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Application of First Order Polynomial Differential Equation for Generating Analytical Solutions to the Three-Dimensional Incompressible Navier-Stokes Equations

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Abstract. The three-dimensional incompressible Navier-Stokes equations are solved in this work with the application of the transformed coordinate which defines as a set of functionals, $h_i(\xi) = k_i x + l_i y + m_i z - c_i t$. The solution is proposed from the base of higher order polynomial first order differential equation, which is firstly reduced into the Riccati equation. The Riccati equation is then implemented into the Navier-Stokes equations to produce the polynomial equation with variable coefficients. The resultant solutions from the system of Riccati and polynomial are then evaluated by the proposed method of integral evaluation. The existence property is analysed and uniqueness of velocities is ensured. It is found that the pressure is not unique.

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1. Introduction

The problem of searching for the classes of analytical solutions of the full Navier-Stokes equations is highly demanding from both theoretical and practical viewpoints, as has been described in the literature [11]. The main difficulty of analytical solution of the Navier-Stokes equations is the contribution of the nonlinear terms representing fluid inertia which then troubled the conventional analysis in general cases. However, there are some works have already been conducted in the literatures [15, 16, 12]. As in the most cases, analytical solutions are examined only in special conditions in which the nonlinearity are weakened or even removed from the analysis.

There are also sophisticated analysis of the Navier-Stokes equations which have been conducted and the results givs more insight to the problems [2]. One of them is the transformation of the Navier-Stokes equations to the Schrödinger equation, performed by application of

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the Riccati equation [1]. It has good prospects since the Schrödinger equation is linear and has well defined solutions. The reduction of the full set of Navier-Stokes equations to be a class of nonlinear ordinary differential equation is also performed [7]. The solution applied to both zero and constant pressure gradient cases. The method of introducing special solutions for velocity has also been investigated in [14, 13].

In this work, we discuss the continuity and three-dimensional incompressible Navier-Stokes equations with forcing functions

$$u_x + v_y + w_z = 0, \tag{1a}$$

$$u_t + uu_x + vu_y + wu_z = -\frac{1}{\rho}p_x + \upsilon u_{xx} + \upsilon u_{yy} + \upsilon u_{zz} + F_1, \tag{1b}$$

$$v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho}p_y + \upsilon v_{xx} + \upsilon v_{yy} + \upsilon v_{zz} + F_2, \tag{1c}$$

$$w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho}p_z + \upsilon w_{xx} + \upsilon w_{yy} + \upsilon w_{zz} + F_3, \tag{1d}$$

The following relation is produced from the continuity equation (1a),

$$u = -\int_x \left(v_y + w_z\right) dx + K_1\left(y, z, t\right). \tag{2}$$

Equation (2) can be substituted into (1d) to give

$$w_t + \left[-\int_x \left(v_y + w_z\right) dx + K_1\right] w_x + vw_y + ww_z = -\frac{1}{\rho}p_z + \upsilon\left(w_{xx} + w_{yy} + w_{zz}\right) + F_3. \tag{3}$$

Then the pressure relation can be determined as

$$\begin{aligned} p =& \rho \int_z \left\{F_3 + \upsilon\left(w_{xx} + w_{yy} + w_{zz}\right) + w_x \left[\int_x \left(v_y + w_z\right) dx - K_1\right] - w_t - vw_y - ww_z\right\} dz \\ &+ K_2\left(x, y, t\right). \end{aligned} \tag{4}$$

The next step is implementing equation (2) and (4) into (1c),

$$\begin{aligned} &v_x \int_x \left(v_y + w_z\right) dx - K_1 v_x = v_t + vv_y + wv_z - \upsilon\left(v_{xx} + v_{yy} + v_{zz}\right) + K_{2y} \\ &+ \frac{\partial}{\partial y}\left[\int_z \left\{F_3 + \upsilon\left(w_{xx} + w_{yy} + w_{zz}\right) + w_x \left[\int_x \left(v_y + w_z\right) dx - K_1\right] - w_t - vw_y - ww_z\right\} dz\right]. \end{aligned} \tag{5}$$

Suppose that the coordinate also satisfy the following set of traveling wave ansatz

$$h_i\left(\xi\right) = k_i x + l_i y + m_i z - c_i t, \tag{6}$$

where k_i, l_i, m_i and c_i are constants. The step is now transforming the cartesian coordinate into ξ-coordinate by setting the subscript $i = 1, 2, 3, 4$ such that we have four equations for the coordinate transformation. The results of $x = \psi_1\left(\xi\right), y = \psi_2\left(\xi\right), z = \psi_3\left(\xi\right)$ and $t = \psi_4\left(\xi\right)$ are then determined.

2. Methodology

Applying one of the traveling wave ansatzs, i.e. $h_1(\xi) = k_1x + l_1y + m_1z - c_1t$ into (5), the considered equation then transformed into

$$\frac{k_1}{h_{1\xi}}v_\xi(klv + kmw) + \frac{k_1}{h_{1\xi}}K_1v_\xi = -\frac{c_1}{h_{1\xi}}v_\xi + \frac{l_1}{h_{1\xi}}vv_\xi + \frac{m_1}{h_{1\xi}}wv_\xi$$
$$-\upsilon\left(\frac{k_1^2}{h_{1\xi}^2} + \frac{l_1^2}{h_{1\xi}^2} + \frac{m_1^2}{h_{1\xi}^2}\right)v_{\xi\xi} + \frac{l_1}{m_1}F_3 + F_2 + \frac{l_1}{h_{1\xi}}K_{2\xi} + \upsilon\left(\frac{k_1^2l_1}{h_{1\xi}^2m_1} + \frac{l_1^3}{h_{1\xi}^2m_1} + \frac{l_1m_1^2}{h_{1\xi}^2m_1}\right)w_{\xi\xi}$$
$$+\frac{k_1}{h_{1\xi}}w_\xi\left[\left(k_1\frac{l_1^2}{m_1}v + k_1l_1w\right) - K_1\right] + \frac{c_1l_1}{h_{1\xi}m_1}w_\xi - \frac{l_1^2}{h_{1\xi}m_1}vw_\xi - \frac{l_1}{h_{1\xi}}ww_\xi. \quad (7)$$

Rearranging (7) to give

$$a_1v_{\xi\xi} + a_2vv_\xi + (a_3 + a_4w)v_\xi + a_5w_\xi v = a_6w_{\xi\xi} + a_7w_\xi + a_8ww_\xi + \frac{l_1}{m_1}F_3 + F_2 + \frac{l_1}{h_{1\xi}}K_{2\xi}, \quad (8)$$

where a_i are variables that depend on ξ.

The previous work shows that equation (8) is transformable into the system of Riccati and polynomial equations with variable coefficients. In this work, we generalize the method to the class of polynomial differential equations as in the following

$$v_\xi = a_1v^n + a_2v^{n-1} + a_3v^{n-2} + a_4v^{n-3} + a_5v^{n-4} + \cdots\cdots + a_{n-1}v^2 + a_nv + a_{n+1}. \quad (9)$$

The equation is also transformable into the Riccati equation as formulated in the following statement.

Theorem 1. *Given q is a function resulted from the factorization of equation* (9). *Set $v = -u$ then the explicit expression for a_{n+1} can be determined as a function of higher order polynomial coefficients such that equation* (9) *is transformable into the Riccati equation as*

$$v_\xi = qv^2 + a_nv + a_{n+1}.$$

Proof. (By induction) Consider the following equation

$$v_\xi = a_1v^7 + a_2v^6 + a_3v^5 + a_4v^4 + a_5v^3 + a_6v^2 + a_7v + a_8. \quad (10)$$

The above equation can be rewritten as

$$\begin{aligned} v_\xi = {} & [v+u]\left[a_1v^6 + (a_2 - a_1u)v^5 + \left(a_3 - a_2u + a_1u^2\right)v^4 + K_3v^3 + K_4v^2\right] \\ & + \left(a_6 - a_5u + a_4u^2 - a_3u^3 - a_2u^4 + a_1u^5\right)v^2 + a_7v + a_8 = 0, \end{aligned} \quad (11)$$

where $K_3 = a_4 - a_3 + a_2u^2 + a_1u^3$ and $K_4 = a_5 - a_4u + a_3u^2 - a_2u^3 + a_1u^4$. Set $v = -u$ and $a_6 - a_5u + a_4u^2 - a_3u^3 - a_2u^4 + a_1u^5 = q$ to give the system of

$$a_6 - a_5u + a_4u^2 - a_3u^3 - a_2u^4 + a_1u^5 = q \text{and } u_\xi = -qu^2 + a_7u - a_8.$$

Lemma 1. *The first equation of* (12) *is reducible to the fourth order equation.*

Proof. The first equation of (12) can be rewritten as

$$\frac{a_6}{a_1} - \frac{a_5}{a_1}u + \frac{a_4}{a_1}u^2 - \frac{a_3}{a_1}u^3 + \frac{a_2}{a_1}u^4 - u^5 = \frac{q}{a_1}. \tag{12}$$

Set $u = z + A$ to give

$$(z+A)^5 - \frac{a_2}{a_1}(z+A)^4 + \frac{a_3}{a_1}(z+A)^3 - \frac{a_4}{a_1}(z+A)^2 + \frac{a_5}{a_1}(z+A) + \frac{q-a_6}{a_1} = 0. \tag{13}$$

