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On R-continuous Functions

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Abstract. In this paper, we introduce a new class of continuous functions as an application of R-closed sets namely R-continuous functions and study their properties in topological spaces. We introduce TR spaces, R-neighbourhood and analyze their properties in this paper. Also we introduce R-compact spaces and R-connected spaces.

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1. Introduction

The study of generalized closed sets in a topological space was initiated by Levine [11] and the concept of $T_{1/2}$ space was introduced. The modified forms of generalized closed sets and generalized continuity were studied by K. Balachandran, P. Sundaram and H. Maki [3]. M. Sheik john introduced ω-closed sets and ω -open sets [8]. As generalizations of closed sets, R-closed were introduced and studied by the same author [9]. The aim of this paper is to introduce a new class of functions called R-continuous functions. Moreover, the relationships and properties of R-continuous functions are obtained.

2. Preliminaries

Throughout this paper (X,τ), (Y,τ) and (Z,τ) will always denote topological spaces on which no separation axioms are assumed, unless otherwise mentioned. When A is a subset of (X,τ), $cl(A)$, $Int(A)$ denote the closure, the interior of A. We recall some known definitions needed in this paper.

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Definition 1. *Let (X,τ) be a topological space. A subset A of the space X is said to be*

(i) Pre open [13] if $A \subseteq Int(cl(A))$ and preclosed if $cl(Int(A)) \subseteq A$.

(ii) Semi open [10] if $A \subseteq cl(Int(A))$ and semiclosed if $Int(cl(A)) \subseteq A$.

(iii) α -open [14] if $A \subseteq Int(cl(Int(A)))$ and α-closed if $cl(Int(cl(A)) \subseteq A$.

(iv) Semi preopen [1] if $A \subseteq cl((Int(cl(A)))$ and semi preclosed if $Int(cl(Int(A))) \subseteq A$.

(v) Regular open [7] if $A = Int(cl(A))$ and regular closed if $A = cl(Int(A))$.

Definition 2. *Let (X,τ) be a topological space.A subset $A \subseteq X$ is said to be*

(i) a generalized closed set [11] (briefly g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ) the complement of a g- closed set is called a g-open set.

(ii) an α-generalized closed set [12] (briefly αg-closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ) the complement of a αg- closed set is called a αg-open set.

(iii) a generalized semi preclosed set [5] (briefly gsp-closed) if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ) the complement of a gsp- closed set is called a gsp-open set.

(iv) an ω-closed set [8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in (X,τ) the complement of a ω-closed set is called a ω-open set.

(v) a generalized preclosed set [15] (briefly gp-closed) if $\alpha cl(A) \subseteq intU$ whenever $A \subseteq U$ and U is α-open in (X,τ) the complement of a gp- closed set is called a gp-open set.

(vi) a generalized pre regular closed set [7] (briefly gpr-closed) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is preopen in (X,τ) the complement of a gpr- closed set is called a gpr-open set.

Definition 3. *A function $f : (X,\tau) \to (Y,\sigma)$ is called*

(i) g-continuous [3] if $f^{-1}(V)$ is g-closed in (X,τ) for every closed set V in (Y,σ),

(ii) ω-continuous [8] if $f^{-1}(V)$ is ω-closed in (X,τ) for every closed set V in (Y,σ),

(iii) gsp-continuous [5] if $f^{-1}(V)$ is gsp-closed in (X,τ) for every closed set V in (Y,σ),

(iv) gp-continuous [2] if $f^{-1}(V)$ is gp-closed in (X,τ) for every closed set V in (Y,σ),

(v) gpr-continuous [7] if $f^{-1}(V)$ is gpr-closed in (X,τ) for every closed set V in (Y,σ),

(vi) semi pre-continuous [1] if $f^{-1}(V)$ is semi pre-open in (X,τ) for every open set V in (Y,σ),

(vii) αg-continuous [12] if $f^{-1}(V)$ is αg-closed in (X,τ) for every closed set V in (Y,σ),

(viii) α-continuous [17] if $f^{-1}(V)$ is α-closed in (X,τ) for every closed set V in (Y,σ),

(ix) Contra-continuous [6] if $f^{-1}(V)$ is closed in (X,τ) for every open set V in (Y,σ),

(x) *ω-irresolute [8] if $f^{-1}(V)$ is ω-closed in (X,τ) for every ω- closed set V in (Y,σ).*

