EUROPEAN JOURNAL OF MATHEMATICAL SCIENCES
Vol. 2, No. 3, 2013, 272-295
ISSN 2147-5512 – www.ejmathsci.com

Coordinated Search for a Conditionally Deterministic Target Motion in the Plane

Abd El-Moneim Anwar Mohamed and Mohamed Abd Allah El-Hadidy*

Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt

---

Abstract. Two unit speed searchers at (0,0) seek for a Conditionally Deterministic moving target in the plane in which any time information of the target position is not available to the searchers. The objective is to find the conditions under which the expected value of the first meeting time for the searchers to return to (0,0) after one of them has met the target is finite. And, to show the existence of an optimal search plan which made the expected value is minimum. In addition, we find the necessary conditions that make the search strategy be optimal.

2010 Mathematics Subject Classifications: 60K30, 90B40

Key Words and Phrases: Conditionally Deterministic motion, search plan, critical search paths, stochastic parameter, searching effort

---

1. Introduction

Searching for a lost target is often a time-critical issue, that is, when the target is very important such as searching for a bomb in the plane or a life raft on the ocean. The prime focus is to find and search for the cast ways in the smallest possible a mount of time.

In an earlier work, the coordinated search technique is used on the line when the located target has symmetric or unsymmetric distribution (see Mohamed et al. [8] and Reyniers [10, 11]. This searching problem is the same as the problem which illustrated by Thomas [13] on the circle with a given radius when the target equally likely to be anywhere on its circumference. Recently, Mohamed et al. [7, 9] have been illustrated this technique in the plane when the located target has symmetric and asymmetrric distribution.

More in an earlier work, the searching problem for a moving target in the plan like missing boats, submarines and missing system has been studied by applying many search techniques such as Bayesian Search and Tracking (SAT). The Bayesian approach would formulate for a target whose prior distribution and probabilistic motion model are known and generalized the approach for coordinated multi-vehicle search [Bourgault et al. 3, 4]. Similarly, the tracking

---

*Corresponding author.

Email addresses: `teamah4@hotmail.com` (A. Mohamed), `melhadidi@eun.eg` (M. El-Hadidy)

study commenced with a simple feedback motion tracking algorithm, and has evolved with the developments of a number of recursive filtering techniques [Furukawa et al. 6].

This paper considers search for target having a rather simple type of motion called Conditionally Deterministic. The target's motion takes place in Euclidean 2−space and depends on a 2−dimensional stochastic parameter. If this parameter were known, then the target's position would be known at all times in the future. Thus, the target's motion is Deterministic Conditioned on knowledge of the parameter.

Conditionally Deterministic motion is more restrictive than Markovian motion. However, the class of Conditionally Deterministic target motions is still rich enough to contain interesting and important search problems. In addition, there are many situations in which optimal search plans can be found explicitly for targets with Conditionally Deterministic motion. This is in contrast to the situation of Markovian motion, where optimal search plans have been found only in the simplest situations.

The primary concern of this paper thus lies in the coordinated search technique which allows two searchers S_1 and S_2 start together and look for a Deterministic moving target with the objective of minimizing the expected time until both of them are met with it at some agreed meeting point. The meeting point is the point where the searchers check back after each period of searching to find out if the other searcher has already found or met the lost target. In addition, S_1 and S_2 do not have to come back to the meeting point to exchange information about the target. In this coordinated system the target reported position is $(0, 0)$ (the meeting point), that is, the center of the plane as indicated in Figure 1.

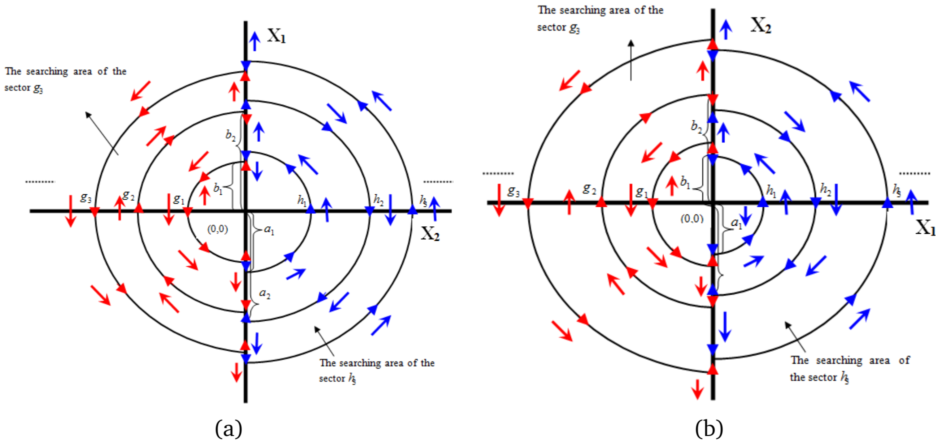

Figure 1: The two search paths that will give us the case of search when we consider all relative positions of the starting point (0, 0) and the moving target.

