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Some Subordination and Superordination for the Wright Generalize Hypergeometric Function

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Abstract. In this paper, we obtain some subordination and superordination results for the Wright generalized hypergeometric function. Sandwich-type theorem for these multivalent function is also obtained.

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1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a,n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$, with $H_0 = H[0,1]$ and $H = H[1,1]$. Let $A(p)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=1+p}^{\infty} a_n z^n \quad \left(p \in \mathbb{N} = \{1,2,3,\ldots\}; z \in U\right). \tag{1}$$

Let f and F be members of $H(U)$. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in U)$, such that $f(z) = F(\omega(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ [see 12, 13].

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp'(z); z\right) \prec h(z), \tag{2}$$

then $p(z)$ is a solution of the differential subordination (2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (2) if $p(z) \prec q(z)$ for

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all $p(z)$ satisfying (2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (2) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp'(z); z\right)$ are univalent in U and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp'(z); z\right), \tag{3}$$

then $p(z)$ is a solution of the differential superordination (3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (3) is called the best subordinant [see 12, 13].

For analytic functions $f(z) \in A(p)$, given by (1) and $\phi(z) \in A(p)$ given by $\phi(z) = z^p + \sum\limits_{n=1+p}^{\infty} b_n z^n \quad (p \in \mathbb{N})$, the Hadamard product (or convolution) of $f(z)$ and $\phi(z)$, is defined by

$$(f * \phi)(z) = z^p + \sum_{n=1+p}^{\infty} a_n b_n z^n = (\phi * f)(z). \tag{4}$$

Let $\alpha_1, A_1, ..., \alpha_q, A_q$ and $\beta_1, B_1, ..., \beta_s, B_s$ $(q, s \in \mathbb{N})$ be positive real parameters such that

$$1 + \sum_{j=1}^{s} B_j - \sum_{j=1}^{q} A_j \geq 0.$$

The Wright generalized hypergeometric function [21] [see also 20].

$${}_q\Psi_s\left[(\alpha_1, A_1), ..., (\alpha_q, A_q); (\beta_1, B_1), ..., (\beta_s, B_s); z\right] = {}_q\Psi_s\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right]$$

is defined by

$${}_q\Psi_s\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right] = \sum_{n=0}^{\infty} \frac{\prod\limits_{i=1}^{q} \Gamma(\alpha_i + nA_i)}{\prod\limits_{i=1}^{s} \Gamma(\beta_i + nB_i)} . \frac{z^n}{n_!} \quad (z \in U).$$

If $A_i = 1 (i = 1, ..., q)$ and $B_i = 1 (i = 1, ..., s)$, we have the relationship:

$$\Omega_q \Psi_s\left[(\alpha_i, 1)_{1,q}; (\beta_i, 1)_{1,s}; z\right] = {}_q F_s\left(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z\right),$$

where ${}_qF_s\left(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z\right)$ is the generalized hypergeometric function [see 20] and

$$\Omega = \frac{\prod\limits_{i=1}^{s} \Gamma(\beta_i)}{\prod\limits_{i=1}^{q} \Gamma(\alpha_i)}. \tag{5}$$

The Wright generalized hypergeometric functions were invoked in the geometric function theory [see 16, 17].

By using the generalized hypergeometric function Dziok and Srivastava [7] introduced a linear operator. In [6] Dziok and Raina and in [2] Aouf and Dziok extended this linear operator by using Wright generalized hypergeometric function.

Aouf et al. [3] considered the following linear operator

$$\theta_{p,q,s}\left[\left(\alpha_i,A_i\right)_{1,q};\left(\beta_i,B_i\right)_{1,s}\right]:A(p)\rightarrow A(p),$$

defined by the following Hadamard product:

$$\theta_{p,q,s}\left[\left(\alpha_i,A_i\right)_{1,q};\left(\beta_i,B_i\right)_{1,s}\right]f\left(z\right)={}_q\Phi_s^p\left[\left(\alpha_i,A_i\right)_{1,q};\left(\beta_i,B_i\right)_{1,s};z\right]*f\left(z\right),$$

where ${}_q\Phi_s^p\left[\left(\alpha_i,A_i\right)_{1,q};\left(\beta_i,B_i\right)_{1,s};z\right]$ is given by

$${}_q\Phi_s^p\left[\left(\alpha_i,A_i\right)_{1,q};\left(\beta_i,B_i\right)_{1,s};z\right]=\Omega z^p{}_q\Psi_s\left[\left(\alpha_i,A_i\right)_{1,q};\left(\beta_i,B_i\right)_{1,s};z\right].$$