Definition 4. *A space (X,τ) is called*

(i) *a $T_{1/2}$ space [11] if every g-closed set is closed.*

(ii) *a T_ω space [16] if every ω-closed set is closed.*

3. R-Continuous Functions

Definition 5. *A subset of a topological space (X,τ) is said to be R-closed in (X,τ) if $\alpha cl(A) \subseteq Int(U)$ whenever $A \subseteq U$ and U is ω-open in (X,τ).*

Definition 6. *A function $f:(X,\tau) \to (Y,\sigma)$ is said to be R-continuous if $f^{-1}(V)$ is R-closed in (X,τ) for every closed set V of (Y,σ).*

Example 1. *Let $X = Y = \{a,b,c\}$, $\tau = \{X,\phi,\{a\},\{a,b\}\}$, $\sigma == \{X,\phi,\{a\},\{a,c\}\}$. We define $f:(X,\tau) \to (Y,\sigma)$ by $f(a) = a, f(b) = c, f(c) = b$. Then 'f' is R-continuous.*

Proposition 1. *Every R-continuous is gp continuous but not conversely.*

Proof. By [theorem 3.6, 9] every R-closed set is gp-closed, the proof follows.
Converse of the above proposition need not be true as seen from the following example.

Example 2. *Let $X = Y = \{a,b,c\}$, $\tau = \{X,\phi,\{a,b\}\}$, $\sigma = \{X,\phi,\{a,c\}\}$. Define the function $f:(X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = a, f(c) = c$. Then f is gp-continuous but not R-continuous.Since for the closed set $U = \{b\}$ in $(Y,\sigma), f^{-1}(U) = \{a\}$ is gp-closed but not R-closed in (X,τ).*

Proposition 2. *Every R-continuous is gpr continuous but not conversely.*

Proof. By [theorem 3.7, 9] every R-closed set is gpr-closed, the proof follows. Converse of the above proposition need not be true as seen from the following example.

Example 3. *Let $X = Y = \{a,b,c\}$, $\tau = \{X,\phi,\{a,b\}\}$, $\sigma = \{X,\phi,\{a,c\}\}$. Define the function $f:(X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. Then f is gp-continuous but not R-continuous.Since for the closed set $U = \{b\} in (Y,\sigma), f^{-1}(U) = \{a\}$ is gpr-closed but not R-closed in (X,τ).*

Proposition 3. *Every R-continuous is gsp continuous but not conversely.*

Proof. By [theorem 3.5, 9] every R-closed set is gsp-closed, the proof follows. Converse of the above proposition need not be true as seen from the following example.

Example 4. *Let $X = Y = \{a,b,c\}$, $\tau = \{X,\phi,\{a,c\}\}$, $\sigma = \{X,\phi,\{a,b\}\}$. Define the function $f:(X,\tau) \to (Y,\sigma)$ by $f(a) = c, f(b) = a, f(c) = b$. Then f is gsp-continuous but not R-continuous.Since for the closed set $U = \{c\}$ in (Y,σ), $f^{-1}(U) = \{a\}$ is gsp-closed but not R-closed in (X,τ).*

Proposition 4. *Every R-continuous is αg-continuous but not conversely.*

Proof. By [theorem 3.3, 9] every R-closed set is αg-closed, the proof follows. Converse of the above proposition need not be true as seen from the following example.

Example 5. *Let $X = Y = \{a,b,c\}$, $\tau = \{X,\phi,\{a\},\{b,c\}\}$, $\sigma = \{X,\phi,\{a,b\}\}$. Define the function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. Then f is αg -continuous but not R-continuous. Since for the closed set $U = \{c\} in (Y,\sigma), f^{-1}(U) = \{b\}$ is αg-closed but not R-closed in (X,τ).*

Proposition 5. *The following example show that R-continuity is independent of continuity. Let $X = Y = \{a,b,c\}$, $\tau = \{X,\emptyset,\{b\},\{a,b\}\}$, $\sigma = \{X,\emptyset,\{a,c\}\}$. Define the function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = a, f(b) = b, f(c) = b$. Since for the closed set $U = \{b\}$ in (Y,σ), $f^{-1}(U) = \{b,c\}$ is R-closed but not closed in (X,τ). Let $X = Y = \{a,b,c\}$, $\tau = \{X,\emptyset,\{b\},\{c\},\{b,c\}\}$, $\sigma = \{X,\emptyset,\{a,b\}\}$. Define the function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Since for the closed set $U = \{c\}$ in (Y,σ), $f^{-1}(U) = \{a\}$ is closed but not R-closed in (X,τ).*

