k+t\beta\right)\left|b_k\right|\overline{z}^k}{z+\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left|a_k\right|z^k-\sum_{k=1}^{\infty}\frac{\mu_k}{k}\left|b_k\right|\overline{z}^k}\right\}\geq 0. \tag{13}$$

The above condition must hold for all values of $z\in U$, so that on taking $z=r<1$, the above inequality reduces to

$$\frac{(\beta-1)-\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-t\beta\right)\left|a_k\right|r^{k-1}-\sum_{k=1}^{\infty}\frac{\mu_k}{k}(k+t\beta)\left|b_k\right|r^{k-1}}{1+\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left|a_k\right|r^{k-1}-\sum_{k=1}^{\infty}\frac{\mu_k}{k}\left|b_k\right|r^{k-1}}\geq 0. \tag{14}$$

If the condition (10) does not hold then the numerator of (14) is negative for r and sufficiently close to 1. Thus there exists a $z_0=r_0$ in $(0,1)$ for which the quotient in (14) is negative. This contradicts the required condition for $f\in\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$. This completes the proof of Theorem. □

Theorem 3. *Let the function $f(z)$ given by (9) be in the class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$ and $A_k\leq\frac{\lambda_k}{k}(k-t\beta)$, $B_k\leq\frac{\mu_k}{k}(k+t\beta)$ for $k\geq 2$, $C=\min\{A_2,B_2\}$. Then for $|z|=r<1$, we have*

$$\left|f(z)\right|\leq(1+\left|b_1\right|)r+\left(\frac{\beta-1}{C}-\frac{\beta+1}{C}\left|b_1\right|\right)r^2, \tag{15}$$

and

$$\left|f(z)\right|\geq(1-\left|b_1\right|)r-\left(\frac{\beta-1}{C}-\frac{\beta+1}{C}\left|b_1\right|\right)r^2. \tag{16}$$

The equalities in (15) and (16) are attained for the functions f given by

$$f(z)=(1+\left|b_1\right|)\overline{z}+\left(\frac{\beta-1}{C}-\frac{\beta+1}{C}\left|b_1\right|\right)\overline{z}^2 \tag{17}$$

and

$$f(z) = (1 - |b_1|)\overline{z} - \left(\frac{\beta - 1}{C} - \frac{\beta + 1}{C}|b_1|\right)\overline{z}^2 \tag{18}$$

where $|b_1| \le \frac{\beta-1}{\beta+1}$.

Proof. Let $f(z) \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta)$, then we have

$$\begin{aligned}
|f(z)| &\le (1 + |b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k \\
&\le (1 + |b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^2 \\
&= (1 + |b_1|)r + \frac{\beta - 1}{C}\sum_{k=2}^{\infty}(\frac{C}{\beta - 1}|a_k| + \frac{C}{\beta - 1}|b_k|)r^2 \\
&\le (1 + |b_1|)r + \frac{\beta - 1}{C}\sum_{k=2}^{\infty}(\frac{\lambda_k}{k}(k - t\beta)|a_k| + \frac{\mu_k}{k}(k + t\beta)|b_k|)r^2 \\
&\le (1 + |b_1|)r + \frac{\beta - 1}{C}\left(1 - \frac{(\beta + 1)|b_1|}{\beta - 1}\right)r^2 \\
&= (1 + |b_1|)r + \left(\frac{\beta - 1}{C} - \frac{(\beta + 1)|b_1|}{C}\right)r^2,
\end{aligned}$$

which proves the assertion (15) of Theorem 3. The proof of the assertion (16) is similar, thus, we omit it. □

Remark 1. *Putting* $\varphi = \psi = z + \sum_{k=2}^{\infty} k\Gamma_k(\alpha_1)z^k$, $\lambda_2 = \mu_2 = 2\Gamma_2(\alpha_1)$, $t = 1$ *and* $C = 2 - \beta$ *in Theorem 3, we improve the result obtained by Pathak et al. [14, Theorem 2.4], by adding the condition* $|b_1| \le \frac{\beta-1}{\beta+1}$.

The following covering result follows the left hand inequality Theorem 3.