We observe that, for a function $f\left(z\right)$ of the form (1), we have

$$\theta_{p,q,s}\left[\left(\alpha_i,A_i\right)_{1,q};\left(\beta_i,B_i\right)_{1,s}\right]f\left(z\right)=z^p+\sum_{n=1+p}^{\infty}\Omega\sigma_{n,p}\left(\alpha_1\right)a_nz^n, \tag{6}$$

where Ω is given by (5) and $\sigma_{n,p}\left(\alpha_1\right)$ is defined by

$$\sigma_{n,p}\left(\alpha_1\right)=\frac{\Gamma\left(\alpha_1+A_1\left(n-p\right)\right)...\Gamma\left(\alpha_q+A_q\left(n-p\right)\right)}{\Gamma\left(\beta_1+B_1\left(n-p\right)\right)...\Gamma\left(\beta_s+B_s\left(n-p\right)\right)\left(n-p\right)!}. \tag{7}$$

If, for convenience, we write

$$\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f\left(z\right)=\theta_{p,q,s}\left[\left(\alpha_1,A_1\right),...,\left(\alpha_q,A_q\right);\left(\beta_1,B_1\right),...,\left(\beta_s,B_s\right)\right]f\left(z\right),$$

then one can easily verify from (6) that

$$\begin{aligned} zA_1\left(\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f\left(z\right)\right)' =&\alpha_1\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]f\left(z\right),\\ &-(\alpha_1-pA_1)\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f\left(z\right)\ (A_1>0). \end{aligned} \tag{8}$$

For $p=1$, $\theta_{1,q,s}\left[\alpha_1,A_1,B_1\right]=\theta\left[\alpha_1\right]$ which was introduced by Dziok and Raina [6] and studied by Aouf and Dziok [2]. We note that, for $f\left(z\right)\in A(p)$, $A_i=1\left(i=1,2,...,q\right)$ and $B_i=1\left(i=1,2,...,s\right)$, we obtain $\theta_{p,q,s}\left[\alpha_1,1,1\right]f\left(z\right)=H_{p,q,s}[\alpha_1]f\left(z\right)$, where $H_{p,q,s}[\alpha_1]$ is the Dziok-Srivastava operator [see 7].

We note also that, for $f\left(z\right)\in A(p), q=2, s=1$ and $A_1=A_2=B_1=1$, we have:

1. $\theta_{p,2,1}\left[a,1;c\right]f\left(z\right)=L_p\left(a,c\right)f\left(z\right)\ \left(a>0,c>0,p\in\mathbb{N}\right)$ [see 18];
2. $\theta_{p,2,1}\left[\mu+p,1;1\right]f\left(z\right)=D^{\mu+p-1}f\left(z\right)\left(\mu>-p,p\in\mathbb{N}\right)$, where $D^{\mu+p-1}f\left(z\right)$ is the $\left(\mu+p-1\right)-$ the order Ruscheweyh derivative [see 8];