The above expression can be expanded as follows

$$\begin{aligned}&\left(z^5 + 5Az^4 + 10A^2z^3 + 10A^3z^2 + 5A^4z + A^5\right) - \frac{a_2}{a_1}\left(z^4 + 4Az^3 + 6A^2z^2 + 4A^3z + A^4\right)\\ &+ \frac{a_3}{a_1}\left(z^3 + 3Az^2 + 3A^2z + A^3\right) - \frac{a_4}{a_1}\left(z^2 + 2Az + A^2\right) + \frac{a_5}{a_1}(z+A) + \frac{q-a_6}{a_1} = 0,\end{aligned} \tag{14}$$

$$\begin{aligned}&z^5 + \left(5A - \frac{a_2}{a_1}\right)z^4 + \left(10A^2 - 4A\frac{a_2}{a_1} + \frac{a_3}{a_1}\right)z^3 + \left(10A^3 - 6A^2\frac{a_2}{a_1} + 3A\frac{a_3}{a_1} - \frac{a_4}{a_1}\right)z^2\\ &\quad + \left(5A^4 - 4A^3\frac{a_2}{a_1} + 3A^2\frac{a_3}{a_1} - 2A\frac{a_4}{a_1} + \frac{a_5}{a_1}\right)z\\ &\quad + \left(A^5 - \frac{a_2}{a_1}A^4 + \frac{a_3}{a_1}A^3 - \frac{a_4}{a_1}A^2 + \frac{a_5}{a_1}A + \frac{q-a_6}{a_1}\right) = 0.\end{aligned} \tag{15}$$

By using the equation $(z+b)(z+c)(z+d)(z+e)\left(z+f\right) = 0$ as

$$\begin{aligned}&z^5 + \left(b+c+d+e+f\right)z^4 + \left(bf+cf+df+ef+be+ce+de+db+dc+bc\right)z^3\\ &\quad + \left(bef+cef+def+dbf+dcf+bcf+dbe+dce+bce\right)z^2\\ &\quad + \left(dbef+dcef+bcef+bcde\right)z + bcdef = 0\end{aligned} \tag{16}$$

with

$$\begin{aligned}&b+c+d+e+f = 5A - \frac{a_2}{a_1},\\ &bf+cf+df+ef+be+ce+de+db+dc+bc = 10A^2 - 4A\frac{a_2}{a_1} + \frac{a_3}{a_1},\\ &bef+cef+def+dbf+dcf+bcf+dbe+dce+bce = 10A^3 - 6A^2\frac{a_2}{a_1} + 3A\frac{a_3}{a_1} - \frac{a_4}{a_1},\\ &dbef+dcef+bcef+bcde = 5A^4 - 4A^3\frac{a_2}{a_1} + 3A^2\frac{a_3}{a_1} - 2A\frac{a_4}{a_1} + \frac{a_5}{a_1},\\ &bcdef = A^5 - \frac{a_2}{a_1}A^4 + \frac{a_3}{a_1}A^3 - \frac{a_4}{a_1}A^2 + \frac{a_5}{a_1}A + \frac{q-a_6}{a_1}.\end{aligned}$$

Evaluating the coefficients in (15) and (16),

$$bf+cf+df+ef+be+ce+de+db+dc+bc=$$

$$10A^2-4A\frac{a_2}{a_1}+\frac{a_3}{a_1}\left(5A-\frac{a_2}{a_1}-f\right)f+be+ce+de+db+dc+bc=10A^2-4A\frac{a_2}{a_1}+\frac{a_3}{a_1}$$

or

$$\left(5A-\frac{a_2}{a_1}\right)f-f^2+be+ce+de+db+dc+bc=10A^2-4A\frac{a_2}{a_1}+\frac{a_3}{a_1}, \tag{17}$$

$$bef+cef+def+dbf+dcf+bcf+dbe+dce+bce=10A^3-6A^2\frac{a_2}{a_1}+3A\frac{a_3}{a_1}-\frac{a_4}{a_1}$$

$$\left(f^2-\left(5A-\frac{a_2}{a_1}\right)f+10A^2-4A\frac{a_2}{a_1}+\frac{a_3}{a_1}\right)f+dbe+dce+bce$$
$$=10A^3-6A^2\frac{a_2}{a_1}+3A\frac{a_3}{a_1}-\frac{a_4}{a_1}$$

or

$$f^3-\left(5A-\frac{a_2}{a_1}\right)f^2+\left(10A^2-4A\frac{a_2}{a_1}+\frac{a_3}{a_1}\right)f+dbe+dce+bce$$
$$=10A^3-6A^2\frac{a_2}{a_1}+3A\frac{a_3}{a_1}-\frac{a_4}{a_1}f^3-\left(5A-\frac{a_2}{a_1}\right)f^2+\left(10A^2-4A\frac{a_2}{a_1}+\frac{a_3}{a_1}\right)f \tag{18}$$
$$+dbe+dce+bce=10A^3-6A^2\frac{a_2}{a_1}+3A\frac{a_3}{a_1}-\frac{a_4}{a_1},$$

$$dbef+dcef+bcef+bcde=5A^4-4A^3\frac{a_2}{a_1}+3A^2\frac{a_3}{a_1}-2A\frac{a_4}{a_1}+\frac{a_5}{a_1},$$

$$\left(-f^3+\left(5A-\frac{a_2}{a_1}\right)f^2-\left(10A^2-4A\frac{a_2}{a_1}+\frac{a_3}{a_1}\right)f+10A^3-6A^2\frac{a_2}{a_1}+3A\frac{a_3}{a_1}-\frac{a_4}{a_1}\right)f$$
$$+bcde=5A^4-4A^3\frac{a_2}{a_1}+3A^2\frac{a_3}{a_1}-2A\frac{a_4}{a_1}+\frac{a_5}{a_1}$$

or

$$-f^4+\left(5A-\frac{a_2}{a_1}\right)f^3-\left(10A^2-4A\frac{a_2}{a_1}+\frac{a_3}{a_1}\right)f^2+\left(10A^3-6A^2\frac{a_2}{a_1}+3A\frac{a_3}{a_1}-\frac{a_4}{a_1}\right)f$$
$$+bcde=5A^4-4A^3\frac{a_2}{a_1}+3A^2\frac{a_3}{a_1}-2A\frac{a_4}{a_1}+\frac{a_5}{a_1}, \tag{19}$$

$$bcdef=A^5-\frac{a_2}{a_1}A^4+\frac{a_3}{a_1}A^3-\frac{a_4}{a_1}A^2+\frac{a_5}{a_1}A+\frac{q-a_6}{a_1},$$

$$\left(f^4-\left(5A-\frac{a_2}{a_1}\right)f^3+K_5f^2-K_6f+5A^4-4A^3\frac{a_2}{a_1}+3A^2\frac{a_3}{a_1}-2A\frac{a_4}{a_1}+\frac{a_5}{a_1}\right)f$$
$$=A^5-\frac{a_2}{a_1}A^4+\frac{a_3}{a_1}A^3-\frac{a_4}{a_1}A^2+\frac{a_5}{a_1}A+\frac{q-a_6}{a_1},$$

where $K_5 = 10A^2 - 4A\frac{a_2}{a_1} + \frac{a_3}{a_1}$ and $K_6 = 10A^3 - 6A^2\frac{a_2}{a_1} + 3A\frac{a_3}{a_1} - \frac{a_4}{a_1}$. The above equation can be rewritten as

$$\begin{aligned} & f^5 - \left(5A - \frac{a_2}{a_1}\right) f^4 + \left(10A^2 - 4A\frac{a_2}{a_1} + \frac{a_3}{a_1}\right) f^3 - \left(10A^3 - 6A^2\frac{a_2}{a_1} + 3A\frac{a_3}{a_1} - \frac{a_4}{a_1}\right) f^2 \\ & + \left(5A^4 - 4A^3\frac{a_2}{a_1} + 3A^2\frac{a_3}{a_1} - 2A\frac{a_4}{a_1} + \frac{a_5}{a_1}\right) f = A^5 - \frac{a_2}{a_1}A^4 + \frac{a_3}{a_1}A^3 - \frac{a_4}{a_1}A^2 + \frac{a_5}{a_1}A \\ & + \frac{q - a_6}{a_1}. \end{aligned} \tag{20}$$

Grouping equation (20) as in the following

$$\begin{aligned} & \left[f^5 - 5Af^4 + 10A^2f^3 - 10A^3f^2 + 5A^4f - A^5\right] + \frac{a_2}{a_1}f^4 + \left(\frac{a_3}{a_1} - 4A\frac{a_2}{a_1}\right) f^3 \\ & + \left(6A^2\frac{a_2}{a_1} - 3A\frac{a_3}{a_1} + \frac{a_4}{a_1}\right) f^2 + \left(3A^2\frac{a_3}{a_1} - 2A\frac{a_4}{a_1} + \frac{a_5}{a_1} - 4A^3\frac{a_2}{a_1}\right) f \\ & + \left(\frac{a_2}{a_1}A^4 - \frac{a_3}{a_1}A^3 + \frac{a_4}{a_1}A^2 - \frac{a_5}{a_1}A + \frac{a_6 - q}{a_1}\right) = 0. \end{aligned} \tag{21}$$

Set $f^5 - 5Af^4 + 10A^2f^3 - 10A^3f^2 + 5A^4f - A^5 = 0$, and $f = g^nA$ to obtain

$$\begin{aligned} & \left(g^{5n} - 5g^{4n} + 10g^{3n} - 10g^{2n} + 5g^n - 1\right) A^5 = 0 \\ \text{or} \quad & g^{5n} - 5g^{4n} + 10g^{3n} - 10g^{2n} + 5g^n - 1 = 0, \end{aligned} \tag{22}$$

where n is an arbitrary constant. Note that equation (22) is a polynomial equation with constant coefficients which the solutions are definable by the existing computer programs. Also, the coefficient a_1 can be recovered by the definition of q as will be explained later. Thus, equation (21) is reduced into,

$$\begin{aligned} & f^4 + \left(\frac{a_3}{a_2} - 4A\right) f^3 + \left(6A^2 - 3A\frac{a_3}{a_2} + \frac{a_4}{a_2}\right) f^2 + \left(3A^2\frac{a_3}{a_2} - 2A\frac{a_4}{a_2} + \frac{a_5}{a_2} - 4A^3\right) f \\ & + \left(A^4 - \frac{a_3}{a_2}A^3 + \frac{a_4}{a_2}A^2 - \frac{a_5}{a_2}A + \frac{a_6 - q}{a_2}\right) = 0. \end{aligned} \tag{23}$$

This proves lemma 1.

