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Certain Generalized Class of Meromorphic Functions of Complex Order Associated with a Differential Operator

Shu-hai Li [1,*], Huo Tang [2],Burenmandula[1]

[1] *School of Mathematics and Statistics, Chifeng University, Chifeng 024000, Inner Mongolia, China*
[2] *School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China*

Abstract. In this paper, we introduce and investigate a new class of meromorphic functions defined by a differential operator. For this class, we obtain coefficient inequality, distortion inequality, radius of close-to-convexity, starlikeness and convexity and extreme points. Further partial sums are considered.

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1. Introduction

Let $\sum^c$ denote the class of functions $f(z)$ of the form

$$f(z) = \frac{a_0}{z-c} + \sum_{n=1}^{\infty} a_n z^n \ \ (a_0 > 0, 0 \leq c < 1) \tag{1}$$

which are analytic in $U^* = U \setminus \{c\} = \{z : |z| < 1\} \setminus \{c\}$. We denote by $\overline{S_c^*}(\mu)$, $\overline{K_c}(\mu)$ and $\overline{C_c}(\mu)$ the subclasses $\sum^c$ consisting of all functions which are, respectively, starlike of order μ, convex of order μ and close-to-convex of order μ in U^* $(0 \leq \mu < 1)$, that is,

$$\overline{S_c^*}(\mu) = \left\{ f \in \Sigma^c : -Re\left\{ \frac{(z-c)f'(z)}{f(z)} \right\} > \mu \ (0 \leq \mu < 1; z \in U^*) \right\} \tag{2}$$

$$\overline{K_c}(\mu) = \left\{ f \in \Sigma^c : -Re\left\{ \frac{(z-c)f''(z)}{f'(z)} + 1 \right\} > \mu \ (0 \leq \mu < 1; z \in U^*) \right\} \tag{3}$$

*Corresponding author.

Email addresses: `lishms66@sina.com` (S-H. Li), `thth2009@tom.com` (H. Tang),`brmdllyc@163.com` (Buremadula)

and

$$\overline{C_c}(\mu)=\Big\{f\in\Sigma^c: if\ there\ exists\ a\ function\ g(z)\in\overline{S_c^*}(0) \text{ such that } -Re\left\{\frac{(z-c)f'(z)}{g(z)}\right\}>\mu\ (0\le\mu<1; z\in U^*)\Big\}. \tag{4}$$

Let $\overline{R_c}(\mu)$ be the subclass of $\sum^c$ consisting of functions $f(z)$ which satisfy the inequality:

$$\left|\frac{(z-c)^2}{a_0}f'(z)+1\right|<1-\mu\ \ (0\le\mu<1) \tag{5}$$

for some μ $(0\le\mu<1)$.

It follows readily from the definitions (2) and (3) that

$$f(z)\in\overline{K_c}(\mu)\Longleftrightarrow(z-c)f'(z)\in\overline{S_c^*}(\mu)\ \ (0\le\mu<1). \tag{6}$$

It follows readily from the definitions (4) and (5) that

$$f(z)\in\overline{R_c}(\mu)\Rightarrow f(z)\in\overline{C_c}(\mu)\ \ (0\le\mu<1). \tag{7}$$

Also, by C_η^ε $(\eta\in R,\ \varepsilon\in\{0,1\})$, we denote the class of functions $f(z)\in\sum^c$ of the form (1) for which [see 10]

$$\arg(a_n)=\varepsilon\pi-(n+1)\eta\ \ (n\in N=\{1,2,\ldots\}). \tag{8}$$

For $\eta=0$, we obtain the classes C_0^0 and C_0^1 of functions with positive coefficients and negative coefficients, respectively.

Motivated by Silverman [see 26], we define the class

$$C^\varepsilon=\bigcup_{\eta\in R}C_\eta^\varepsilon.$$

For two functions $f(z)$ and $g(z)$, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write $f(z)\prec g(z)$, if there exists a Schwarz function ω, which is analytic in U with $\omega(0)=0$ and $|\omega(z)|<1$ for all $z\in U$, such that $f(z)=g(\omega(z))$ $(z\in U)$.