3. $\theta_{p,2,1}\left[\nu+p,1;\nu+p+1\right]f(z)=F_{\nu,p}(f)(z)\quad(\nu>-p,p\in\mathbb{N})$, where $F_{\nu,p}\left(f\right)(z)$ is the generalized Bernardi-Libera-Livingston-integral operator [see 5];
4. $\theta_{p,2,1}\left[c,1;a\right]f(z)=I^{a}_{c,p}f(z)\left(a\in\mathbb{R},c\in\mathbb{C}\backslash\mathbb{Z}^{-}_{0},p\in\mathbb{N}\right)$, where the operator $I^{a}_{c,p}$ was introduced and studied by AL-Kharasani and Al-Hajiry [see 1];
5. $\theta_{p,2,1}\left[p+1,1;n+p\right]f(z)=I_{n,p}f(z)\left(n\in\mathbb{Z};n>-p,p\in\mathbb{N}\right)$, where the operator $I_{n,p}$ was introduced and studied by Liu and Noor [see 9];
6. $\theta_{p,2,1}\left[\lambda+p,c;a\right]f(z)=I^{\lambda}_{p}(a,c)f(z)\quad\left(a,c\in\mathbb{R}\backslash\mathbb{Z}^{-}_{o};\lambda>-p,p\in\mathbb{N}\right)$, where $I^{\lambda}_{p}(a,c)$ is the Cho-Kwon-Srivastava operator [see 4];
7. $\theta_{p,2,1}\left[1+p,1;1+p-\mu\right]f(z)=\Omega^{(\mu,p)}_{z}f(z)\left(-\infty<\mu<1+p,p\in\mathbb{N}\right)$, where the operator $\Omega^{(\mu,p)}_{z}$ was introduced and studied by Patel and Mishra [see 14] and studied by Srivastava and Aouf [19] when $\left(0\le\mu<1\right)$.

To prove our results, we need the following definitions and lemmas.

Definition 1. *[12] Denote by $\mathcal{F}$ the set of all functions $q(z)$ that are analytic and injective on $\bar{U}\backslash E(q)$ where*

$$E(q)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}q(z)=\infty\right\},$$

and are such that $q^{'}(\zeta)\neq 0$ for $\zeta\in\partial U\backslash E(q)$. Further let the subclass of $\mathcal{F}$ for which $q(0)=a$ be denoted by $\mathcal{F}(a),\mathcal{F}(0)\equiv\mathcal{F}_0$ and $\mathcal{F}(1)\equiv\mathcal{F}$.

Definition 2. *[13] A function $L(z,t)(z\in U,t\ge 0)$ is said to be a subordination chain if $L(0,t)$ is analytic and univalent in U for all $t\ge 0$, $L(z,0)$ is continuously differentiable on $[0;1)$ for all $z\in U$ and $L(z,t_1)\prec L(z,t_2)$ for all $0\le t_1\le t_2$.*

Lemma 1. *[15] Let $L(z,t)=a_1(t)z+a_2(t)z^2+...$, with $a_1(t)\neq 0$ for all $t\ge 0$ and $\lim_{t\to\infty}\left|a_1(t)\right|=\infty$ Suppose that $L(.;t)$ is analytic in U for all $t\ge 0$, $L(z;.)$ is continuously differentiable on $[0;+\infty)$ for all $z\in U$. If $L(z;t)$ satisfies*

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\}>0\quad(z\in U,t\ge 0).$$

and

$$|L(z;t)|\le K_0\left|a_1(t)\right|,|z|<r_0<1,t\ge 0$$

for some positive constants K_0 and r_0, then $L(z;t)$ is a subordination chain.

Lemma 2. *[10] Suppose that the function $H:\mathbb{C}^2\to\mathbb{C}$ satisfies the condition*

$$Re\left\{H(is;t)\right\}\le 0$$

for all real s and for all $t\le -n\left(1+s^2\right)/2, n\in\mathbb{N}$. If the function $p(z)=1+p_nz^n+p_{n+1}z^{n+1}+...$ is analytic in U and

$$Re\left\{H\left(p(z);zp^{'}(z)\right)\right\}>0\quad(z\in U),$$

then $Re\left\{p(z)\right\}>0$ for $z\in U$.

Lemma 3. *[11] Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in H(U)$ with $h(0) = c$. If $Re\left\{\kappa h(z) + \gamma\right\} > 0\,(z \in U)$, then the solution of the following differential equation:*

$$q(z) + \frac{zq^{'}(z)}{\kappa q(z) + \gamma} = h(z) \quad \left(z \in U; q(0) = c\right)$$

is analytic in U and satisfies $Re\left\{\kappa q(z) + \gamma\right\} > 0$ for $z \in U$.