Substituting for $f = g^nA$, then the fourth order polynomial equation is produced which then solvable by radical solution. Therefore, take the root of $A = \alpha(\xi)$, meanwhile, $v = -u$, $v = z + A$, $z = -f$ and $f = g^nA$, the solution for u will become

$$v = -u = -z - A = f - A = \left(g^n - 1\right) A = \left(g^n - 1\right) \alpha. \tag{24}$$

Lemma 2. *Consider the second equation of* (12). *There exists a function β such that $a_8 = \beta u$, which generate the Bernoulli equation. The equation then has closed-form exact solutions when β is solvable.*

Proof. Setting $a_8 = \beta u$, the original problem is transformed into

$$u_\xi = -qu^2 + \left(a_7 - \beta\right) u. \tag{25}$$

The solution of (25) is expressed as

$$u = \frac{a_8}{\beta} = \frac{e^{\int_\xi (a_7-\beta) d\xi}}{\int_\xi q e^{\int_\xi (a_7-\beta) d\xi} d\xi} \tag{26}$$

Let $\int_\xi q e^{\int_\xi (a_7-\beta)d\xi} d\xi = D$, then $e^{\int_\xi (a_7-\beta)d\xi} = \frac{1}{q} D_\xi$. Solving for D,

$$D = \int_\xi q e^{\int_\xi (a_7-\beta)d\xi} d\xi = e^{\int_\xi \frac{q a_8}{\beta} d\xi}. \tag{27}$$

Set $\beta = \frac{A}{B}$ and differentiate (27) once to give

$$A e^{\int_\xi \left(a_7 - \frac{A}{B}\right) d\xi} = a_8 B e^{\int_\xi \frac{q a_8 B}{A} d\xi} = g. \tag{28}$$

Thus, A and B are given by

$$A = -\frac{g e^{-\int_\xi a_7 d\xi}}{\int_\xi \frac{g}{B} e^{-\int_\xi a_7 d\xi} d\xi} \quad \text{and } B = \frac{g}{a_8 \int_\xi \frac{qg}{A} d\xi}. \tag{29}$$

The function α can be determined as

$$\beta = \frac{A}{B} = -\frac{a_8 e^{-\int_\xi a_7 d\xi} \int_\xi \frac{qg}{A} d\xi}{\int_\xi \frac{g}{B} e^{-\int_\xi a_7 d\xi} d\xi}. \tag{30}$$

Without loss of generality suppose that, $g e^{-\int_\xi a_7 d\xi} = A$ to produce

$$\frac{A}{B} = -\frac{a_8 e^{-\int_\xi a_7 d\xi} \int_\xi q e^{\int_\xi a_7 d\xi} d\xi}{\int_\xi \frac{A}{B} d\xi}. \tag{31}$$

The solution for β is then

$$\beta = \frac{A}{B} = \left\{ \left[C_1 \int_\xi \left(a_8 e^{-\int_\xi a_7 d\xi} \int_\xi q e^{\int_\xi a_7 d\xi} d\xi \right) d\xi \right]^{\frac{1}{2}} \right\}_\xi . \tag{32}$$

The above equation is combined with (26) to form the solution of the general Riccati equation. This proves lemma 2.

The next step is combining lemma 1 and lemma 2 by equating equations (24) and (26),

$$v = \left(g^n - 1\right)\alpha = C_2 \frac{\left[\int_\xi \left(a_8 e^{-\int_\xi a_7 d\xi} \int_\xi q e^{\int_\xi a_7 d\xi} d\xi\right) d\xi\right]^{\frac{1}{2}}}{\int_\xi q e^{\int_\xi a_7 d\xi} d\xi}. \tag{33}$$

Since the function $\alpha = \alpha\left(q\right)$, it is difficult to determine q from (33). Thus, take q as an arbitrary function, the solution for a_8 is then

$$a_8 = C_3 \left(\frac{e^{\int_\xi a_7 d\xi}}{\int_\xi q e^{\int_\xi a_7 d\xi} d\xi}\right) \left[\alpha\left(\left(\int_\xi q e^{\int_\xi a_7 d\xi} d\xi\right)^2\right]_\xi \tag{34}$$

which then reduce equation (11) into the Riccati equation with variable coefficients. According to the previous consideration of a_1, the function, $q = q\left(a_1\right)$ can be implemented to recover a_1.

Note that the procedure of (12- 23) can be repeated and iterated by induction to reduce the higher order polynomial equation until the fourth order equation is achieved. Expanding the result to any order as in (9) and combined with (26) to represent the Riccati equation, $v_\xi = qv^2 + a_n v + a_{n+1}$. The expression for a_{n+1} is thus defined as

$$a_{n+1} = C_3 \left(\frac{e^{\int_\xi a_n d\xi}}{\int_\xi q e^{\int_\xi a_n d\xi} d\xi}\right) \left[\alpha\left(\left(\int_\xi q e^{\int_\xi a_n d\xi} d\xi\right)^2\right]_\xi .$$

where $q = q\left(a_1, a_2, a_3, \cdots\cdots\right)$ following the reduction process and we are done.

3. Analytical Solutions to the 3D Incompressible Navier-Stokes Equations

The results from theorem 1 can be implemented into the Navier-Stokes equations with the application of Riccati equation as a representation of higher order polynomial equations.

Lemma 3. *Equation* (8) *is transformable into the system of Riccati and third order polynomial equations.*

Proof. Consider the Riccati equation

$$v_\xi = b_1 v^2 + b_2 v + b_3 \tag{35}$$

such that the following relation is satisfied

$$v_{\xi\xi} = b_{1\xi} v^2 + 2b_1 v v_\xi + b_{2\xi} v + b_2 v_\xi + b_{3\xi},$$

$$v_{\xi\xi} = 2b_1^2 v^3 + \left(b_{1\xi} + 3b_1 b_2\right) v^2 + \left(b_{2\xi} + 2b_1 b_3 + b_2^2\right) v + b_{3\xi} + b_2 b_3. \tag{36}$$

The equivalent coefficients are, $b_1 = q, b_2 = a_n$ and $b_3 = a_{n+1}$. Substituting into (8) to get the following polynomial equation

$$\begin{aligned}&\left(2a_1 b_1^2 + a_2 b_1\right) v^3 + \left(a_1 b_{2\xi} + 2a_1 b_1 b_3 + a_1 b_2^2 + a_2 b_3 + a_3 b_2 + a_5 w_\xi + a_4 b_2 w\right) v \\ &+ \left(a_1 b_{1\xi} + 3a_1 b_1 b_2 + a_2 b_2 + a_3 b_1 + a_4 b_1 w\right) v^2 \\ &= a_6 w_{\xi\xi} + a_7 w_\xi + a_8 w w_\xi - a_4 b_3 w - a_1 b_{3\xi} - a_1 b_2 b_3 - a_3 b_3 + \frac{l_1}{m_1} F_3 + F_2 + \frac{l_1}{h_{1\xi}} K_{2\xi}.\end{aligned} \tag{37}$$

Therefore, equation (8) is transformed to the system of (35) and (37). This proves lemma 3.

Set $\left(2a_1 b_1^2 + a_2 b_1\right) = 0$ and $\left(a_1 b_{1\xi} + 3a_1 b_1 b_2 + a_2 b_2 + a_3 b_1 + a_4 b_1 w\right) = 0$, thus the expression for b_1 and b_2 are defined by

$$b_1 = -\frac{a_2}{2a_1}, \tag{38}$$

$$b_2 = -\frac{\left(a_1 b_{1\xi} + a_3 b_1 + a_4 b_1 w\right)}{3a_1 b_1 + a_2}. \tag{39}$$

Therefore, equation (37) is reduced into

$$v = \frac{a_6 w_{\xi\xi} + a_7 w_\xi + a_8 w w_\xi - a_4 b_3 w - a_1 b_{3\xi} - a_1 b_2 b_3 - a_3 b_3 + \frac{l_1}{m_1} F_3 + F_2 + \frac{l_1}{h_{1\xi}} K_{2\xi}}{a_1 b_{2\xi} + 2a_1 b_1 b_3 + a_1 b_2^2 + a_2 b_3 + a_3 b_2 + a_5 w_\xi + a_4 b_2 w}. \tag{40}$$

The expression for b_3 will be determined later as a requirement of unique solutions under general initial-boundary values. Also note that the solution for the system (35) and (40) will be similar to that of the velocity in z direction as derived in the subsequent paragraph.