Various subclasses of $\sum^c$ when $c=0$ were introduced and studied by many authors [see 23, 24, 20, 5, 21, 8, 1, 6, 2, 29, 28]. In recent years, some subclasses of meromorphic functions associated with several families of integral operators and derivative operators were introduced and investigated [see, for example 7, 9, 30, 18, 4]). In [14], Frasin and Darus introduced a differential operator defined by

$$I^0f(z)=f(z),\ I^1f(z)=zf'(z)+\frac{a_0(2z-c)}{(z-c)^2},\ I^2f(z)=z(I^1f(z))'+\frac{a_0(2z-c)}{(z-c)^2},$$

and for $k=1,2,\ldots$, we can write as

$$I^kf(z)=z(I^{k-1}f(z))'+\frac{a_0(2z-c)}{(z-c)^2},$$

where $k \in N_0 = N \cup \{0\},\ z \in U^*$.

If f is given by (1), then from the definition of the operator I^k, it is easy to see that

$$I^k f(z) = \frac{a_0}{z-c} + \sum_{n=1}^{\infty} n^k a_n z^n \quad (z \in U^*). \tag{9}$$

By using the operator I^k, some authors have established many subclasses of meromorphic functions, for example [15, 16, 11, 17]. But second positive coefficient of some subclasses was fixed in these papers. In this paper, we avoid the situation effectively. With the help of the differential operator and making use of the method in [19], we define the following new class of analytic functions and obtain some interesting results.

Let $K_{i,j}(\delta,\gamma,A,B)$ denote the subclass of $\sum^c$ consisting of function $f(z)$ which satisfies the following condition:

$$1 + \frac{1}{\delta}\left\{\frac{I^i f(z)}{I^j f(z)} + \gamma\left|\frac{I^i f(z)}{I^j f(z)} - 1\right| - 1\right\} \prec \frac{1+Az}{1+Bz} \quad (z \in U^*), \tag{10}$$

where $\delta,\gamma \in C, \delta \neq 0$, real number $A, B, |A| \leq 1, |B| \leq 1, A \neq B, i \in N, j \in N_0$.

Obviously, from (10), we have

$$f(z) \in K_{i,j}(\delta,\gamma,A,B) \Leftrightarrow \frac{I^i f(z)}{I^j f(z)} + \gamma\left|\frac{I^i f(z)}{I^j f(z)} - 1\right| \prec \frac{1+(B+\delta(A-B))z}{1+Bz}. \tag{11}$$

Moreover, let us define

$$CK_\eta^\varepsilon(i,j;\delta,\gamma,A,B) = C_\eta^\varepsilon \cap K_{i,j}(\delta,\gamma,A,B). \tag{12}$$

The object of the present paper is to investigate coefficient estimates, distortion properties, radius of close-to-convexity, starlikeness and convexity, extreme points and partial sums for the classes of meromorphic functions with varying argument of coefficients.

2. Coefficient Inequality

Theorem 1. *A function $f \in \sum^c$ is in the class $CK_\eta^\varepsilon(i,j;\delta,\gamma,A,B)$ if and only if*

$$\sum_{n=1}^{\infty} \phi_n |a_n| \leq |(A-B)\delta| a_0, \tag{13}$$

where

$$\phi_n = \phi_n(i,j,c,\delta,\gamma,A,B) = [(1+(1+|B|)|\gamma|)|n^i - n^j| + |Bn^i - (B+\delta(A-B))n^j|](1+c), \tag{14}$$

$$(\delta,\gamma \in C, \delta \neq 0, \textit{ real number } A,B, |A| \leq 1, |B| \leq 1, A \neq B, i \in N, j \in N_0, z \in U^*).$$

Proof. Suppose that (13) is true for $\delta, \gamma \in C$, $\delta \neq 0$, real number $A, B, |A| \leq 1$, $|B| \leq 1$, $A \neq B$. For $f \in \sum^c$, let us define the function $p(z)$ by

$$p(z) = \frac{I^i f(z)}{I^j f(z)} + \gamma \left| \frac{I^i f(z)}{I^j f(z)} - 1 \right|.$$