Lemma 4. *[10] Let $p \in \mathscr{F}(a)$and let $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$ be analytic in U with $q(z) \neq a$ and $n \geq 1$. If q is not subordinate to p, then there exists two points $z_0 = r_0 e^{i\theta} \in U$ and $\zeta_0 \in \partial U \backslash E(q)$ such that*

$$q(U_{r_0}) \subset p(U); \quad q(z_0) = p(\zeta_0) \quad and \quad z_0 p^{'}(z_0) = m\zeta_0 p^{'}(\zeta_0) \quad (m \geq n).$$

Lemma 5. *[13] Let $q \in H[a,1]$ and $\phi : \mathbb{C}^2 \to \mathbb{C}$. Also set $\phi\left(q(z), zq^{'}(z)\right) = h(z)$. If $L(z,t) = \phi\left(q(z), tzq^{'}(z)\right)$ is a subordination chain and $q \in H[a,1] \cap \mathscr{F}(a)$ then*

$$h(z) \prec \varphi\left(p(z), zp^{'}(z)\right)$$

implies that $q(z) \prec p(z)$.Furthermore, if $\varphi\left(q(z), zq^{'}(z)\right) = h(z)$ has a univalent solution $q \in \mathscr{F}(a)$, then q is the best subordinant.

2. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters, $\eta > 0$, $\alpha_1, A_1, ..., \alpha_q, A_q$ and $\beta_1, B_1, ..., \beta_s, B_s$ $\left(q, s \in \mathbb{N}\right)$ are positive real numbers and $z \in U$.

Theorem 1. *Let $f, g \in A(p)$ and*

$$Re\left\{1 + \frac{z\phi^{''}(z)}{\phi^{'}(z)}\right\} > -\delta \qquad \left(\phi(z) = \left(\frac{\theta_{p,q,s}\left[\alpha_1+1, A_1, B_1\right]g(z)}{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right]g(z)}\right)\left(\frac{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right]g(z)}{z^p}\right)^{\eta}\right), \tag{9}$$

where δ is given by

$$\delta = \frac{A_1^2 + (\eta\alpha_1)^2 - \left|A_1^2 - (\eta\alpha_1)^2\right|}{4\eta\alpha_1 A_1} \qquad (z \in U). \tag{10}$$

Then the subordination condition:

$$\left(\frac{\theta_{p,q,s}\left[\alpha_1+1, A_1, B_1\right]f(z)}{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right]f(z)}\right)\left(\frac{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right]f(z)}{z^p}\right)^{\eta} \prec \left(\frac{\theta_{p,q,s}\left[\alpha_1+1, A_1, B_1\right]g(z)}{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right]g(z)}\right)\left(\frac{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right]g(z)}{z^p}\right)^{\eta}, \tag{11}$$

implies that

$$\left(\frac{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right]f(z)}{z^p}\right)^{\eta} \prec \left(\frac{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right]g(z)}{z^p}\right)^{\eta}, \tag{12}$$

where $\left(\frac{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right]g(z)}{z^p}\right)^{\eta}$ is the best dominant.

Proof. Let us define the functions $F(z)$ and $G(z)$ in U by

$$F(z)=\left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)}{z^p}\right)^{\eta} \text{ and } G(z)=\left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]g(z)}{z^p}\right)^{\eta}, \tag{13}$$

we assume here, without loss of generality, that $G(z)$ is analytic, univalent on $\bar{U}$ and

$$G^{'}(\zeta)\neq 0 \quad (|\zeta|=1).$$

If not, then we replace $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0<\rho<1$. These new functions have the desired properties on $\bar{U}$, so we can use them in the proof of our result and the results would follow by letting $\rho\to 1$.

We first show that, if

$$q(z)=1+\frac{zG^{''}(z)}{G^{'}(z)}, \tag{14}$$

then

$$Re\left\{q(z)\right\}>0.$$

From (8) and the definition of the functions G,ϕ, we obtain that

$$\phi(z)=G(z)+\frac{A_1}{\eta\alpha_1}zG^{'}(z). \tag{15}$$

Differentiating both sides of (15) with respect to z yields

$$\phi^{'}(z)=\frac{A_1+\eta\alpha_1}{\eta\alpha_1}G^{'}(z)+\frac{A_1}{\eta\alpha_1}zG^{''}(z). \tag{16}$$

Combining (14) and (16), we easily get

$$1+\frac{z\phi^{''}(z)}{\phi^{'}(z)}=q(z)+\frac{zq^{'}(z)}{q(z)+\frac{\eta\alpha_1}{A_1}}=h(z) \quad (z\in U). \tag{17}$$

It follows from (9) and (17) that

$$Re\left\{h(z)+\frac{\eta\alpha_1}{A_1}\right\}>0 \quad (z\in U). \tag{18}$$

Moreover, by using Lemma 3, we conclude that the differential equation (17) has a solution $q(z)\in H(U)$ with $h(0)=q(0)=1$. Let

$$H(u,v)=u+\frac{v}{u+\frac{\eta\alpha_1}{A_1}}+\delta,$$

where δ is given by (10). From (17) and (18), we obtain $Re\left\{H\left(q(z);zq^{'}(z)\right)\right\}>0 \ (z\in U)$.