3.1. The Formulation for w Velocity

The step now is performing equations (2) and (4) into (1b),

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\int_x \left(v_y + w_z\right) dx - K_1\right] + \left[-\int_x \left(v_y + w_z\right) dx + K_1\right]\left(v_y + w_z\right) \\ &+ v\frac{\partial}{\partial y}\left[\int_x \left(v_y + w_z\right) dx - K_1\right] + w\frac{\partial}{\partial z}\left[\int_x \left(v_y + w_z\right) dx - K_1\right] \\ &= \frac{\partial}{\partial x}\left[\int_z \left\{F_3 + \upsilon\left(w_{xx} + w_{yy} + w_{zz}\right) + w_x\left[\int_x \left(v_y + w_z\right) dx - K_1\right] - w_t - v w_y - w w_z\right\} dz\right] \\ &+ K_{2x} + \upsilon\left(\left(v_{xy} + w_{xz}\right) + \frac{\partial^2}{\partial y^2}\left[\int_x \left(v_y + w_z\right) dx - K_1\right] + \frac{\partial^2}{\partial z^2}\left[\int_x \left(v_y + w_z\right) dx - K_1\right]\right) + F_1.\end{aligned} \tag{41}$$

Performing the coordinate transformation (6),

$$
\begin{aligned}
&-\frac{c_1}{h_{1\xi}}\left(k_1 l_1 v_\xi + k_1 m_1 w_\xi - K_1\right) + \left[-\left(k_1 l_1 v + k_1 m_1 w\right) + K_1\right]\left(\frac{l_1}{h_{1\xi}} v_\xi + \frac{m_1}{h_{1\xi}} w_\xi\right) \\
&+ v\frac{l_1}{h_{1\xi}}\left(k_1 l_1 v_\xi + k_1 m_1 w_\xi - K_1\right) + w\frac{m_1}{h_{1\xi}}\left(k_1 l_1 v_\xi + k_1 m_1 w_\xi - K_1\right) \\
&= \frac{k_1}{h_{1\xi}} K_{2\xi} + \upsilon\left(\frac{k_1 l_1}{h_{1\xi}^2} v_{\xi\xi} + \frac{k_1 m_1}{h_{1\xi}^2} w_{\xi\xi}\right) + \upsilon\frac{l_1^2}{h_{1\xi}^2}\left(k_1 l_1 v_{\xi\xi} + k_1 m_1 w_{\xi\xi} - K_1\right) \\
&+ \upsilon\frac{m_1^2}{h_{1\xi}^2}\left(k_1 l_1 v_{\xi\xi} + k_1 m_1 w_{\xi\xi} - K_1\right) + \frac{k_1}{m_1}\left\{\frac{k_1 c_1}{h_{1\xi} m_1} w_\xi - \frac{k_1 l_1}{h_{1\xi} m_1} v w_\xi - \frac{k_1}{h_{1\xi}} w w_\xi\right\} \\
&+ \frac{k_1}{m_1}\left\{F_3 + \upsilon\left(\frac{k_1^3}{h_{1\xi}^2 m_1} + \frac{k_1 l_1^2}{h_{1\xi}^2 m_1} + \frac{k_1 m_1}{h_{1\xi}^2}\right) w_{\xi\xi} + \frac{k_1}{h_{1\xi}} w_\xi\left[\left(\frac{k_1^2 l_1}{m_1} v + k_1^2 w\right) - K_1\right]\right\} + F_1.
\end{aligned}
\tag{42}
$$

Thus by grouping the above equation, the momentum in z direction is given by

$$
\begin{aligned}
&a_9 w_{\xi\xi} + a_{10} w w_\xi + a_{11} w_\xi + \frac{k_1}{m_1} F_3 + F_1 + \frac{k_1}{h_{1\xi}} K_{2\xi} - \left(\frac{c_1}{h_{1\xi}} + \upsilon\frac{l_1^2}{h_{1\xi}^2} + \upsilon\frac{m_1^2}{h_{1\xi}^2}\right) K_1 \\
&\qquad = a_{12} v_{\xi\xi} + a_{13} v v_\xi + \left(a_{14} + a_{15} w\right) v_\xi + a_{16} w_\xi v.
\end{aligned}
\tag{43}
$$

Since equation (43) is similar to that of (37), it is reasonable to state that the solution for v is also similar to (40). Repeating the procedure described by (35-40) with different variables, b_4, b_5 and b_6 yielding

$$
v = \frac{a_9 w_{\xi\xi} + a_{10} w w_\xi + a_{11} w_\xi - a_{15} b_6 w - a_{12} b_{6\xi} - a_{12} b_5 b_6 - a_{14} b_6 + \frac{k_1}{m_1} F_3 + F_1 + \frac{k_1}{h_{1\xi}} K_{2\xi} - K_7}{a_{12} b_{5\xi} + 2 a_{12} b_4 b_6 + a_{12} b_5^2 + a_{13} b_6 + a_{14} b_5 + a_{16} w_\xi + a_{15} b_5 w},
\tag{44}
$$

where $K_7 = \left(\frac{c_1}{h_{1\xi}} + \upsilon\frac{l_1^2}{h_{1\xi}^2} + \upsilon\frac{m_1^2}{h_{1\xi}^2}\right) K_1$. Equating (40) and (44) to get

$$
\begin{aligned}
v &= \frac{a_6 w_{\xi\xi} + a_7 w_\xi + a_8 w w_\xi - a_4 b_3 w - a_1 b_{3\xi} - a_1 b_2 b_3 - a_3 b_3 + \frac{l_1}{m_1} F_3 + F_2 + \frac{l_1}{h_{1\xi}} K_{2\xi}}{a_1 b_{2\xi} + 2 a_1 b_1 b_3 + a_1 b_2^2 + a_2 b_3 + a_3 b_2 + a_5 w_\xi + a_4 b_2 w} \\
&= \frac{a_9 w_{\xi\xi} + a_{10} w w_\xi + a_{11} w_\xi - a_{15} b_6 w - a_{12} b_{6\xi} - a_{12} b_5 b_6 - a_{14} b_6 + \frac{k_1}{m_1} F_3 + F_1 + \frac{k_1}{h_{1\xi}} K_{2\xi} - K_7}{a_{12} b_{5\xi} + 2 a_{12} b_4 b_6 + a_{12} b_5^2 + a_{13} b_6 + a_{14} b_5 + a_{16} w_\xi + a_{15} b_5 w}
\end{aligned}
\tag{45}
$$

which then can be performed algebraically to form a single expression.

In this research, we are interested to the class of solutions that is physically important [3, 4],

$$
w = r(\xi) e^{i h_1(\xi)}. \tag{46}
$$

Substituting (46) into (45) to get

$$r_\xi\left(a_{17}r_{\xi\xi}+a_{18}rr_\xi+a_{19}r_\xi+a_{20}r+a_{21}\right)+r\left(a_{22}r_{\xi\xi}+a_{23}rr_\xi+a_{24}r_\xi+a_{25}r+a_{26}\right)$$
$$+a_{27}r_{\xi\xi}+a_{28}rr_\xi+a_{29}r_\xi+a_{30}r+a_{31}=0. \tag{47}$$

By setting,

$$r_\xi=b_7r^2+b_8r+b_9 \tag{48}$$

such that the following relation is fulfilled

$$r_{\xi\xi}=2b_7^2r^3+\left(b_{7\xi}+3b_7b_8\right)r^2+\left(b_{8\xi}+2b_7b_9+b_8^2\right)r+b_{9\xi}+b_8b_9. \tag{49}$$

Note that equation (48) is also an equivalent form of higher order equation as stated in theorem 1. Thus, applying (48) and (49) into (47), the resulting polynomial equation is defined by

$$a_{32}r^5+a_{33}r^4+a_{34}r^3+a_{35}r^2+a_{36}r+a_{37}=0, \tag{50}$$

where a_{37} is given by

$$a_{37}=a_{17}\left(b_{9\xi}b_9+b_8b_9^2\right)+a_{19}b_9^2+a_{21}b_9+a_{27}\left(b_{9\xi}+b_8b_9\right)+a_{29}b_9+a_{31}. \tag{51}$$

Lemma 1 can be implemented to reduce equation (50) to the fourth order polynomial which the root is solvable by radicals, say be written as $r=\varepsilon(\xi)$. Meanwhile, b_7 will be defined later by the similar condition as applied to b_3.