It suffices to show that

$$\left| \frac{p(z) - 1}{(B + \delta(A - B)) - Bp(z)} \right| < 1 \quad (z \in U^*). \tag{15}$$

We note that

$$\left| \frac{p(z) - 1}{(B + \delta(A - B)) - Bp(z)} \right| = \left| \frac{I^i f(z) + \gamma e^{i\theta} |I^i f(z) - I^j f(z)| - I^j f(z)}{(B + \delta(A - B)) I^j f(z) - B(I^i f(z) + \gamma e^{i\theta} |I^i f(z) - I^j f(z)|)} \right|$$

$$= \left| \frac{\sum_{n=1}^{\infty} (n^i - n^j) a_n z^n + \gamma e^{i\theta} |\sum_{n=1}^{\infty} (n^i - n^j) a_n z^n|}{(A - B)\delta \frac{a_0}{z - c} - \sum_{n=1}^{\infty} (Bn^i - (B + \delta(A - B) n^j) a_n z^n + B\gamma e^{i\theta} |\sum_{n=1}^{\infty} (n^i - n^j) a_n z^n|} \right|$$

$$= \left| \frac{\sum_{n=1}^{\infty} (n^i - n^j) a_n (z^{n+1} - cz^n) - \gamma e^{i\theta} e^{i\varphi} |\sum_{n=1}^{\infty} (n^i - n^j) a_n (z^{n+1} - cz^n)|}{(A - B)\delta a_0 - \sum_{n=1}^{\infty} (Bn^i - (B + \delta(A - B) n^j) a_n (z^{n+1} - cz^n) + B\gamma e^{i\theta} e^{i\varphi} |\sum_{n=1}^{\infty} (n^i - n^j) a_n (z^{n+1} - cz^n)|} \right|$$

$$\leq \frac{\sum_{n=1}^{\infty} |n^i - n^j| |a_n| (|z|^{n+1} + c|z|^n) + |\gamma| \sum_{n=1}^{\infty} |n^i - n^j| |a_n| (|z|^{n+1} + c|z|^n)}{|(A - B)\delta| a_0 - \sum_{n=1}^{\infty} |Bn^i - (B + \delta(A - B) n^j| |a_n| (|z|^{n+1} + c|z|^n) - |B| |\gamma| \sum_{n=1}^{\infty} |n^i - n^j| |a_n| (|z|^{n+1} + c|z|^n)}$$

$$\leq \frac{\sum_{n=1}^{\infty} |n^i - n^j| |a_n| (1 + c) + |\gamma| \sum_{n=1}^{\infty} |n^i - n^j| |a_n| (1 + c)}{|(A - B)\delta| a_0 - \sum_{n=1}^{\infty} |Bn^i - (B + \delta(A - B) n^j| |a_n| (1 + c) - |B| |\gamma| \sum_{n=1}^{\infty} |n^i - n^j| |a_n| (1 + c)}.$$

The last expression is bounded above by 1, if

$$\sum_{n=1}^{\infty} |n^i - n^j| |a_n| (1 + c) + |\gamma| \sum_{n=1}^{\infty} |n^i - n^j| |a_n| (1 + c)$$
$$\leq |(A - B)\delta| a_0 - \sum_{n=1}^{\infty} |Bn^i - (B + \delta(A - B) n^j| |a_n| (1 + c) - |B| |\gamma| \sum_{n=1}^{\infty} |n^i - n^j| |a_n| (1 + c)$$

which is equivalent to our condition (13).