To verify the condition

$$Re\{H(iu;t)\} \leq 0 \quad \left(u \in \mathbb{R}; t \leq -\frac{1+u^2}{2}\right), \tag{19}$$

we proceed as follows:

$$Re\{H(iu;t)\} = Re\left\{iu + \frac{t}{iu + \frac{\eta\alpha_1}{A_1}} + \delta\right\} = \frac{\frac{\eta\alpha_1}{A_1}t}{u^2 + \left(\frac{\eta\alpha_1}{A_1}\right)^2} + \delta$$
$$\leq -\frac{\Omega\left(\alpha_1, A_1, u, \delta\right)}{u^2 + \left(\frac{\eta\alpha_1}{A_1}\right)^2},$$

where

$$\Omega\left(\alpha_1, A_1, u, \delta\right) = \left[\frac{\eta\alpha_1}{A_1} - 2\delta\right]u^2 - 2\left(\frac{\eta\alpha_1}{A_1}\right)^2\delta + \frac{\eta\alpha_1}{A_1}. \tag{20}$$

For δ given by (10), the coefficient of u^2 in the quadratic expression $\Omega\left(\alpha_1, A_1, u, \delta\right)$ given by (20) is positive, which implies that (19) holds. Thus, by using Lemma 2, we conclude that

$$Re\left\{q(z)\right\} > 0 \quad (z \in U),$$

that is, that G defined by (13) is convex (univalent) in U. To prove $F \prec G$, where F and G given by (13), let the function $L(z;t)$ be defined by

$$L(z,t) = G(z) + \frac{A_1(1+t)}{\eta\alpha_1} zG'(z) \quad (0 \leq t < \infty; z \in U). \tag{21}$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(\frac{A_1(1+t) + \eta\alpha_1}{\eta\alpha_1}\right) \neq 0 \quad (0 \leq t < \infty; z \in U).$$

This show that the function

$$L(z,t) = a_1(t)z + \ldots,$$

satisfies the condition $a_1(t) \neq 0 \ (0 \leq t < \infty)$. Further, we have

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = Re\left\{\frac{\eta\alpha_1}{A_1} + (1+t)q(z)\right\} > 0 \quad (0 \leq t < \infty; z \in U).$$

Therefore, by using Lemma 1, $L(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{A_1}{\eta\alpha_1} zG'(z) = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \quad (0 \le t < \infty),$$

which implies that

$$L(\zeta,t) \notin L(U,0) = \phi(U) \quad (0 \le t < \infty; \zeta \in \partial U). \tag{22}$$

If F is not subordinate to G, by using Lemma 4, we know that there exist two points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that

$$F(z_0) = G(\zeta_0) \text{ and } z_0 F^{'}(z_0) = (1+t)\zeta_0 G^{'}(\zeta_0) \quad (0 \le t < \infty). \tag{23}$$

Hence, by using (13), (21), (23) and (11), we have

$$L(\zeta_0,t) = G(\zeta_0) + \frac{A_1(1+t)}{\eta\alpha_1}\zeta_0 G^{'}(\zeta_0)$$

$$= F(z_0) + \frac{A_1}{\eta\alpha_1} z_0 F^{'}(z_0)$$

$$= \left(\frac{\theta_{p,q,s}[\alpha_1+1,A_1,B_1]f(z_0)}{\theta_{p,q,s}[\alpha_1,A_1,B_1]f(z_0)}\right)\left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1]f(z_0)}{z_0^p}\right)^{\eta} \in \phi(U).$$

This contradicts (22). Thus, we deduce that $F \prec G$. Considering $F = G$, we see that the function G is the best dominant. This completes the proof of Theorem 1.