Proposition 6. *The following example show that R-continuity is independent of g-continuity. Let $X = Y = \{a,b,c\}$, $\tau = \{X,\emptyset,\{a\},\{a,c\}\}$, $\sigma = \{X,\emptyset,\{b,c\}\}$. Define the function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = a, f(c) = c$. Since for the closed set $U = \{a\}$ in (Y,σ), $f^{-1}(U) = \{b\}$ is g-closed but not R-closed in (X,τ). Let $X = Y = \{a,b,c\}$, $\tau = \{X,\emptyset,\{a\},\{a,b\}\}, \sigma = \{X,\emptyset,\{b,c\}\}$. Define the function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Since for the closed set $U = \{a\}$ in (Y,σ), $f^{-1}(U) = \{b\}$ is R-closed but not g-closed in (X,τ).*

Proposition 7. *The following example show that R-continuity is independent of pre-continuity. Let $X = Y = \{a,b,c\}$, $\tau = \{X,\emptyset,\{a\}\}$, $\sigma = \{X,\emptyset,\{b\}\}$. Define the function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = a, f(b) = c, f(c) = b$. Since for the closed set $U = \{a,c\}$ in (Y,σ), $f^{-1}(U) = \{a,b\}$ is R-closed but not pre-closed in (X,τ). Let $X = Y = \{a,b,c\}$, $\tau = \{X,\emptyset,\{a,b\}\}$, $\sigma = \{X,\emptyset,\{a,c\}\}$. Define the function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. Since for the closed set $U = \{b\}$ in (Y,σ), $f^{-1}(U) = \{a\}$ is pre-closed but not R-closed in (X,τ).*

Proposition 8. *The following example show that R-continuity is independent of α-continuity. Let $X = Y = \{a,b,c\}$, $\tau = \{X,\emptyset,\{a\}\}$, $\sigma = \{X,\emptyset,\{c\}\}$. Define the function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b, f(b) = a, f(c) = c$. Since for the closed set $U = \{a,b\}$ in (Y,σ), $f^{-1}(U) = \{a,b\}$ is R-closed but not α-closed in (X,τ). Let $X = Y = \{a,b,c\}$, $\tau = \{X,\emptyset,\{b,c\}\}$, $\sigma = \{X,\emptyset,\{a,b\}\}$. Define the function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = c, f(b) = a, f(c) = b$. Since for the closed set $U = \{c\}$ in (Y,σ), $f^{-1}(U) = \{a\}$ is α-closed but not R-closed in (X,τ). From the above discussions and known results we have the following implications. $A \to B$ ($A \leftrightarrow B$) represents A implies B but not conversely (A and B are independent of each other).*

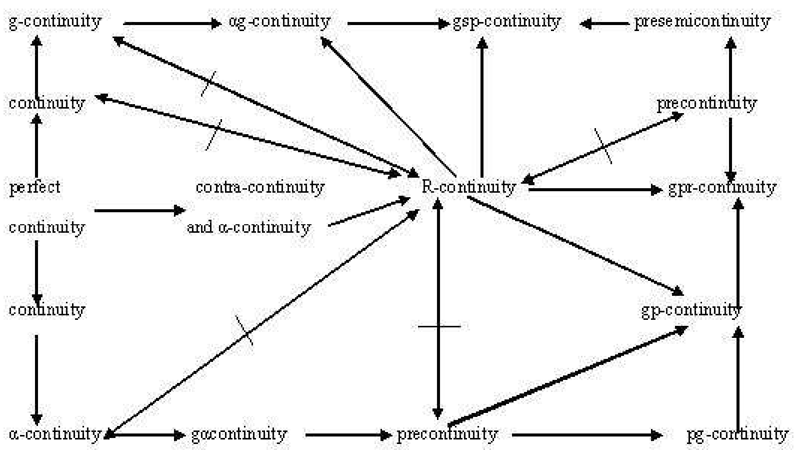

Figure 1: Functional relationships.

4. Characterizations of R-continuous Functions.

Now we shall obtain characterizations of R-continuous functions in the sense of definition 1.