3.2. The Solution for the Riccati Equation

In order to solve the system of (48) and (50), the following step is necessary [5],

Lemma 4. *Consider equation* (48) *and set* $b_8=f_1-\frac{f_{2\xi}}{f_2}$ *to generate*

$$Z_\xi=\frac{b_7}{f_2}Z^2+f_1Z+f_2b_9,$$

where $Z=f_2r$. *Set a relation*$f_2b_9=\phi Z$, *and according to the result of lemma 2, the function* ϕ *is expressed as,*

$$\phi=\frac{C_2}{\Phi}\left\{\left[\int_\xi\left(f_2\Phi b_9e^{-\int_\xi f_1d\xi}\int_\xi\frac{b_7}{f_2}\Phi e^{\int_\xi f_1d\xi}d\xi\right)d\xi\right]^{\frac{1}{2}}\right\}_\xi.$$

Proof. Set $b_8=f_1-\frac{f_{2\xi}}{f_2}$ to rearrange equation (48) as

$$r_\xi+\frac{f_{2x}}{f_2}r=\frac{1}{f_2}\left(f_2r\right)_\xi=b_7r^2+f_1r+b_9. \tag{52}$$

Suppose that $Z = f_2 r$, then the following equation is produced

$$Z_\xi = \frac{b_7}{f_2} Z^2 + f_1 Z + f_2 b_9. \tag{53}$$

Set $f_2 b_9 = \phi Z$, the original problem is transformed into

$$Z_\xi = \frac{b_7}{f_2} Z^2 + \left(f_1 + \phi\right) Z. \tag{54}$$

The solution of (8c) is expressed as

$$Z = \frac{f_2 b_9}{\phi} = -\frac{e^{\int_\xi (f_1+\phi) d\xi}}{\int_\xi \frac{b_7}{f_2} e^{\int_\xi (f_1+\phi) d\xi} d\xi}. \tag{55}$$

Applying the result of lemma 2 and define the function, $f_2 g e^{-\int_\xi f_1 d\xi} = A\Phi$ to produce

$$\frac{A}{B} = -\frac{f_2 b_9 e^{-\int_\xi f_1 d\xi} \int_\xi \frac{b_7}{f_2} \Phi e^{\int_\xi f_1 d\xi} d\xi}{\int_\xi \frac{A}{B} \Phi d\xi}, \tag{56}$$

where Φ will be defined later. The solution for ϕ is then

$$\phi = \frac{A}{B} = \frac{C_4}{\Phi} \left\{ \left[\int_\xi \left(f_2 \Phi b_9 e^{-\int_\xi f_1 d\xi} \int_\xi \frac{b_7}{f_2} \Phi e^{\int_\xi f_1 d\xi} d\xi \right) d\xi \right]^{\frac{1}{2}} \right\}_\xi. \tag{57}$$

The above equation is combined with (55) to form the solution of the general Riccati equation. This proves lemma 4.

3.3. The Complete Solution

The system which is represented by (48) and (50) will have a simultaneous solution as it is driven by equation (52- 57) and equation (50). The claim is concluded in the following theorem:

Theorem 2. *Consider the solution of the Riccati equation as described by* (55) *and* (57). *By combining with the root of* (50), $r = \varepsilon(\xi)$, *then the expressions of* f_2 *and* Φ *can be determined. The resulting expressions thus complete the solution of the system defined by* (48) *and* (50).

Proof. Equating the results from (50) and lemma 4 as follows

$$\frac{b_9}{\alpha} = \varepsilon \text{ or } b_9 = \varepsilon\phi = \varepsilon \frac{C_4}{\Phi} \left\{ \left[\int_\xi \left(f_2 \Phi b_9 e^{-\int_\xi f_1 d\xi} \int_\xi \frac{b_7}{f_2} \Phi e^{\int_\xi f_1 d\xi} d\xi \right) d\xi \right]^{\frac{1}{2}} \right\}_\xi. \tag{58}$$

Integrate the above equation and perform the algebraic calculations to give

$$\int_\xi \frac{b_7}{f_2}\Phi e^{\int_\xi f_1 d\xi} d\xi = C_5 \frac{e^{\int_\xi f_1 d\xi}}{f_2 b_9 \Phi}\left[\left(\int_\xi \frac{\Phi b_9}{\varepsilon} d\xi\right)^2\right]_\xi = \frac{\varphi}{f_2} e^{\int_\xi f_1 d\xi} \tag{59}$$

with $\varphi = \frac{C_5}{\Phi b_9}\left[\left(\int_\xi \frac{\Phi b_9}{\varepsilon} d\xi\right)^2\right]_\xi$. Differentiate (59) once to get the relations of f_1 and f_2 as in the following

$$\left(\frac{1}{f_2}\right)_\xi = \frac{1}{f_2}\left(\frac{\Phi b_7}{\varphi} - \frac{\varphi_\xi}{\varphi} - f_1\right). \tag{60}$$

The solution for f_2 is then

$$f_2 = \varphi e^{\int_\xi f_1 d\xi} e^{-\int_\xi \frac{\Phi b_7}{\varphi} d\xi}. \tag{61}$$

Equating the above equation with $b_8 = f_1 - \frac{f_{2\xi}}{f_2}$ to get the following expression

$$b_8 = f_1 + \left(\frac{\Phi b_7}{\varphi} - \frac{\varphi_\xi}{\varphi} - f_1\right) \tag{62}$$

which has also solved φ as in the following,

$$\varphi = \frac{C_5}{\Phi b_9}\left[\left(\int_\xi \frac{\Phi b_9}{\varepsilon} d\xi\right)^2\right]_\xi = e^{-\int_\xi b_8 d\xi}\int_\xi \Phi b_7 e^{\int_\xi b_8 d\xi} d\xi. \tag{63}$$

It is important to mention that the solution for f_1 does not exist. This condition is consistent with equation (61), since f_1 will vanish when (61) is substituted into (57). The above equation can be rearranged as

$$\frac{2C_5}{\varepsilon} e^{\int_\xi b_8 d\xi}\left(\int_\xi \frac{\Phi b_9}{\varepsilon} d\xi\right) = \int_\xi \Phi b_7 e^{\int_\xi b_8 d\xi} d\xi. \tag{64}$$

Equation (64) forms the linear first order equation in Φ, the solution is then expressed as

$$\begin{aligned} \Phi &= \frac{\varepsilon}{b_9}\left[\frac{\left(\frac{2C_5}{\varepsilon}\right)_\xi + \frac{2b_8 C_5}{\varepsilon}}{\left(\frac{\varepsilon b_7}{b_9} - \frac{2C_5}{\varepsilon}\right)}\right] \exp\int_\xi \left[\frac{\left(\frac{2C_5}{\varepsilon}\right)_\xi + \frac{2b_8 C_5}{\varepsilon}}{\left(\frac{\varepsilon b_7}{b_9} - \frac{2C_5}{\varepsilon}\right)}\right] d\xi \\ &= \frac{1}{b_9}\left(\frac{2b_8\varepsilon C_5 - 2\varepsilon_\xi C_5}{\varepsilon^2 b_7 - 2C_5}\right) e^{\int_\xi \left(\frac{2b_8\varepsilon C_5 - 2\varepsilon_\xi C_5}{\varepsilon^3 b_7 - 2\varepsilon C_5}\right) d\xi}. \end{aligned} \tag{65}$$

Therefore, the explicit solution of the system (48) and (50) is given by

$$r = \frac{b_9}{\phi} = -\frac{e^{\int_\xi (f_1+\phi) d\xi}}{f_2 \int_\xi \frac{b_7}{f_2} e^{\int_\xi (f_1+\phi) d\xi} d\xi} = -\frac{e^{\int_\xi \left(\frac{\Phi b_7}{\varphi}+\phi\right) d\xi}}{\varphi \int_\xi \frac{b_7}{\varphi} e^{\int_\xi \left(\frac{\Phi b_7}{\varphi}+\phi\right) d\xi} d\xi}, \tag{66}$$

where the function ϕ is determined by

$$\begin{aligned}\phi =&\frac{C_4}{\Phi}\left\{\left[\int_\xi \left(f_2\Phi b_9 e^{-\int_\xi f_1 d\xi}\int_\xi \frac{b_7}{f_2}\Phi e^{\int_\xi f_1 d\xi}d\xi\right)d\xi\right]^{\frac{1}{2}}\right\}_\xi \\ =&\frac{C_4}{\Phi}\left\{\left[\int_\xi \left(\varphi\Phi b_9 e^{-\int_\xi \frac{\Phi b_7}{\varphi}d\xi}\int_\xi \frac{b_7}{\varphi}\Phi e^{\int_\xi \frac{\Phi b_7}{\varphi}d\xi}d\xi\right)d\xi\right]^{\frac{1}{2}}\right\}_\xi \\ =&\frac{C_4}{\Phi}\left[\left(\int_\xi \varphi\Phi b_9 d\xi\right)^{\frac{1}{2}}\right]_\xi\end{aligned} \tag{67}$$

This completes the proof of theorem 1.

Note that the procedure that explained by equation (52-57) and equation (58-67) can also be applied to system (35) and (40). Therefore, by substituting (66) into (46) the solution of mass and momentum equations, w, v, p and u are defined by (46), (40) or (44), (4) and (2). Therefore, based on the obtained solutions, the result then can be generalised as in the following

$$\begin{aligned} w =& \; w_1 + \; w_2 + \; \cdots \\ v =& \; v_1 + \; v_2 + \; \cdots \\ u =& \; u_1 + \; u_2 + \; \cdots \\ p =& \; p_1 + \; p_2 + \; \cdots \end{aligned} \tag{68}$$

which can also be solved by the same procedure.