Conversely, we need only show that each function $f(z)$ from the class $CK_\eta^\varepsilon(i, j; \delta, \gamma, A, B)$ satisfies the coefficient inequality (13). Let $f(z) \in CK_\eta^\varepsilon(i, j; \delta, \gamma, A, B)$, then by (15) and (1), we obtain

$$\left| \frac{\sum_{n=1}^{\infty} (n^i - n^j) a_n (z^{n+1} - cz^n) - \gamma e^{i\theta} e^{i\varphi} |\sum_{n=1}^{\infty} (n^i - n^j) a_n (z^{n+1} - cz^n)|}{(A - B)\delta a_0 - \sum_{n=1}^{\infty} (Bn^i - (B + \delta(A - B) n^j) a_n (z^{n+1} - cz^n) + B\gamma e^{i\theta} e^{i\varphi} |\sum_{n=1}^{\infty} (n^i - n^j) a_n (z^{n+1} - cz^n)|} \right| < 1$$

for $(z \in U^*)$. Therefore putting $z = re^{i\eta}$ $(0 \leq r < 1)$, and applying (8), we have

$$\frac{\sum_{n=1}^{\infty} |n^i - n^j| |a_n| (r^{n+1} + cr^n) + |\gamma| \sum_{n=1}^{\infty} |n^i - n^j| |a_n| (r^{n+1} + cr|^n)}{|(A - B)\delta| a_0 - \sum_{n=1}^{\infty} |Bn^i - (B + \delta(A - B) n^j| |a_n| (r^{n+1} + cr^n) - |B| |\gamma| \sum_{n=1}^{\infty} |n^i - n^j| |a_n| (r^{n+1} + cr^n)} < 1. \tag{16}$$

It is clear that the denominator of the left hand side cannot vanish for $0 \le r < 1$. Moreover, it is positive for $r = 0$, and in consequence for $0 \le r < 1$. Thus, by (16), we have

$$\sum_{n=1}^{\infty} \phi_n |a_n| r^{n+1} < |(A-B)\delta| a_0 \tag{17}$$

which, upon letting $r \to 1^-$, readily yields the assertion (13).

From Theorem 1, we obtain coefficient estimates for the class $CK_\eta^\varepsilon(i,j;\delta,\gamma,A,B)$.

Corollary 1. *If a function $f(z)$ of the form (1) belongs to the class $CK_\eta^\varepsilon(i,j;\delta,\gamma,A,B)$, then*

$$|a_n| < \frac{|(A-B)\delta| a_0}{\phi_n} \quad (n \in N), \tag{18}$$

where ϕ_n is defined by (14). The result is sharp. The functions of the form

$$f_{n,\eta}(z) = \frac{a_0}{z-c} + \frac{|(A-B)\delta| a_0}{e^{i\{(n+1)\eta - \varepsilon\pi\}}\phi_n} z^n \quad (z \in U^*;\ n \in N) \tag{19}$$

are the extremal functions.

Also, from Theorem 1, we have the following result.

Corollary 2. *If $f(z) \in CK_\eta^\varepsilon(i,j;\delta,\gamma,A,B)$, and*

$$\phi_n \ge \phi_1 \tag{20}$$

then we have

$$\sum_{n=1}^{\infty} |a_n| \le \frac{a_0}{1+c}.$$

Moreover, if

$$\phi_n \ge n\phi_1 \tag{21}$$

then we have

$$\sum_{n=1}^{\infty} n|a_n| \le \frac{a_0}{1+c}.$$

3. Distortion Theorem

Theorem 2. *If $f(z)$ defined by (1) is in the class $K_\eta^\varepsilon(i,j;\delta,\gamma,A,B)$ for $|z| = r < 1$, and ϕ_n defined by (14) satisfies (20), then we have*

$$\frac{a_0}{|z-c|} - \frac{a_0}{1+c} r \le |f(z)| \le \frac{a_0}{|z-c|} + \frac{a_0}{1+c} r. \tag{22}$$

Moreover, if (21) holds, then

$$\frac{a_0}{|z-c|^2} - \frac{a_0}{1+c} \le |f'(z)| \le \frac{a_0}{|z-c|^2} + \frac{a_0}{1+c}. \tag{23}$$

Proof. Let $f(z)$ is given by (1). For $|z| = r < 1$, we have

$$|f(z)| \leq \frac{a_0}{|z-c|} + \sum_{n=1}^{\infty} |a_n||z|^n \leq \frac{a_0}{|z-c|} + |z| \sum_{n=1}^{\infty} |a_n| = \frac{a_0}{|z-c|} + r \sum_{n=1}^{\infty} |a_n|,$$

and

$$|f(z)| \geq \frac{a_0}{|z-c|} - \sum_{n=1}^{\infty} |a_n||z|^n \geq \frac{a_0}{|z-c|} - |z| \sum_{n=1}^{\infty} |a_n| = \frac{a_0}{|z-c|} - r \sum_{n=1}^{\infty} |a_n|.$$

Then by Corollary 2, we get (22). Analogously we can prove (23). This completes the proof of our theorem.