Corollary 1. *Let the function* $f(z)$ *given by* (9) *be in the class* $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta)$, *where* $|b_1| < \frac{C-(\beta-1)}{C-(\beta+1)}$ *and* $A_k \le \frac{\lambda_k}{k}(k - t\beta)$, $B_k \le \frac{\mu_k}{k}(k + t\beta)$ *for* $k \ge 2$, $C = \min\{A_2, B_2\}$. *Then for* $|z| = r < 1$, *we have*

$$\left\{w : |w| < \frac{C - (\beta - 1)}{C} - \frac{C - (\beta + 1)}{C}|b_1|\right\} \subset f(U).$$

3. Extreme Points

In this section we determine the extreme points of the closed convex hull of the class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta)$ denoted by $clco\ \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta)$.

Theorem 4. *Let $f(z)$ given by (9), Then $f(z) \in clco\ \overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty} \left[X_k h_k(z) + Y_k g_k(z)\right], \tag{19}$$

where

$$h_1(z) = z, \tag{20}$$

$$h_k(z) = z + \frac{(\beta-1)k}{\lambda_k(k-t\beta)} z^k \ (k \geq 2), \tag{21}$$

and

$$g_k(z) = z - \frac{(\beta-1)k}{\mu_k(k+t\beta)} \overline{z}^k \ (k \geq 1), \tag{22}$$

where $\sum_{k=1}^{\infty} \left(X_k + Y_k\right) = 1, X_k \geq 0$ and $Y_k \geq 0$. In particular, the extreme points of the class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$ are $\{h_k\}$ $(k \geq 2)$ and $\{g_k\}$ $(k \geq 1)$, respectively.

Proof. For a function $f(z)$ of the form (19), we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} \left[X_k h_k(z) + Y_k g_k(z)\right] \\ &= \sum_{k=1}^{\infty} X_k \left(z + \frac{(\beta-1)k}{\lambda_k(k-t\beta)} z^k\right) + Y_k \left(z - \frac{(\beta-1)k}{\mu_k(k+t\beta)} \overline{z}^k\right) \\ &= z + \sum_{k=2}^{\infty} \frac{(\beta-1)k}{\lambda_k(k-t\beta)} X_k z^k - \sum_{k=1}^{\infty} \frac{(\beta-1)k}{\mu_k(k+t\beta)} Y_k \overline{z}^k. \end{aligned}$$

But,

$$\begin{aligned} &\sum_{k=2}^{\infty} \left(\frac{\lambda_k(k-t\beta)}{k(1-\beta)} \cdot \frac{k(\beta-1)}{\lambda_k(k-t\beta)} X_k\right) \\ &\qquad + \sum_{k=1}^{\infty} \left(\frac{\mu_k(k+t\beta)}{k(1-\beta)} \cdot \frac{k(\beta-1)}{\mu_k(k+t\beta)} Y_k\right) \\ &= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1. \end{aligned}$$

Thus $f(z) \in clco\ \overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$.

Conversely, assume that $f(z) \in clco\ \overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$. Set

$$X_k = \frac{\lambda_k(k-t\beta)}{k(\beta-1)} \left|a_k\right| \ (0 \leq X_k \leq 1; k \geq 2),$$

$$Y_k = \frac{\mu_k(k+t\beta)}{k(\beta-1)} \left|b_k\right| \ (0 \le Y_k \le 1; k \ge 1)$$

and $X_1 = 1 - \sum\limits_{k=2}^{\infty} X_k - \sum\limits_{k=1}^{\infty} Y_k$. Therefore,

$$\begin{aligned} f(z) &= z + \sum_{k=2}^{\infty} \left|a_k\right| z^k - \sum_{k=1}^{\infty} \left|b_k\right| \overline{z^k} \\ &= z + \sum_{k=2}^{\infty} \frac{(\beta-1)k}{\lambda_k(k-t\beta)} X_k z^k - \sum_{k=1}^{\infty} \frac{(\beta-1)k}{\mu_k(k+t\beta)} Y_k \overline{z}^k \\ &= z + \sum_{k=2}^{\infty} (h_k(z)-z) X_k + \sum_{k=1}^{\infty} (g_k(z)-z) Y_k \\ &= z\left(1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k\right) + \sum_{k=2}^{\infty} h_k(z) X_k + \sum_{k=1}^{\infty} g_k(z) Y_k \\ &= \sum_{k=1}^{\infty} \left(h_k(z) X_k + g_k(z) Y_k\right). \end{aligned}$$