4. Remarks on Integral Evaluation of Analytical Solutions

It is important to note that the integrals which appear in the analytical solutions are usually approximated in series forms [17], by which the solution is then no longer exact. In order to resolve the problem, now the following integral is considered [10],

$$A = \int_\xi \lambda e^{\int_\xi f d\xi} d\xi. \tag{69}$$

By setting

$$A = \int_\xi \lambda e^{\int_\xi f d\xi} d\xi = (R+Q)\,\eta e^{\int_\xi g d\xi}, \tag{70}$$

where C_3 is a constant, equation (70) can be differentiated once to give

$$\lambda e^{\int_\xi f d\xi} = \left(R_\xi + Q_\xi\right)\eta e^{\int_\xi g d\xi} + (R+Q)\,\eta_\xi e^{\int_\xi g d\xi} + (R+Q)\,\eta g e^{\int_\xi g d\xi}.$$

Rearranging the above equation as

$$R_\xi + \left(\frac{\eta_\xi}{\eta} + g\right) R = \frac{\lambda}{\eta} e^{\int_\xi f - g d\xi} - \left\{ Q_\xi + \left(\frac{\eta_\xi}{\eta} + g\right) Q \right\}. \tag{71}$$

The solution of p is then expressed by

$$R = \frac{1}{\eta} e^{-\int_\xi g d\xi} \int_\xi \eta e^{\int_\xi g d\xi} \left[\frac{\lambda}{\eta} e^{\int_\xi f - g d\xi} - \left\{ Q_\xi + \left(\frac{\eta_\xi}{\eta} + g\right) Q \right\} \right] d\xi. \tag{72}$$

Let

$$\frac{\lambda}{\eta} e^{\int_\xi f - g d\xi} - \left\{ Q_\xi + \left(\frac{\eta_\xi}{\eta} + g\right) Q \right\} = f_3. \tag{73}$$

Then, R is evaluated in the following

$$R = \frac{1}{\eta} e^{-\int_\xi g d\xi} \left\{ \left(\int_\xi f_3 \eta d\xi \right) e^{\int_\xi g d\xi} - \int_\xi \left(\int_\xi f_3 \eta d\xi \right) g e^{\int_\xi g d\xi} d\xi \right\}. \tag{74}$$

Suppose that from equation (73),

$$\frac{\lambda}{\eta} e^{\int_\xi f - g d\xi} = C_6,$$

where C_4 is also a constant. The expression for $e^{\int_\xi g d\xi}$ is written as

$$\frac{\lambda}{C_6 \eta} e^{\int_\xi f d\xi} = e^{\int_\xi g d\xi}. \tag{75}$$

Thus, equation (74) will become

$$R = \frac{1}{C_6 \eta} e^{-\int_\xi g d\xi} \left\{ \left(\int_\xi f_3 \eta d\xi \right) \frac{\lambda}{\eta} e^{\int_\xi f d\xi} - \int_\xi \left(\int_\xi f_3 \eta d\xi \right) \left(\frac{\lambda}{\eta} e^{\int_\xi f d\xi} \right)_\xi d\xi \right\}. \tag{76}$$

Without loss of generality, set $\int_\xi f_3 \eta d\xi = \ln\left(\frac{\lambda}{\eta} e^{\int_\xi f d\xi}\right)$, and the expression of f_3 is obtained as

$$f_3 = \frac{1}{\eta} \left\{ \ln\left(\frac{\lambda}{\eta} e^{\int_\xi f d\xi} \right) \right\}_\xi. \tag{77}$$

The solution for Q is consequently obtained from (73) as in the following relation

$$Q = \frac{1}{\eta} e^{-\int_\xi g d\xi} \int_\xi \left(C_4 - f_3 \right) \eta e^{\int_\xi g d\xi} d\xi.$$

Substituting (75) to get

$$Q = \frac{1}{C_6 \eta} e^{-\int_\xi g d\xi} \int_\xi \left(C_6 - f_3 \right) \lambda e^{\int_\xi f d\xi} d\xi. \tag{78}$$

Equations (70), (76) and (78) will give the evaluation as

$$\int_{\xi} \lambda e^{\int_{\xi} f d\xi} d\xi = (R+Q)\,\eta e^{\int_{\xi} g d\xi} = \frac{1}{C_6}\frac{\lambda}{\eta} e^{\int_{\xi} f d\xi} + \frac{1}{C_6}\int_{\xi} (C_6 - f_3)\,\lambda e^{\int_{\xi} f d\xi} d\xi, \tag{79}$$

where f_3 is determined by (77).

Equation (79) can be differentiated once and rearranged to be

$$\int_{\xi} \lambda e^{\int_{\xi} f d\xi} d\xi = \int_{\xi} \frac{1}{f_3}\left(\frac{\lambda}{\eta} e^{\int_{\xi} f d\xi}\right)_{\xi} d\xi. \tag{80}$$

Now suppose that $\frac{\lambda}{\eta} e^{\int_{\xi} f d\xi} = L$ and $f_3 = \frac{1}{\eta}\left\{\ln\left(\frac{\lambda}{\eta} e^{\int_{\xi} f d\xi}\right)\right\}_{\xi} = L^n$, with n is an arbitrary constant. The relation of η is then given by

$$\lambda^n e^{n\int_{\xi} f d\xi} \eta^{1-n} = -\frac{\eta_{\xi}}{\eta} + \left(\frac{\lambda_{\xi}}{\lambda} + f\right). \tag{81}$$

Let $\eta = (\gamma + \chi)^{\frac{1}{1-n}}$, equation (81) will then produce

$$\begin{aligned} \chi_{\xi} =& (n-1)\,\lambda^n e^{n\int_{\xi} f d\xi} \chi^2 + (1-n)\left\{\left(\frac{\lambda_{\xi}}{\lambda} + f\right) - 2\lambda^n e^{n\int_{\xi} f d\xi}\gamma\right\}\chi \\ &+ (1-n)\left(\frac{\lambda_{\xi}}{\lambda} + f\right)\gamma + (n-1)\,\lambda^n e^{n\int_{\xi} f d\xi}\gamma^2 - \gamma_{\xi}. \end{aligned} \tag{82}$$

The solution for χ is similar to that of formulation described by the method in lemma 4,

$$\chi = \frac{(1-n)\left(\frac{\lambda_{\xi}}{\lambda} + f\right)\gamma + (n-1)\,\lambda^n e^{n\int_{\xi} f d\xi}\gamma^2 - \gamma_{\xi}}{\left\{\left[C_7 \int_{\xi}\left(\left\{(1-n)\left(\frac{\lambda_{\xi}}{\lambda} + f\right)\gamma + (n-1)\,\lambda^n e^{n\int_{\xi} f d\xi}\gamma^2 - \gamma_{\xi}\right\} K_9\right) d\xi\right]^{\frac{1}{2}}\right\}_{\xi}}, \tag{83}$$

where $K_9 = \lambda^{(n-1)} e^{(n-1)\int_{\xi} f d\xi} e^{-2\int_{\xi} \lambda^n e^{n\int_{\xi} f d\xi}\gamma d\xi} \int_{\xi} \lambda e^{\int_{\xi} f d\xi} e^{2\int_{\xi} \lambda^n e^{n\int_{\xi} f d\xi}\gamma d\xi} d\xi$. Without loss of rigor, set $\gamma = -\frac{1}{2}\left(\frac{\lambda_{\xi}}{\lambda} + f\right)\lambda^{-n} e^{-n\int_{\xi} f d\xi}$ to produce a more simplified solution

$$\chi = \frac{C_8\left\{\int_{\xi}\left((n-1)\frac{3}{4}\left(\frac{\lambda_{\xi}}{\lambda} + f\right)^2 x + \left(\frac{1}{2}\left(\frac{\lambda_{\xi}}{\lambda} + f\right)\lambda^{-n} e^{-n\int_{\xi} f d\xi}\right)_{\xi} \lambda^n e^{n\int_{\xi} f d\xi}\xi\right) d\xi\right\}^{\frac{1}{2}}}{\lambda^n e^{n\int_{\xi} f d\xi}\xi}. \tag{84}$$

The step is now performing the integration of (80) to give

$$\int_{\xi} \lambda e^{\int_{\xi} f d\xi} d\xi = \frac{1}{1-n} L^{1-n} = \frac{1}{1-n}\left(\frac{\lambda}{\eta} e^{\int_{\xi} f d\xi}\right)^{1-n}. \tag{85}$$

Substitute the relation $\eta = (\gamma+\chi)^{\frac{1}{1-n}}$ into (85) and rearranging the result to produce

$$\left(C_9\left\{\int_\xi K_{10}d\xi\right\}^{\frac{1}{2}} - \frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\int_\xi \lambda e^{\int_\xi f d\xi}d\xi = \frac{\xi}{1-n}\lambda e^{\int_\xi f d\xi}$$

with $K_{10} = (n-1)\frac{3}{4}\left(\frac{\lambda_\xi}{\lambda}+f\right)^2\xi + \left(\frac{1}{2}\left(\frac{\lambda_\xi}{\lambda}+f\right)\lambda^{-n}e^{-n\int_\xi f d\xi}\right)_\xi \lambda^n e^{n\int_\xi f d\xi}\xi$. Rearrange and differentiate the above equation once to get

$$\begin{aligned} D =& C_{10}\left(\frac{\xi\lambda e^{\int_\xi f d\xi}}{\int_\xi \lambda e^{\int_\xi f d\xi}d\xi} + \frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right) \\ &\times\left(\frac{\left(\xi\lambda e^{\int_\xi f d\xi}\right)_\xi}{\int_\xi \lambda e^{\int_\xi f d\xi}d\xi} - \frac{\xi\lambda^2 e^{2\int_\xi f d\xi}}{\left(\int_\xi \lambda e^{\int_\xi f d\xi}d\xi\right)^2} + \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi\right), \end{aligned} \tag{86}$$

where D is expressed by $D = (n-1)\frac{3}{4}\left(\frac{\lambda_\xi}{\lambda}+f\right)^2\xi + \left(\frac{1}{2}\left(\frac{\lambda_\xi}{\lambda}+f\right)\lambda^{-n}e^{-n\int_\xi f d\xi}\right)_\xi \lambda^n e^{n\int_\xi f d\xi}\xi$. It is not hard to see that equation (86) will produce a polynomial equation as in the following

$$\begin{aligned} &\left(\int_\xi \lambda e^{\int_\xi f d\xi}d\xi\right)^3 D = C_{10}\left(E + F\left(\int_\xi \lambda e^{\int_\xi f d\xi}d\xi\right)\right) \\ &\times\left\{\left(\int_\xi \lambda e^{\int_\xi f d\xi}d\xi\right)\left(\xi\lambda e^{\int_\xi f d\xi}\right)_\xi + \left(-\xi\lambda^2 e^{2\int_\xi f d\xi}\right) + \left(\int_\xi \lambda e^{\int_\xi f d\xi}d\xi\right)^2\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi\right\} \end{aligned} \tag{87}$$

with $E = \xi\lambda e^{\int_\xi f d\xi}$and $F = \frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)$. Thus, the integral of $\int_\xi \lambda e^{\int_\xi f d\xi}d\xi$ can be defined by the root of equation (87). By taking one root of (87) as

$$\begin{aligned} \int_\xi \lambda e^{\int_\xi f d\xi}d\xi =& \frac{C_{10}}{3}\frac{\left(\xi\lambda e^{\int_\xi f d\xi}\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi + \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\xi\lambda e^{\int_\xi f d\xi}\right)_\xi}{\left[K_{10} - \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi\right]} \\ &+ \left[G + \left(H^3+G^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[G - \left(H^3+G^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}. \end{aligned} \tag{88}$$