Theorem 1. *A function $f:(X,\tau)\to(Y,\sigma)$ is R-continuous if and only if $f^{-1}(U)$ is R-open in (X,τ) for every open set U in (Y,σ).*

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be R-continuous and U be an open set in (Y,σ). Hence U^c is closed in (Y,σ). Since f is R-continuous $f^{-1}(U^c)$ is R-closed in (X,τ). Therefore $[f^{-1}(U^c)]^c = f^{-1}(U)$ is R-open in (X,τ). Converse is similar.

Remark 1. *The composition of two R-continuous functions need not be R-continuous and this can be shown by the following example. By [remark 6.3, 9], composition of two R-continuous functions need not be R-continuous.*

Definition 7. *A space (X,τ) is said to be T_R-space if every R-closed set is closed.*

Theorem 2. *If (X,τ) and (Z,ζ) be topological spaces and (Y,σ) be T_R-space then the composition $g\circ f:(X,\tau)\to(Z,\zeta)$ of R-continuous functions $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\zeta)$ is R-continuous.*

Proof. Let G be any closed set of (Z,ζ). Then by assumption $g^{-1}(G)$ is closed in (Y,σ). Hence $f^{-1}(g^{-1}(G)) = (g\circ f)^{-1}(G)$ is R-closed in (X,τ). Thus $g\circ f$ is R-continuous.

Theorem 3. *Let (X,τ) and (Z,ζ) be any topological spaces and (Y,σ) be $T_{1/2}$ space (respectively $T\omega$ space). Then the composition $g\circ f:(X,\tau)\to(Z,\zeta)$ of R-continuous $f:(X,\tau)\to(Y,\sigma)$ and g-continuous function $g:(Y,\sigma)\to(Z,\zeta)$ (respectivelyω-continuous) is R-continuous.*

Proof. Let G be any closed set of (Z,ζ). Then $g^{-1}(G)$ is g-closed in (Y,σ) and by assumption, $g^{-1}(G)$ is closed in (Y,σ). Since f is R-continuous $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is R-closed in (X,τ).Thus $g \circ f$ is R-continuous.

Theorem 4. *If $f : (X,\tau) \to (Y,\sigma)$ is R-continuous and $g : (Y,\sigma) \to (Z,\zeta)$ is continuous. Then their composition $g \circ f : (X,\tau) \to (Z,\zeta)$ is R-continuous.*

Proof. Let G be closed in (Z,ζ). Thus $g^{-1}(G)$ is closed in (Y,σ). Since f is R-continuous $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is R-closed in (X,τ). Thus $g \circ f$ is R-continuous.

Theorem 5. *If $f : (X,\tau) \to (Y,\sigma)$ is continuous and contra α-continuous and $g : (Y,\sigma) \to (Z,\zeta)$ is contra continuous, then $g \circ f : (X,\tau) \to (Z,\zeta)$ is R-continuous.*

Proof. Let G be any closed set of (Z,ζ). Since g is contra continuous $g^{-1}(G)$ is open in (Y,σ). Since f is continuous and contra α-continuous, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is open and α-closed in (X,τ). Then by [theorem 3.2, 9] we get the proof.

Theorem 6. *If $f : (X,\tau) \to (Y,\sigma)$ is R-irresolute and $g : (Y,\sigma) \to (Z,\zeta)$ is R-continuous then $g \circ f : (X,\tau) \to (Z,\zeta)$ is R-continuous.*

Proof. Let G be any closed set of (Z,ζ). Since g is R-continuous $g^{-1}(G)$ is R-closed in (Y,σ) . Since f is R-irresolute $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is R-closed in (X,τ). Thus $g \circ f$ is R-continuous.

Definition 8. *Let x be a point of (X,τ) and V be subset of X. Then V is called a R-neighbourhood of x in (X,τ) if there exist a R-open set U of (X,τ) such that $x \in U \subseteq V$.*

Theorem 7. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function. Then the following statements are equivalent.*

(i) *The function f is R continuous.*

(ii) *The inverse of each open set in (Y,σ) is R-open in (X,τ).*

(iii) *The inverse of each closed set in (Y,σ) is R-closed in (X,τ).*

(iv) *For each x in (X,τ) the inverse of every neighbourhood of $f(x)$ is a R-neighbourhood of x.*

(v) *For each x in (X,τ) and each neighbourhood N of $f(x)$ there is a R-neighbourhood W of x such that $f(W) \subseteq N$.*

(vi) *For each subset A of (X,τ), $f(Rcl(A)) \subseteq cl(f(A))$.*

(vii) *For each subset B of (Y,σ), $Rcl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.*