This completes the proof of Theorem. □

4. Convolution and Convex Combination

In this section, we determine the convolution properties and convex combination. Let the functions $f_m(z)$ define by

$$f_m(z) = z + \sum_{k=2}^{\infty} \left|a_{k,m}\right| z^k - \sum_{k=1}^{\infty} \left|b_{k,m}\right| \overline{z}^k \quad (m = 1, 2), \tag{23}$$

are in the class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$, we denote by $(f_1 * f_2)(z)$ the convolution or (Hadamard Product) of the function $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} \left|a_{k,1}\right| \left|a_{k,2}\right| z^k - \sum_{k=1}^{\infty} \left|b_{k,1}\right| \left|b_{k,2}\right| \overline{z}^k, \tag{24}$$

while the integral convolution is defined by

$$(f_1 \Diamond f_2)(z) = z + \sum_{k=2}^{\infty} \frac{\left|a_{k,1}\right| \left|a_{k,2}\right|}{k} z^k - \sum_{k=1}^{\infty} \frac{\left|b_{k,1}\right| \left|b_{k,2}\right|}{k} \overline{z}^k. \tag{25}$$

We first show that the class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$ is closed under convolution.

Theorem 5. *For $1 < \beta \leq \delta \leq \frac{4}{3}$, let the functions $f_1 \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta)$ and $f_2 \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \delta)$. Then*

$$(f_1 * f_2)(z) \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta) \subset \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \delta), \tag{26}$$

$$(f_1 \Diamond f_2)(z) \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta) \subset \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \delta). \tag{27}$$

Proof. Let $f_m(z)(m = 1, 2)$ are given by (23), where $f_1(z)$ be in the class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta)$ and $f_2(z)$ be in the class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \delta,)$. We wish to show that the coefficients of $(f_1 * f_2)(z)$ satisfy the required condition given in (10). For $f_2 \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \delta)$, we note that $\left|a_{k,2}\right| < 1$ and $\left|b_{k,2}\right| < 1$. Now for the convolution functions $(f_1 * f_2)(z)$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left(\frac{k - t\delta}{\delta - 1} \right) \left|a_{k,1}\right| \left|a_{k,2}\right| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left(\frac{k + t\delta}{\delta - 1} \right) \left|b_{k,1}\right| \left|b_{k,2}\right| \\
&\leq \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left(\frac{k - t\delta}{\delta - 1} \right) \left|a_{k,1}\right| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left(\frac{k + t\delta}{\delta - 1} \right) \left|b_{k,1}\right| \\
&\leq \sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left(\frac{k - t\beta}{\beta - 1} \right) \left|a_{k,1}\right| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left(\frac{k + t\beta}{\beta - 1} \right) \left|b_{k,1}\right| \leq 1,
\end{aligned}$$

since $1 < \beta \leq \delta \leq \frac{4}{3}$ and $f_1 \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta)$. Thus

$$(f_1 * f_2)(z) \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta) \subset \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \delta).$$

The proof of the assertion (27) is similar, thus, we omit it. This completes the proof of Theorem. □

Next we show that $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta)$ is closed under convex combinations of its members.

Theorem 6. *The class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, \ldots$, let $f_i \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi, \psi; t, \beta)$, where

$$f_i(z) = z + \sum_{k=2}^{\infty} \left|a_{k,i}\right| z^k - \sum_{k=1}^{\infty} \left|b_{k,i}\right| \overline{z}^k \ (z \in U; i = 1, 2, \ldots), \tag{28}$$

then from (10), for $\sum_{i=1}^{\infty} m_i = 1$, $0 \leq m_i < 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} m_i f_i(z) = z + \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} m_i \left|a_{k,i}\right| \right) z^k - \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} m_i \left|b_{k,i}\right| \right) \overline{z}^k. \tag{29}$$

Then by (29), we have

$$\sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left(\frac{k - t\beta}{\beta - 1} \right) \left(\sum_{i=1}^{\infty} m_i \left|a_{k,i}\right| \right) + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left(\frac{k + t\beta}{\beta - 1} \right) \left(\sum_{i=1}^{\infty} m_i \left|b_{k,i}\right| \right)$$

$$=\sum_{i=1}^{\infty} m_i \left[\sum_{k=2}^{\infty} \frac{\lambda_k}{k}\left(\frac{k-t\beta}{\beta-1}\right)\left|a_{k,i}\right| + \frac{\mu_k}{k}\left(\frac{k+t\beta}{\beta-1}\right)\left|b_{k,i}\right|\right]$$
$$\leq \sum_{i=1}^{\infty} m_i \leq 1.$$

This completes the proof of Theorem. □

5. Integral Operator

In this section we examine a closure property of the class $\overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$ under the generalized Bernardi -Libera-Livingston integral operator (see [5, 12]) $L_c(f(z))$ which is defined by

$$L_c(f(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt, \quad c > -1. \tag{30}$$