Meanwhile, G and H are defined by

$$G = \frac{C_{10}^2}{6}\left\{\frac{\left[K_{11}\left(\xi\lambda e^{\int_\xi f d\xi}\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi + \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\xi\lambda e^{\int_\xi f d\xi}\right)_\xi\right]}{\left[K_{10} - \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi\right]^2}\right\}$$
$$+ \frac{C_{10}}{2}\frac{\left(\xi\lambda e^{\int_\xi f d\xi}\right)\left(-\xi\lambda^2 e^{2\int_\xi f d\xi}\right)}{\left[K_{10} - \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi\right]} \quad (89)$$
$$+ \frac{C_{10}^3}{27}\frac{\left[\left(\xi\lambda e^{\int_\xi f d\xi}\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi + \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\xi\lambda e^{\int_\xi f d\xi}\right)_\xi\right]^3}{\left[K_{10} - \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi\right]^3},$$

and

$$H = \frac{C_{10}}{3}\frac{\left[\left(\xi\lambda e^{\int_\xi f d\xi}\right)\left(\xi\lambda e^{\int_\xi f d\xi}\right)_\xi + \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(-\xi\lambda^2 e^{2\int_\xi f d\xi}\right)\right]}{\left[K_{10} - \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi\right]}$$
$$+ \frac{C_{10}^2}{2}\frac{\left[\left(\xi\lambda e^{\int_\xi f d\xi}\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi + \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\xi\lambda e^{\int_\xi f d\xi}\right)_\xi\right]^2}{\left[K_{10} - \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi\right]^2} \quad (90)$$

with $K_{11} = \left[\left(\xi\lambda e^{\int_\xi f d\xi}\right)\left(\xi\lambda e^{\int_\xi f d\xi}\right)_\xi + \left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(-\xi\lambda^2 e^{2\int_\xi f d\xi}\right)\right]$. This will solve the integral in (69). Therefore, the following theorem is just proved.

Theorem 3. *Consider the following integral equation*

$$A = \int_\xi \lambda(\xi) e^{\int_\xi f(\xi) d\xi} d\xi.$$

There exists a functional $\eta = \gamma + \chi$, with $\gamma$and χ are given by

$$\gamma = -\frac{1}{2}\left(\frac{\lambda_\xi}{\lambda}+f\right)\lambda^{-n} e^{-n\int_\xi f d\xi}, \chi$$
$$= \frac{C_8\left\{\int_\xi\left((n-1)\frac{3}{4}\left(\frac{\lambda_\xi}{\lambda}+f\right)^2 x + \left(\frac{1}{2}\left(\frac{\lambda_\xi}{\lambda}+f\right)\lambda^{-n}e^{-n\int_\xi f d\xi}\right)_\xi \lambda^n e^{n\int_\xi f d\xi}\xi\right)d\xi\right\}^{\frac{1}{2}}}{\lambda^n e^{n\int_\xi f d\xi}\xi}$$

such that the integral A can be evaluated as

$$\int_{\xi} \lambda e^{\int_{\xi} f d\xi} d\xi = \frac{C_{10}}{3} \frac{\left(\xi \lambda e^{\int_{\xi} f d\xi}\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_{\xi}}{\lambda}+f\right)\right)_{\xi} + \left(\frac{1}{2}\xi\left(\frac{\lambda_{\xi}}{\lambda}+f\right)\right)\left(\xi \lambda e^{\int_{\xi} f d\xi}\right)_{\xi}}{\left[K_{10} - \left(\frac{1}{2}\xi\left(\frac{\lambda_{\xi}}{\lambda}+f\right)\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_{\xi}}{\lambda}+f\right)\right)_{\xi}\right]}$$
$$+ \left[G + \left(H^3 + G^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[G - \left(H^3 + G^2\right)^{\frac{1}{2}}\right]^{\frac{1}{3}},$$

where C_8, C_{10} and n are arbitrary constants and G and H are defined by (89) *and* (90).

5. Remarks on the Properties of the Solutions

Now we are at step to answer and proof the questions of existence and uniqueness of smooth solutions with the result from integral evaluation. Since some functions of the solution are arbitrary, they can become powerful objects to justify the properties under general initial-boundary conditions [6].

5.1. Uniqueness Property

Consider that the velocity vectors and pressure have already been described by two solutions with identical boundary conditions i.e. $(u_1, u_2), (v_1, v_2), (w_1, w_2)$ and (p_1, p_2). Substituting the solution pairs into the continuity and Navier-Stokes equations (1a-1d) and set the following relations

$$\begin{aligned} P &= p_1 - p_2 \\ U &= u_1 - u_2 \\ V &= v_1 - v_2 \\ W &= w_1 - w_2. \end{aligned} \tag{91}$$

Suppose that the solutions are also supplemented by cauchy conditions as follows

$$\begin{aligned} &u_1(x_i, 0) = u_2(x_i, 0) = u_0(x_i) \\ &v_1(x_i, 0) = v_2(x_i, 0) = 0 \\ &w_1(x_i, 0) = w_2(x_i, 0) = 0. \end{aligned} \tag{92}$$

Hence, the following theorem is justified.

Theorem 4. *Implements equation* (92) *into* (91) *in order to satisfy the following initial conditions*

$$\begin{aligned} &U(x_i, 0) = u_1(x_i, 0) - u_2(x_i, 0) = 0 \\ &V(x_i, 0) = v_1(x_i, 0) - v_2(x_i, 0) = 0 \\ &W(x_i, 0) = w_1(x_i, 0) - w_2(x_i, 0) = 0 \end{aligned}.$$

The condition $\frac{\lambda_\xi}{\lambda}+f=0$ *then can be applied such that the expression* $\frac{b_9}{\varepsilon}=C_{12}\Phi$ *and* $b_3=-e^{-\int_\xi\left(b_2+\frac{a_3}{a_1}\right)d\xi}\int_\xi\left[\frac{1}{a_1}\left(\frac{l_1}{m_1}I_3+I_2-\frac{l_1}{h_{1\xi}}K_{2\xi}\right)e^{\int_\xi\left(b_2+\frac{a_3}{a_1}\right)d\xi}\right]d\xi$ *are produced. The continuity and incompressible Navier-Stokes equations will have at most one solution for velocities and more than one for pressure.*

Proof. The following equations are produced by substraction and by utilisation of (91),

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0, \tag{93}$$

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\upsilon\frac{\partial^2 U}{\partial x^2}+\upsilon\frac{\partial^2 U}{\partial y^2}+\upsilon\frac{\partial^2 U}{\partial z^2}-I_1, \tag{94}$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\upsilon\frac{\partial^2 V}{\partial x^2}+\upsilon\frac{\partial^2 V}{\partial y^2}+\upsilon\frac{\partial^2 V}{\partial z^2}-I_2, \tag{95}$$

$$\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\upsilon\frac{\partial^2 W}{\partial x^2}+\upsilon\frac{\partial^2 W}{\partial y^2}+\upsilon\frac{\partial^2 W}{\partial z^2}-I_3, \tag{96}$$

where I_1, I_2 and I_3 are defined by

$$I_1=(u_1+2u_2)\frac{\partial u_2}{\partial x}+(v_1+2v_2)\frac{\partial u_2}{\partial y}+(w_1+2w_2)\frac{\partial u_2}{\partial z}+u_2\frac{\partial u_1}{\partial x}+v_2\frac{\partial u_1}{\partial y}+w_2\frac{\partial u_1}{\partial z}, \tag{97}$$

$$I_2=(u_1+2u_2)\frac{\partial v_2}{\partial x}+(v_1+2v_2)\frac{\partial v_2}{\partial y}+(w_1+2w_2)\frac{\partial v_2}{\partial z}+u_2\frac{\partial v_1}{\partial x}+v_2\frac{\partial v_1}{\partial y}+w_2\frac{\partial v_1}{\partial z}, \tag{98}$$

$$I_3=(u_1+2u_2)\frac{\partial w_2}{\partial x}+(v_1+2v_2)\frac{\partial w_2}{\partial y}+(w_1+2w_2)\frac{\partial w_2}{\partial z}+u_2\frac{\partial w_1}{\partial x}+v_2\frac{\partial w_1}{\partial y}+w_2\frac{\partial w_1}{\partial z}. \tag{99}$$