Proof. i⟺ii

This follows from theorem 1.

ii⟺iii. This proof is clear from the result $f^{-1}(A^c) = (f^{-1}(A))^c$.

ii⟺iv. Assume ii. For $x \in (X,\tau)$, let N be a neighbourhood of $f(x)$. Then there exist an open set V in (Y,σ) such that $f(x) \in V \subseteq N$. Consequently $f^{-1}(V)$ is R-open set in (X,τ) and $x \in f^{-1}(V) \subseteq f^{-1}(N)$. Thus $f^{-1}(N)$ is an R-neighbourhood of x.

Assume iv. Let U be open in (Y,σ). Let x $\in U$. Then by assumption $f^{-1}(U)$ is a R-neighbourhood of X. Thus $f^{-1}(U)$ is open in (X,τ).

iv⟺v. Let $x \in (X,\tau)$. Let N be a neighbourhood of $f(x)$. $\Leftrightarrow W = f^{-1}(N)$ is a R-neighbourhood of x and $f(W) = f(f^{-1}(N)) \subseteq N$.

vi⟺iii. Suppose iii holds. Let A be a subset of (X,τ). Since $A \subseteq f^{-1}(f(A)), A \subseteq f^{-1}(cl(f(A)))$. But $f^{-1}(cl(f(A)))$ is a closed set, by assumption $f^{-1}(cl(f(A)))$ is a R-closed set containing A.

Consequently, $R(cl(A)) \subseteq f^{-1}(cl(f(A)))$. Thus $f(R(cl(A)) \subseteq cl(f(A))$. Conversely iv holds. Let F be a closed subset of (Y,σ). Hence $f(R(cl(f^{-1}(F)))) \subseteq cl(f(f^{-1}(F))) \subseteq cl(F) = F$. Hence $R(cl(f^{-1}(F))) \subseteq f^{-1}(F)$. Thus $f^{-1}(F)$ is a R-closed set in (X,τ).

vi⟺vii. Suppose vi holds. Let B be any subset of (Y,σ). Replacing A by $f^{-1}(B)$ in vi we get, $f(R - cl(f^{-1}(B)) \subseteq cl(f(f^{-1}(B)) \subseteq cl(B)$. Thus $R - cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$. Suppose vii holds. Let $B = f(A)$ where A is a subset of (X,τ).

Then $R - cl(A) \subseteq R - cl(f^{-1}(B)) \subseteq f^{-1}(cl(f(A))$. Thus $f(R - cl(A) \subseteq clf(A)$.

This completes the proof of the theorem.

Proposition 9. *If A is any R-closed set in (X,τ) and if $f : (X,\tau) \to (Y,\sigma)$ is ω-irresolute, open and α-closed then $f(A)$ is R-closed in (Y,σ).*

Proof. Let U be any ω-open in (Y,σ) such that $f(A) \subseteq U$. Then $A \subseteq f^{-1}(U)$. Since f is ω-irresolute and A is R-closed in (X,τ), $\alpha - cl(A) \subseteq int(f^{-1}(U))$. Since f is open, $f(int(f^{-1}(U))) \subseteq int(U)$. Thus $f(\alpha cl(A)) \subseteq int(U)$. Hence $\alpha cl(f(A)) \subseteq \alpha cl(f(\alpha cl(A))) = f(\alpha cl(A) \subseteq int(U)$. Thus $f(A)$ is R-closed in (Y,σ).

Theorem 8. *If A is R-closed(respectively R-open) subset of (Y,σ), $f : (X,\tau) \to (Y,\sigma)$ is a bijection,continuous and ω-open mappings,then $f^{-1}(A)$ is R-closed (respectively R-open) in (X,τ).*

Proof. Let U be ω-open in (X,τ) such that $f^{-1}(A) \subseteq U$. Then $A \subseteq f(U)$. Since A is R-closed in (Y,σ), $\alpha cl(A \subseteq) int(f(U))$. Since f is a bijection and continuous, $f^{-1}(\alpha cl(A)) \subseteq f^{-1}(int(f(U))) \subseteq int(f^{-1}(f(U)) = int(U)$. Now $\alpha cl(f^{-1}(A)) \subseteq \alpha cl(f^{-1}(\alpha cl(A))) = f^{-1}(\alpha cl(A)) \subseteq int(U)$. Thus $f^{-1}(A)$ is R-closed in (X,τ). By taking complements we can show that if A is R-open in $(Y,\sigma), f^{-1}(A)$ is R-open in (X,τ).