The plane is divided by two roads and they intersect at the center of this plane, one of these roads is vertical (x_2−axis) and the other is horizontal (x_1−axis). The searchers go on x_2-axis ($+ve$ and $-ve$ parts) as in the following parts with equal speeds ($u_1 = u_2 = 1$), they search the sectors and its tracks with "regular speed" β and they return to $(0, 0)$ after searching successively through the two-axis ($+ve$ and $-ve$ parts) until the target be met. Any track i

has width $a_i - a_{i-1}$ and $b_i - b_{i-1}$ in the right and the left part of x_2-axis, respectively. Our aim is to obtain the expected value of the first meeting time for detecting the target; also we wish to find the optimal search plan (O.S.P.) to met it.

The rest of the paper is organized as follows. In Section 2, we describe the search problem. In Section 3 we discuss the assumptions that model the search problem and we find the condition which make the expected value of the first meeting time be finite. Section 4, studies the existence of optimal search plan. Section 5 gives the necessary conditions which make the search plan be optimal. Finally, conclusions and ongoing research directions are highlighted in Section 6, and the appendix contains the mathematical proofs.

2. Description of the Search Problem

Let X be the plane. At time 0 the target's location is given by a probability density function p. The target's motion is a simple translation described by a function,

$$\eta : X \times [0,\infty) \longrightarrow X,$$

such that $\eta(x,t)$ gives the target's position at time t given that it was at point x at time 0; that is,

$$\eta(x,t) = (x_1 + vt, x_2) \text{ for } x = (x_1,x_2) \in X, \qquad t \geq 0, \tag{1}$$

where v gives the constant speed of the translation [Stone 12]. This situation might occur on the ocean if the target is a life raft known to be drifting with a constant velocity but his initial position is unknown. In this case an optimal search may be obtained by first solving the problem for the stationary case and then moving, the optimal allocation of effort along with the drift of the target distribution.

To be more precise, let the cumulative effort function M be given and suppose $\phi \in \Phi(M)$. Let

$$\dot{\phi}(x,t) = \frac{\partial \phi(x,t)}{\partial t} \text{ for } x \in X, \qquad t \geq 0,$$

when the partial derivative on the right exists, that is, $\dot{\phi}$ indicates the derivative of ϕ with respect to time. Assume $\dot{\phi}$ exists and

$$\phi(x,t) = \int_0^t \dot{\phi}(x,s)ds \text{ for } x \in X, \qquad t \geq 0.$$

Therefore, $\dot{\phi}(x,t)$ gives the rate at which effort density is accumulating at x at time t. If the target motion is given by (1) and ϕ is uniformly optimal for the stationary problem, then at time t we want to apply effort at the rate $\dot{\phi}(x,t)$ to the point $(x_1 + vt, x_2)$. From this it is clear that if one fixes a point $y \in X$, then $\psi(y,t)$, the rate at which effort density is applied at $y = (y_1, y_2)$ at time t in the uniformly optimal plan is given by,

$$\psi(y,t) = \dot{\phi}\left((y_1 - vt, y_2), t\right) = \dot{\phi}(\eta_t^{-1}(y), t) \text{ for } y \in X, \qquad t \geq 0,$$

where η_t denotes the transformation $\eta(.,t)$ and η_t^{-1} is the inverse map of η_t.

Observing that if the target starts at point x at time 0, then $\psi(\eta_s(x),s)$ is the rate at which effort density is accumulating on the target at time s and $\int_0^t \psi(\eta_s(x),s)ds$ gives the total effort density accumulated by time t. As a result, the probability of detecting the target by time t is

$$\int_X p(x)\Lambda\left(x,\int_0^t \psi(\eta_s(x),s)ds\right)dx. \tag{2}$$

Definition 1. *Assume that the target motion take place in the space E, which is a copy of Euclidean* $2-$*space. The space X, also a copy of Euclidean* $2-$*space, is the parameter space. To prescribe the target motion, this is a Borel function* $\eta : X \times T \longrightarrow E$*, where T is an interval of nonnegative real numbers containing* 0 *as its left-hand endpoint. The target motion is characterized by a stochastic parameter* ξ *that takes values in X, that is, if* $\xi = x$*, then* $\eta(x,t)$ *gives the target's position at time t. The distribution of the parameter* ξ *is given by the probability density function p, so that:*

$$Pr\{\xi \in S\} = \int_S p(x)dx,$$

for any Borel set $S \subset X$*. In this case, we call that a target has Conditionally Deterministic motion [Stone 12].*

Often one takes $X = E$ and considers ξ as the position of the target at time 0, which has a probability distribution specified by p.