Theorem 7. *Let $f(z) \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$. Then $L_c(f(z)) \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$.*

Proof. From (30), it follows that

$$L_c(f(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(t)+\overline{g(t)}\right]dt$$
$$= \frac{c+1}{z^c}\left[\int_0^z t^{c-1}\left(t+\sum_{k=2}^{\infty} a_k t^k\right)dt - \int_0^z \overline{(t^{c-1}\sum_{k=1}^{\infty} b_k t^k)}dt\right]$$
$$= z + \sum_{k=2}^{\infty} A_k z^k - \sum_{k=1}^{\infty} B_k z^k,$$

where

$$A_k = \left(\frac{c+1}{c+k}\right)a_k, B_k = \left(\frac{c+1}{c+k}\right)b_k.$$

Therefore,

$$\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(\frac{k-t\beta}{\beta-1}\right)\left(\frac{c+1}{c+k}\right)\left|a_k\right| + \sum_{k=1}^{\infty}\frac{\mu_k}{k}\left(\frac{k+t\beta}{\beta-1}\right)\left(\frac{c+1}{c+k}\right)\left|b_k\right|$$
$$\leq \sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(\frac{k-t\beta}{\beta-1}\right)\left|a_k\right| + \sum_{k=1}^{\infty}\frac{\mu_k}{k}\left(\frac{k+t\beta}{\beta-1}\right)\left|b_k\right| \leq 1.$$

Since $f(z) \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$, by using Theorem 1, then $L_c(f(z)) \in \overline{\mathcal{M}}_{\mathcal{H}}(\varphi,\psi;t,\beta)$. This completes the proof of Theorem. □

For suitable choose of $h(z)$ and $g(z)$ we can obtain the following remarks.

Remark 2.

i) *Putting* $\varphi = \psi = \frac{z}{(1-z)^2}$ *and* $t = \sigma = 1$ *in the above results, we obtain the corresponding results obtained by Porwal [15];*

ii) *Putting* $\varphi = \psi = z + \sum_{k=2}^{\infty} k\Gamma_k(\alpha_1)z^k$ *and* $t = \sigma = 1$ *in the above results, we obtain the corresponding results obtained by Pathak et al. [14];*

iii) *Putting* $\varphi = \psi = \frac{z+z^2}{(1-z)^3}$, $t = 1$ *and* $\sigma = -1$ *in the above results, we obtain new results of the class* $\mathcal{N}_{\mathcal{H}}(\beta)$*;*

iv) *Putting* $\varphi = \psi = z + \sum_{k=2}^{\infty} k^{n+1}z^k$, $t = 1$, $n \in \mathbb{N}_0$ *and* $\sigma = (-1)^n$ *in the above results, we obtain new results of the class* $\mathcal{M}_{\mathcal{H}}(\beta, n)$*;*

v) *Putting* $\varphi = \psi = z + \sum_{k=2}^{\infty} k^{-n}z^k$, $t = 1$, $n \in \mathbb{N}_0$ *and* $\sigma = (-1)^{n+1}$ *in the above results, we obtain new results of the class* $\mathcal{L}_{\mathcal{H}}(\beta, n)$*;*

vi) *Putting* $\varphi = \psi = z + \sum_{k=2}^{\infty} k[1 + \lambda(k-1)]^n z^k$, $t = 1$, $\lambda \geq 0$, $n \in \mathbb{N}_0$ *and* $\sigma = (-1)^n$ *in the above results, we obtain new results of the class* $\mathcal{M}_{\mathcal{H}}(\beta, n, \lambda)$*;*

vii) *Putting* $\varphi = \psi = z + \sum_{k=2}^{\infty} k[1 + \lambda(k-1)]^{-n} z^k$, $t = 1$, $\lambda \geq 0$, $n \in \mathbb{N}_0$ *and* $\sigma = (-1)^n$ *in the above results, we obtain new results of the class* $\mathcal{L}_{\mathcal{H}}(\beta, n, \lambda)$*;*

viii) *Putting* $\varphi = \psi = z + \sum_{k=2}^{\infty} k \left(\frac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m z^k$, $t = 1$, $\ell, \lambda \geq 0$, $m \in \mathbb{N}_0$ *and* $\sigma = (-1)^m$ *in the above results, we obtain new results of the class* $\mathcal{M}_{\mathcal{H}}(\beta, m, \ell, \lambda)$*.*

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