We now derive the following superordination result.

Theorem 2. *Let $f, g \in A(p)$ and*

$$Re\left\{1 + \frac{z\phi^{''}(z)}{\phi^{'}(z)}\right\} > -\delta \quad \left(\phi(z) = \left(\frac{\theta_{p,q,s}[\alpha_1+1,A_1,B_1]g(z)}{\theta_{p,q,s}[\alpha_1,A_1,B_1]g(z)}\right)\left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1]g(z)}{z^p}\right)^{\eta}\right), \tag{24}$$

where δ is given by (10). If the function $\left(\frac{\theta_{p,q,s}[\alpha_1+1,A_1,B_1]f(z)}{\theta_{p,q,s}[\alpha_1,A_1,B_1]f(z)}\right)\left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\eta}$ is univalent in U and $\left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\eta} \in \mathscr{F}$, then the superordination condition

$$\left(\frac{\theta_{p,q,s}[\alpha_1+1,A_1,B_1]g(z)}{\theta_{p,q,s}[\alpha_1,A_1,B_1]g(z)}\right)\left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1]g(z)}{z^p}\right)^{\eta} \prec \left(\frac{\theta_{p,q,s}[\alpha_1+1,A_1,B_1]f(z)}{\theta_{p,q,s}[\alpha_1,A_1,B_1]f(z)}\right)\left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\eta}, \tag{25}$$

implies that

$$\left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1]g(z)}{z^p}\right)^{\eta} \prec \left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\eta}, \tag{26}$$

where $\left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1]g(z)}{z^p}\right)^{\eta}$ is the best subordinant.

Proof. Suppose that the functions F, G and q are defined by (13) and (14), respectively. By applying similar method as in the proof of Theorem 1, we get

$$Re\left\{q(z)\right\} > 0 \quad (z \in U).$$

Next, to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L(z,t)$ be defined by (21). Since G is convex, by applying a similar method as in Theorem 1, we deduce that $L(z,t)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G \prec F$. Moreover, since the differential equation

$$\phi(z) = G(z) + \frac{A_1}{\eta\alpha_1} zG^{'}(z) = \varphi\left(G(z), zG^{'}(z)\right)$$

has a univalent solution G, it is the best subordinant. This completes the proof of Theorem 2.

Combining Theorem 1 and Theorem 2, we obtain the following "sandwich-type result".

Theorem 3. *Let $f, g_j \in A(p)$ and*

$$Re\left\{1 + \frac{z\phi_j^{''}(z)}{\phi_j^{'}(z)}\right\} > -\delta,$$

$$\left(\phi_j(z) = \left(\frac{\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]g_j(z)}{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]g_j(z)}\right)\left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]g_j(z)}{z^p}\right)^{\eta} \quad (j=1,2)\right),$$

where δ is given by (10). If the function $\left(\frac{\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]f(z)}{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)}\right)\left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)}{z^p}\right)^{\eta}$ is univalent in U and $\left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)}{z^p}\right)^{\eta} \in \mathscr{F}$, then the condition

$$\left(\frac{\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]g_1(z)}{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]g_1(z)}\right)\left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]g_1(z)}{z^p}\right)^{\eta} \prec \left(\frac{\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]f(z)}{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)}\right)\left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)}{z^p}\right)^{\eta}$$
$$\prec \left(\frac{\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]g_2(z)}{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]g_2(z)}\right)\left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]g_2(z)}{z^p}\right)^{\eta}, \tag{27}$$

implies that

$$\left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]g_1(z)}{z^p}\right)^{\eta} \prec \left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)}{z^p}\right)^{\eta} \prec \left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]g_2(z)}{z^p}\right)^{\eta}, \tag{28}$$

where $\left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]g_1(z)}{z^p}\right)^{\eta}$ and $\left(\frac{\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]g_2(z)}{z^p}\right)^{\eta}$ are , respectively, the best subordinant and the best dominant.

Remark 1. *Specializing $q, s, \alpha_1, A_1, ..., \alpha_q, A_q$ and $\beta_1, B_1, ...\beta_s, B_s$, in the above results, we obtain the corresponding results for different classes associated with the operators (1-7) defined in the introduction.*

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