Definition 9. *The intersection of all R-closed sets each containing a set A in a topological space X is called the R-closure of A and is denoted by $R - cl(A)$.*

Theorem 9. *Let A be a subset of (X,τ). Then $x \epsilon R - cl(A)$ if and only if for any R-neighbourhood N_x of x in (X,τ), $A \bigcap N_x = \phi$.*

Proof. Assume $x \in R - cl(A)$. Suppose that there exists a neighbourhood N_x of x such that $A \bigcap N_x \neq \emptyset$. Since N_x is a R-neighbourhood of x in (X, τ), there exist a R-open set V_x such that $x \in V_x \subseteq N_x$. Hence $A \bigcap V_x = \emptyset$. Thus $A \subseteq V_x^c$. Since V_x^c is a R-closed set containing A, we get $R - cl(A) \subseteq V_x^c$. $\Rightarrow x \notin R - cl(A)$. Which is a contradiction. Assume that for each R-neighbourhood N_x of x in (X, τ), $A \bigcap N_x = \emptyset$. Suppose $x \in R - cl(A)$ then there exist a R-closed set V of (X, τ) such that $A \subseteq V$ and $x \in V$. Thus $x \in V^c$ and V^c is R-open in (X, τ). But $A \in V^c = \emptyset$. Which is a contradiction.

Theorem 10. *(i) If $f : (X, \tau) \to (Y, \sigma)$ is α-continuous and contra continuous then f is R-continuous.*

(ii) If $f : (X, \tau) \to (Y, \sigma)$ is αg-continuous and contra continuous then f is R-continuous.

Proof. Let V be closed in (Y, σ). Since f is α-continuous and contra continuous $f^{-1}(V)$ is α-closed and open in (X, τ). By [theorem 3.2, 9] $f^{-1}(V)$ is R-closed in (X, τ). Let V be closed in (Y, σ). Since f is αg-continuous and contra continuous $f^{-1}(V)$ is αg closed and open in (X, τ). By [theorem 2.5.28, 8] $f^{-1}(V)$ is α-closed and open in (X, τ). Hence $f^{-1}(V)$ is R-closed.

Theorem 11. *If $f : (X, \tau) \to (Y, \sigma)$ is R-irresolute and $g : (Y, \sigma) \to (Z, \zeta)$ is R-irresolute then $g \circ f : (X, \tau) \to (Z, \zeta)$ is R-irresolute.*

Proof. Let G be R-closed in (Z, ζ). Since g is R-irresolute, $g^{-1}(G)$ is R-closed in (Y, σ). Since f is R-irresolute $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is R-closed in (X, τ). Thus $g \circ f$ is R-irresolute.

5. R-Compact and R-Connected Spaces

Definition 10. *A topological space (X, τ) is R-compact if every R-open cover of X has a finite subcover.*

Definition 11. *A topological space (X, τ) is R-connected if X cannot be written as the disjoint union of two nonempty R-open sets.*

Theorem 12. *Let $f : (X, \tau) \to (Y, \sigma)$ is surjective, R-continuous functions. If X is compact then Y is compact.*

Proof. Let $\{(A_i)/i \subsetneq I\}$ be an open cover of Y. Then $\{f^{-1}(A_i)/i \epsilon I\}$ is a R-open cover of X. Since X is R-compact, it has a finite subcover say $\{f^{-1}(A_1), f^{-1}(A_2), f^{-1}(A_3), \ldots \{f^{-1}(A_n)\}$. Since f is surjective, $\{A_1, A_2, \ldots A_n\}$ is a finite subcover of Y and Y is compact.

Theorem 13. *Let $f : (X, \tau) \to (Y, \sigma)$ is surjective, R-continuous (R-irresolute) functions. If X is R-connected then Y is connected (R-connected).*

Proof. Suppose Y is not connected (R-connected).Then $Y = AUB$ where $AnB \neq \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and A, B are open(R-open)sets in Y. Since f is surjective, $f(X) = Y$ and since f is R-continuous(R-irresolute), $X = f^{-1}(A)\ Uf^{-1}(B)$ is disjoint non empty R-open sets of X. Thus a contradiction that X is R-connected.