4. Radius of Close-to-Convexity, Starlikeness and Convexity

We concentrate upon getting the radius of close-to-convexity, starlikeness and convexity.

Theorem 3. *Let the function $f(z)$ given by (1) be in the class $K_{\eta}^{\varepsilon}(i, j; \delta, \gamma, A, B)$. Then $f(z)$ is close-to-convex of order μ $(0 \leq \mu < 1)$ in $|z-c| < |z| < r_1$, where*

$$r_1 = \inf_n \left\{ \frac{(1-\mu)\phi_n}{n|(A-B)\delta|} \right\}^{\frac{1}{n+1}} \ (n \geq 1),$$

and ϕ_n is given by (14).

Proof. We must show that

$$\left| \frac{(z-c)^2}{a_0} f'(z) + 1 \right| \leq 1 - \mu \ \ (0 \leq \mu < 1) \tag{24}$$

for $|z| < r_1$. Note that

$$\left| \frac{(z-c)^2}{a_0} f'(z) + 1 \right| = \left| \sum_{n=1}^{\infty} \frac{na_n}{a_0} z^{n-1}(z-c)^2 \right| \leq \sum_{n=1}^{\infty} \frac{|na_n|}{a_0} |z|^{n-1} |z-c|^2.$$

Thus for $|z-c| < |z| < r_1$, (24) holds true if

$$\sum_{n=1}^{\infty} \frac{nr^{n+1}}{a_0(1-\mu)} |a_n| \leq 1. \tag{25}$$

By Theorem 1, we have

$$\sum_{n=1}^{\infty} \frac{\phi_n}{|(A-B)\delta| a_0} |a_n| \leq 1. \tag{26}$$

Hence (25) will be true, if

$$\frac{nr^{n+1}}{a_0(1-\mu)} \leq \frac{\phi_n}{|(A-B)\delta| a_0},$$

or equivalently, if

$$r \leq \left\{ \frac{(1-\mu)\phi_n}{n|(A-B)\delta|} \right\}^{\frac{1}{n+1}} \quad (n \geq 1). \tag{27}$$

The theorem follows from (27).

Theorem 4. *Let the function $f(z)$ given by (1) be in the class $K_\eta^\varepsilon(i,j;\delta,\gamma,A,B)$. Then $f(z)$ is starlike of order μ $(0 \leq \mu < 1)$ in $|z-c| < |z| < r_2$, where*

$$r_2 = \inf_n \left\{ \frac{(1-\mu)\phi_n}{(n+2-\mu)|(A-B)\delta|} \right\}^{\frac{1}{n+1}} \quad (n \geq 1),$$

and ϕ_n is given by (14).

Proof. We must show that

$$\left| \frac{(z-c)f'(z)}{f(z)} + 1 \right| \leq 1-\mu \quad (0 \leq \mu < 1) \tag{28}$$

for $|z| < r_2$. Note that

$$\begin{aligned} \left| \frac{(z-c)f'(z)}{f(z)} + 1 \right| &= \left| \frac{(z-c)\left(-\frac{a_0}{(z-c)^2} + \sum_{n=1}^{\infty} n a_n z^{n-1}\right) + \frac{a_0}{z-c} + \sum_{n=1}^{\infty} a_n z^n}{\frac{a_0}{z-c} + \sum_{n=1}^{\infty} a_n z^n} \right|, \\ &= \left| \frac{\sum_{n=1}^{\infty} n a_n z^{n-1}(z-c)^2 + \sum_{n=1}^{\infty} a_n z^n (z-c)}{a_0 + \sum_{n=1}^{\infty} a_n z^n (z-c)} \right|, \\ &\leq \frac{\sum_{n=1}^{\infty} n|a_n||z|^{n-1}|z-c|^2 + \sum_{n=1}^{\infty} |a_n||z|^n|z-c|}{a_0 - \sum_{n=1}^{\infty} |a_n||z|^n|z-c|}. \end{aligned}$$