Note that the external forces F_1, F_2 and F_3 are vanish by the substraction procedure. Applying the same method as derived before, the solution of (93-96) are can be rewritten as

$$W=\frac{b_9}{\phi}e^{ih_1(\xi)}=C_{11}e^{ih_1(\xi)}\frac{1}{\varphi}\left(\int_\xi \varphi\Phi b_9 d\xi\right)^{\frac{1}{2}}, \tag{100}$$

$$V=\frac{a_9W_{\xi\xi}+a_{10}WW_\xi+a_{11}W_\xi-a_{15}b_6W-a_{12}b_{6\xi}-a_{12}b_5b_6-a_{14}b_6}{a_{12}b_{5\xi}+2a_{12}b_4b_6+a_{12}b_5^2+a_{13}b_6+a_{14}b_5+a_{16}W_\xi+a_{15}b_5W}$$
$$+\frac{-\frac{k_1}{m_1}I_3-I_1+\frac{k_1}{h_{1\xi}}K_{2\xi}-\left(\frac{c_1}{h_{1\xi}}+\upsilon\frac{l_1^2}{h_{1\xi}^2}+\upsilon\frac{m_1^2}{h_{1\xi}^2}\right)K_1}{a_{12}b_{5\xi}+2a_{12}b_4b_6+a_{12}b_5^2+a_{13}b_6+a_{14}b_5+a_{16}W_\xi+a_{15}b_5W}$$

or

$$V=\frac{a_6W_{\xi\xi}+a_7W_\xi+a_8WW_\xi-a_4b_3W-a_1b_{3\xi}-a_1b_2b_3-a_3b_3-\frac{l_1}{m_1}I_3-I_2+\frac{l_1}{h_{1\xi}}K_{2\xi}}{a_1b_{2\xi}+2a_1b_1b_3+a_1b_2^2+a_2b_3+a_3b_2+a_5W_\xi+a_4b_2W}, \tag{101}$$

$$P=\rho\int_z\left\{\upsilon\left(W_{xx}+W_{yy}+W_{zz}\right)+W_x\left[\int_x\left(V_y+W_z\right)dx-C_1\right]-W_t-VW_y-WW_z-I_3\right\}dz$$
$$+K_2, \tag{102}$$

$$U=-\int_x\left(V_y+W_z\right)dx+K_1. \tag{103}$$

In order to ensure uniqueness, it is necessary to set, for example, $W=w_1-w_2=0$ with initial value $W\left(x_i,0\right)=w_1\left(x_i,0\right)-w_2\left(x_i,0\right)=0$ such that $w_1=w_2$ with $w_1\left(x_i,0\right)=w_2\left(x_i,0\right)$ are fulfilled [8]. Thus, the denumerator of (100) can be considered as, $\varphi=\frac{2C_5}{\varepsilon}\int_\xi\frac{\Phi b_9}{\varepsilon}d\xi=\infty$ which from the integral evaluation (88), (89) and (90), it is not hard to verify that the following expression must be fulfilled

$$\begin{aligned}(n-1)\frac{3}{4}\left(\frac{\lambda_\xi}{\lambda}+f\right)^2\xi+\left(\frac{1}{2}\left(\frac{\lambda_\xi}{\lambda}+f\right)\lambda^{-n}e^{-n\int_\xi fd\xi}\right)_\xi\lambda^ne^{n\int_\xi fd\xi}\xi\\-\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi=0,\end{aligned} \tag{104}$$

where $\lambda=\frac{b_9}{\varepsilon}$ and $f=\frac{\Phi_\xi}{\Phi}$. It is save to suppose that, $\frac{\left(b_9/\varepsilon\right)_\xi}{b_9/\varepsilon}+\frac{\Phi_\xi}{\Phi}=0$ such that the following relation is fulfilled

$$\frac{b_9}{\varepsilon}=C_{12}\Phi, \tag{105}$$

where C_{12} is a constant. Note that equation (105) satisfies both initial condition and uniqueness since the criteria are the same, i.e. $W=0$ and $W\left(x_i,0\right)=0$. Substituting $W=0$ into equation (101) with $V=0$ and $V\left(x_i,0\right)=v_1\left(x_i,0\right)-v_2\left(x_i,0\right)=0$, yielding

$$0=\frac{a_6W_{\xi\xi}+a_7W_\xi+a_8WW_\xi-a_4b_3W-a_1b_{3\xi}-a_1b_2b_3-a_3b_3-\frac{l_1}{m_1}I_3-I_2+\frac{l_1}{h_{1\xi}}K_{2\xi}}{a_1b_{2\xi}+2a_1b_1b_3+a_1b_2^2+a_2b_3+a_3b_2+a_5W_\xi+a_4b_2W}. \tag{106}$$

Dropping the term involving W, the first order differential is produced as in the following

$$b_{3\xi}+\left(b_2+\frac{a_3}{a_1}\right)b_3=-\frac{1}{a_1}\left(\frac{l_1}{m_1}I_3+I_2-\frac{l_1}{h_{1\xi}}K_{2\xi}\right). \tag{107}$$

In this case, b_2 is defined by $b_2=-\frac{\left(a_1b_{1\xi}+a_3b_1\right)}{3a_1b_1+a_2}$, where $b_1=-\frac{a_2}{2a_1}$. The solution of (107) is then

$$b_3=-e^{-\int_\xi\left(b_2+\frac{a_3}{a_1}\right)d\xi}\int_\xi\left[\frac{1}{a_1}\left(\frac{l_1}{m_1}I_3+I_2-\frac{l_1}{h_{1\xi}}K_{2\xi}\right)e^{\int_\xi\left(b_2+\frac{a_3}{a_1}\right)d\xi}\right]d\xi. \tag{108}$$

Consider a special case of equation (103), which will generate $K_1\left(y,z,0\right)=0$ due to $U\left(x_i,0\right)=0$. The result of $U=0$ will then be produced since V and W are zero. Following (91), we have that $u_1=u_2, v_1=v_2$ and $w_1=w_2$which show the uniqueness of velocities as in [9]. Meanwhile, since the relation, $K_2\left(x,y,t\right)\neq\int_z I_3dz$ of (102) is always hold, the result is consequently, $P\neq 0$, or $p_1\neq p_2$ which imply the non uniqueness of pressure. This proves theorem 3.

5.2. Existence and Regularity Properties

Apart from uniqueness, the existence and regularity properties depend on the chosen function of the coordinate transformation, $h_1(\xi)$, and the variable coefficients of Riccati equations, b_i . Suppose that we take a velocity in z direction after the evaluation of initial-boundary conditions as

$$w=\frac{b_9}{\phi}e^{ih_1(\xi)}=C_{11}e^{ih_1(\xi)}\frac{1}{\varphi}\left(\int_\xi\varphi\Phi b_9d\xi\right)^{\frac{1}{2}}. \tag{109}$$

Then, the following statement is produced.

Theorem 5. *Let $K_1\left(y,z,t\right)$,$K_2\left(x,y,t\right)$ and external forces are smooth functions, the following relations*

$$f_2=\varphi e^{\int_\xi f_1d\xi}e^{-\int_\xi\frac{\Phi b_7}{\varphi}d\xi}\ \text{and}\ \Phi=\frac{1}{b_9}\left(\frac{2b_8\varepsilon C_5-2\varepsilon_\xi C_5}{\varepsilon^2b_7-2C_5}\right)e^{\int_\xi\left(\frac{2b_8\varepsilon C_5-2\varepsilon_\xi C_5}{\varepsilon^3b_7-2\varepsilon C_5}\right)d\xi}$$

are satisfied such that there exist global solutions for u,v,w and p if b_7,b_8,b_9 and $h_1(\xi)$ are bounded.

Proof. The solution becomes bounded if $\varphi=\frac{2C_5}{\varepsilon}\int_\xi\frac{\Phi b_9}{\varepsilon}d\xi\neq 0$, which resulted to the following requirement

$$\begin{aligned}(n-1)\frac{3}{4}\left(\frac{\lambda_\xi}{\lambda}+f\right)^2\xi+\left(\frac{1}{2}\left(\frac{\lambda_\xi}{\lambda}+f\right)\lambda^{-n}e^{-n\int_\xi fd\xi}\right)_\xi\lambda^ne^{n\int_\xi fd\xi}\xi\\-\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)\left(\frac{1}{2}\xi\left(\frac{\lambda_\xi}{\lambda}+f\right)\right)_\xi<\infty,\end{aligned} \tag{110}$$

where $\lambda=\frac{b_9}{\varepsilon}$ and $f=\frac{\Phi_\xi}{\Phi}$. Therefore, by considering (65) and (110) it is clear that if b_7,b_8,b_9 and $h_1(\xi)$ are bounded, the solution will then also bounded. Applying the result into (40) or (49), (4) and (2), the bounded solution for velocities in x and y directions including pressure are also produced. Also note that the solutions of u,v,w and p are twice differentiable if the external forces and arbitrary functions, $K_1\left(y,z,t\right)$ and $K_2\left(x,y,t\right)$ are smooth enough. Thus, theorem 5 is proved.

6. Conclusions

The method for producing the analytical solutions of the three-dimensional incompressible Navier-Stokes equations is introduced in this paper. The solution is constructed under the implementation of the higher polynomial differential equation such as in the following

$$v_\xi = a_1 v^n + a_2 v^{n-1} + a_3 v^{n-2} + a_4 v^{n-3} + a_5 v^{n-4} + \cdots\cdots + a_{n-1} v^2 + a_n v + a_{n+1} \tag{111}$$

which then reduce to the Riccati equation. The solution that is generated from the system of Riccati and polynomial equations is then expanded by the utilisation of proposed integral evaluation.The existence is proved and the uniqueness property is ensured for velocities, meanwhile the pressure is not unique.

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