Definition 12. *A subset of a space X is called R-compact relative to X if every collection $\{U_i/i \subset I\}$ of R-open subsets of X such that $A \subset \bigcup_{i\in i} U_i$ there exist a finite subset I_0 of I such that $A \subset \bigcup_{i\in i_0} U_i$.*

Theorem 14. *Every R-closed subset of a R-compact space X is R-compact relative to X.*

Proof. Let A be a R-closed subset of a R-compact space X. Let $\{U_i/i \in I\}$ be a cover of A by R-open subsets of X. Hence $A \subseteq \bigcup_{i\in i} U_i$ and then $(X\backslash A)\bigcup(\bigcup_{i\in i} U_i) = X$. Since X is R-compact,there exist a finite subset I_0 of I such that $(X\backslash A)\bigcup(\bigcup_{i\in i_0} U_i) = X$. Thus $A \subseteq \bigcup_{i\in i_0} U_i$. Hence A is R-compact relative to X.

Proposition 10. *An R-closed subset of αGO-compact space is αGO-compact relative to (X, τ).*

Proof. By [theorem 3.3, 9] every R-closed set is αg-closed and since a g-closed subset of a αGO-compact space is αGO-compact relative to (X, τ) [4], the result follows.

Proposition 11. *If a map $f : (X, \tau) \to (Y, \sigma)$ is R-irresolute and a subset B is R-compact relative to (X, τ), then the image $f(B)$ is R-compact relative to (Y, σ)*

Proof. Let $\{A_i/i \in I\}$ be any collection of R-open sets of (Y, σ) such that $f(B) \subseteqq \bigcup_{i\in i} A_i$. Then $B \subseteq \bigcup_{i\in i} f^{-1}(A_i)$. By hypothesis,there exists a finite subset I_0 of I such that $B \subseteq \bigcup_{i\in i_0} f^{-1}(A_i)$ and so $f(B)$ is R-compact relative to (Y, σ).

Proposition 12. *If (X, τ) is a T_R-space and connected then (X, τ) is R-connected.*

Proof. If (X, τ) is not R-connected,then $X = A\bigcup B$ where A and B are disjoint non empty R-open sets. Since (X, τ) is a T_R-space, we get a contradiction to the connectedness of (X, τ).

Proposition 13. *If $f : (X, \tau) \to (Y, \sigma)$ is an R-continuous surjection and (X, τ) is R-connected,then (Y, σ) is connected.*

Proof. Suppose that $Y = AUB$, where A and B are disjoint nonempty open sets of (Y, σ). Since f is R-continuous and onto, $X = f^{-1}(A)Uf^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nonempty R-open sets in (X, τ). This contradicts the fact that (X, τ) is R-connected and so (Y, σ) is connected.

Theorem 15. *For a topological space (X, τ), the following are equivalent:*

(i) (X, τ) *is R-connected.*

(ii) The only subsets of (X, τ) which are both R-open and R-closed are the empty set $\emptyset$ and X.

(iii) *Each R-continuous map of* (X,τ) *into a discrete space* (Y,σ) *with at least two points is a constant map.*

Proof. i ⇒ii. Let U be an R-open and R-closed subsets of (X,τ). Then U^c is both R-open and R-closed in (X,τ). Hence (X,τ) is the disjoint union of R-open sets U and U^c, by assumption one of these must be empty. Thus $U=\emptyset$ or $U=X$.

ii⇒i. Suppose $X=AUB$ where A and B are disjoint nonempty R-open subsets of (X,τ). Then A is both R-open and R-closed subsets of (X,τ). Hence by assumption $A=\emptyset$ or X. Thus (X,τ) is R-connected.

ii⇒iii. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be an R-continuous map. Then (X,τ) is covered by R-open and R-closed covering $\{f^{-1}(y)/y\in Y\}$. By assumption $f^{-1}(y)=\emptyset$ or X for each $y\in Y$. If $f^{-1}(y)=\emptyset$ for each $y\in Y$,then f fails to be a map which shows that f is a constant map.

iii⇒ii. Let U be both R-open and R-closed in (X,τ). Suppose that $U\neq\emptyset$. Define $f:(X,\tau)\rightarrow(Y,\sigma)$ by $f(U)=\{y_1\}$ and $f(U^c)=\{y_2\}$ for some distinct points y_1 and y_2 in (Y,σ) then f is an R-continuous map.By assumption f is a constant map. Thus $y_1=y_2$ and $U=X$.

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