Thus for $|z-c| < |z| < r_2$, (28) holds true if

$$\sum_{n=1}^{\infty}(n+1)|a_n|r^{n+1} \leq 1-\mu(a_0 - \sum_{n=1}^{\infty}|a_n|r^{n+1}),$$

or

$$\sum_{n=1}^{\infty} \frac{(n+2-\mu)r^{n+1}}{(1-\mu)a_0}|a_n| \leq 1.$$

By using (26) and (28), we have

$$\frac{(n+2-\mu)r^{n+1}}{(1-\mu)a_0} \leq \frac{\phi_n}{|(A-B)\delta|a_0},$$

or equivalently,

$$r \leq \left\{ \frac{(1-\mu)\phi_n}{(n+2-\mu)|(A-B)\delta|} \right\}^{\frac{1}{n+1}} \quad (n \geq 1). \tag{29}$$

The theorem follows from (29).

Using (2) and Theorem 4, we obtain

Theorem 5. *Let the function $f(z)$ given by (1) be in the class $K_{\eta}^{\varepsilon}(i,j;\delta,\gamma,A,B)$. Then $f(z)$ is convex of order μ $(0 \le \mu < 1)$ in $|z-c| < |z| < r_3$, where*

$$r_3 = \inf_n \left\{ \frac{(1-\mu)\phi_n}{(n^2+2n-n\mu)|(A-B)\delta|} \right\}^{\frac{1}{n+1}} \quad (n \ge 1),$$

and ϕ_n is given by (14).

5. Extreme points

Theorem 6. *Let the function $f(z)$ be defined by (1). We define*

$$f_0(z) = \frac{a_0}{z-c}, \quad f_n(z) = \frac{a_0}{z-c} + \frac{|(A-B)\delta|a_0}{\phi_n} z^n, \ (n \in N) \tag{30}$$

where ϕ_n is defined by (14), then $f(z) \in K_{\eta}^{\varepsilon}(i,j;\delta,\gamma,A,B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n > 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$.

Proof. Suppose that

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) = \frac{a_0}{z-c} + \sum_{n=0}^{\infty} \lambda_n \frac{|(A-B)\delta|a_0}{\phi_n} z^n.$$

Then

$$\sum_{n=0}^{\infty} \phi_n \left| \lambda_n \frac{|(A-B)\delta|a_0}{\phi_n} \right| = |(A-B)\delta|a_0 \sum_{n=1}^{\infty} \lambda_n = |(A-B)\delta|a_0(1-\lambda_0) < |(A-B)\delta|a_0.$$

Thus $f(z) \in K_{\eta}^{\varepsilon}(i,j;\delta,\gamma,A,B)$.

Conversely, suppose that $f(z) \in K_{\eta}^{\varepsilon}(i,j;\delta,\gamma,A,B)$. By using Corollary 1, we get

$$|a_n| < \frac{|(A-B)\delta|a_0}{\phi_n} \quad (n \in N),$$

we may set

$$\lambda_n = \frac{\phi_n}{|(A-B)\delta|a_0} |a_n| \quad (n \in N),$$

and

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n \quad (n \in N).$$

Then

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z).$$

This completes the proof of the theorem.

6. Partial Sums

Let $f \in \sum^c$ be a function of the form (1). Motivated by Silverman [25] and Silvia [27] [see also 3, 12, 13, 22], we define the partial sums $f_m(z)$ defined by

$$f_m(z) = \frac{a_0}{z-c} + \sum_{n=1}^{m} a_n z^n \quad (m \in N; z \in U^*). \tag{31}$$

In this section, we consider partial sums of functions from the class $K_\eta^\varepsilon(i,j;\delta,\gamma,A,B)$ and obtain sharp lower bounds for the real part of ratios of $f(z)$ to $f_m(z)$ and $f'(z)$ to $f_m^{'}(z)$.

Theorem 7. *Let $m \in N$ and let the sequence $\{\phi_n\}$, defined by (14), satisfy the inequalities*

$$\phi_n \geq \begin{cases} (1+c)|(A-B)\delta|, & \text{if } n = 1,2,\ldots,m, \\ \frac{1+c}{a_0}\phi_{m+1}, & \text{if } n = m+1, m+2, \ldots \end{cases}. \tag{32}$$

If a function $f(z)$ belongs to the class $K_\eta^\varepsilon(i,j;\delta,\gamma,A,B)$, then

$$Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq 1 - \frac{a_0|(A-B)\delta|}{\phi_{m+1}}, \ (z \in U^*) \tag{33}$$

$$Re\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{\phi_{m+1}}{a_0|(A-B)\delta| + \phi_{m+1}}, \ (z \in U^*) \tag{34}$$

The bounds are sharp, with the extremal function $f_{m+1,\eta}$ of the form (19).

Proof. Let a function f of the form (1) belong to the class $K_\eta^\varepsilon(i,j;\delta,\gamma,A,B)$. Define the function $\omega(z)$ by

$$\begin{aligned}\frac{\omega(z)-1}{\omega(z)+1} &= \frac{\phi_n}{a_0|(A-B)\delta|}\left[\frac{f(z)}{f_m(z)} - \left(1 - \frac{a_0|(A-B)\delta|}{\phi_{m+1}}\right)\right] \\ &= \frac{1 + \sum_{n=1}^m a_n z^n \left(\frac{z-c}{a_0}\right) + \frac{\phi_{m+1}}{a_0|(A-B)\delta|}\sum_{n=m+1}^\infty a_n z^n \left(\frac{z-c}{a_0}\right)}{1 + \sum_{n=1}^m a_n z^n \left(\frac{z-c}{a_0}\right)}.\end{aligned} \tag{35}$$

It suffices to show that $|\omega(z)| \leq 1$. Now, from (35) we can write

$$\omega(z) = \frac{\frac{\phi_{m+1}}{a_0|(A-B)\delta|}\sum_{n=m+1}^\infty a_n z^n \left(\frac{z-c}{a_0}\right)}{2 + 2\sum_{n=1}^m a_n z^n \left(\frac{z-c}{a_0}\right) + \frac{\phi_{m+1}}{a_0|(A-B)\delta|}\sum_{n=m+1}^\infty a_n z^n \left(\frac{z-c}{a_0}\right)}. \tag{36}$$

Hence we obtain

$$|\omega(z)| \leq \frac{\frac{\phi_{m+1}}{a_0|(A-B)\delta|}\sum_{n=m+1}^\infty |a_n| \left(\frac{1+c}{a_0}\right)}{2 - 2\sum_{n=1}^m |a_n| \left(\frac{1+c}{a_0}\right) - \frac{\phi_{m+1}}{a_0|(A-B)\delta|}\sum_{n=m+1}^\infty |a_n| \left(\frac{1+c}{a_0}\right)}. \tag{37}$$

Now $|\omega(z)| \leq 1$ if and only if

$$2\frac{\phi_{m+1}}{a_0|(A-B)\delta|}\sum_{n=m+1}^{\infty}|a_n|\left(\frac{1+c}{a_0}\right) \leq 2-2\sum_{n=1}^{m}|a_n|\left(\frac{1+c}{a_0}\right), \tag{38}$$

or equivalently,

$$\sum_{n=1}^{m}\left(\frac{1+c}{a_0}\right)|a_n| + \sum_{n=m+1}^{\infty}\frac{\phi_{m+1}}{a_0|(A-B)\delta|}\left(\frac{1+c}{a_0}\right)|a_n| \leq 1. \tag{39}$$

From the condition (13), it is sufficient to show that

$$\sum_{n=1}^{m}\left(\frac{1+c}{a_0}\right)|a_n| + \sum_{n=m+1}^{\infty}\frac{\phi_{m+1}}{a_0|(A-B)\delta|}\left(\frac{1+c}{a_0}\right)|a_n| \leq \sum_{n=1}^{\infty}\frac{\phi_n}{a_0|(A-B)\delta|}|a_n|,$$

which is equivalent to

$$\sum_{n=1}^{m}\left(\frac{\phi_n-(1+c)|(A-B)\delta|}{a_0|(A-B)\delta|}\right)|a_n| + \sum_{n=m+1}^{\infty}\left(\frac{\phi_n-\phi_{m+1}\left(\frac{1+c}{a_0}\right)}{a_0|(A-B)\delta|}\right)|a_n| \geq 0. \tag{40}$$

The bound in (33) is sharp for each $m \in N$. In order to see that $f = f_{m+1,\eta}$ given the result sharp, we observe that for $z = re^{i\eta}$, we have

$$\frac{f(z)}{f_m(z)} = 1-\frac{a_0|(A-B)\delta|r^{n+2}}{\phi_{m+1}} \to^{r\to 1^-} 1-\frac{a_0|(A-B)\delta|}{\phi_{m+1}}. \tag{41}$$

Similarly, we write

$$\begin{aligned}\frac{\omega(z)-1}{\omega(z)+1} &= \frac{a_0|(A-B)\delta|}{a_0|(A-B)\delta|+\phi_{m+1}}\left[\frac{f_m(z)}{f(z)} - \frac{\phi_{m+1}}{a_0|(A-B)\delta|+\phi_{m+1}}\right] \quad (z\in U^*)\\ &= \frac{1+\sum_{n=1}^{m}a_nz^n\left(\frac{z-c}{a_0}\right) - \frac{\phi_{m+1}}{a_0|(A-B)\delta|}\sum_{n=m+1}^{\infty}a_nz^n\left(\frac{z-c}{a_0}\right)}{1+\sum_{n=1}^{\infty}a_nz^n\left(\frac{z-c}{a_0}\right)},\end{aligned} \tag{42}$$

where

$$|\omega(z)| \leq \frac{\frac{\phi_{m+1}+a_0|(A-B)\delta|}{a_0|(A-B)\delta|}\sum_{n=m+1}^{\infty}|a_n|\left(\frac{1+c}{a_0}\right)}{2-2\sum_{n=1}^{m}|a_n|\left(\frac{1+c}{a_0}\right) - \frac{\phi_{m+1}+a_0|(A-B)\delta|}{a_0|(A-B)\delta|}\sum_{n=m+1}^{\infty}|a_n|\left(\frac{1+c}{a_0}\right)} \leq 1. \tag{43}$$

This last inequality is equivalent to

$$\sum_{n=1}^{m}\left(\frac{1+c}{a_0}\right)|a_n| + \sum_{n=m+1}^{\infty}\frac{\phi_{m+1}+a_0|(A-B)\delta|}{a_0|(A-B)\delta|}\left(\frac{1+c}{a_0}\right)|a_n| \leq 1. \tag{44}$$

We are making use of (13) to get (34). Finally, equality holds in (34) for the extremal function given by (19).

Theorem 8. *Let $m \in N$ and let the sequence $\{\phi_n\}$, defined by (14), satisfy the inequalities*

$$\phi_n \geq \begin{cases} n(1+c)|(A-B)\delta|, & \text{if } n = 1,2,\ldots,m, \\ \frac{n(1+c)}{a_0}\frac{\phi_{m+1}}{m+1}, & \text{if } \ n = m+1, m+2,\ldots \end{cases}. \tag{45}$$

If a function $f(z)$ belongs to the class $K_\eta^\varepsilon(i,j;\delta,\gamma,A,B)$, then

$$Re\left\{\frac{f'(z)}{f_m'(z)}\right\} \geq 1 - \frac{a_0(m+1)|(A-B)\delta|}{\phi_{m+1}},\ (z \in U^*) \tag{46}$$

$$Re\left\{\frac{f_m'(z)}{f'(z)}\right\} \geq \frac{\phi_{m+1}}{a_0(m+1)|(A-B)\delta| + \phi_{m+1}},\ (z \in U^*). \tag{47}$$

The bounds are sharp, with the extremal function $f_{m+1,\eta}$ of the form (19).

Proof. The proof is analogous to that of Theorem 8, and we omit the details.

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