3. Modeling of Search Problem and Formulation

Here, assumptions of the coordinated search problem with S_1 and S_2 for a Conditionally Deterministic moving target are described and the problem is mathematically formulated as an allocation of searching effort that is the expected value of the first meeting time between any one of the searchers and the target. The surface of the plane be a "*Standard Euclidean 2-space E*", with points designated by ordered pairs $(x_1,\ x_2)$. We shall divide the plane to many sectors as in Figure 1.

Figure 1 gives an illustration of such search paths. The search process is continuous space and time. It is clear that, the two searchers go different distances on x_2-axis because the target moves randomly in the plane. Then, the two searchers should go different distances through x_2-axis and search the two parts as in the following: The searcher S_1 would conduct his search in the right part as in the following manner:

I. Start at $(0,0)$ and go to the $-ve$ part of x_2-axis until reach the point $(0,-a_1)$.

II. Search the sector h_1 and its track until the search reaches the point $(0,a_1)$ on x_2-axis. Then S_1 returns to $(0,0)$ to tell S_2 if the target be met or not.

III. If the target did not meet, S_1 would repeat the steps I and II. But, in II when the search reaches the point $(0, a_1)$, S_1 will go to a distance $a_2 - a_1$ through x_2-axis to the point $(0, a_2)$ and search the sector h_2 until reaches the point $(0, -a_2)$. Then S_1 returns to $(0, 0)$ to tell S_2 if the target be met or not.

IV. If the target is still not met by any one of the searchers, the steps I-III will be repeated. However, in III when the search reaches the point $(0, -a_2)$ S_1 will go to a distance $a_3 - a_2$ through x_2-axis to the point $(0, -a_3)$ and search the sector h_3 until reaches the point $(0, a_3)$. Then S_1 returns to $(0, 0)$ to tell S_2 if the target be met or not, and etc.

Therefore, the search path of S_1 is given as follows:

$$\begin{aligned} e_1 &= |a_1| + h_1 + |a_1|. \\ e_2 &= |a_1| + h_1 + |a_2 - a_1| + h_2 + |a_2|. \\ e_3 &= |a_1| + h_1 + |a_2 - a_1| + h_2 + |a_3 - a_2| + h_3 + |a_3|. \\ &\dots \\ e_n &= |a_1| + h_1 + |a_2 - a_1| + h_2 + |a_3 - a_2| + h_3 + |a_4 - a_3| + \dots + |a_n - a_{n-1}| + h_n + |a_n|. \\ &\dots \end{aligned}$$

and it is defined by a sequence,

$$e = \left\{ e_i = |a_1| + \sum_{j=1}^{i-1} \left[h_j + |a_{j+1} - a_j| \right] + h_i + |a_i|, \text{ for } i \geq 0 \text{ and } i \text{ an integer} \right\}.$$

By the same method, S_2 will go in the first step to the $+ve$ part of x_2-axis as far as b_1 to the point $(0, b_1)$, and conduct his search in the left part with search path given by,

$$\begin{aligned} f_1 &= |b_1| + g_1 + |b_1|. \\ f_2 &= |b_1| + g_1 + |b_2 - b_1| + g_2 + |b_2|. \\ f_3 &= |b_1| + g_1 + |b_2 - b_1| + g_2 + |b_3 - b_2| + g_3 + |b_3|. \\ &\dots \\ f_n &= |b_1| + g_1 + |b_2 - b_1| + g_2 + |b_3 - b_2| + g_3 + |b_4 - b_3| + \dots + |b_n - b_{n-1}| + g_n + |b_n|. \\ &\dots \end{aligned}$$

and it is defined by a sequence,

$$f = \left\{ f_i = |b_1| + \sum_{j=1}^{i-1} \left[g_j + |b_{j+1} - b_j| \right] + g_i + |b_i|, \text{ for } i \geq 0 \text{ and } i \text{ an integer} \right\}.$$

Since, S_1 and S_2 search on the x_2-axis with speed equal to one and on the sectors and its tracks with regular speed β in the two parts. Then, S_1 and S_2 start from $(0,0)$ following search plans which are the functions:

$\gamma^{(1)}(t):\mathbb{R}\longrightarrow\mathbb{R}^+$ and $\gamma^{(2)}(t):\mathbb{R}\longrightarrow\mathbb{R}^+$, respectively. And, they are completely defined by the sequences,

$$\gamma^{(1)}(t)=\left\{\sum_{j=1}^{i}\left[\sum_{k=1}^{n}a_j\theta_k\right]+2a_i,\quad i\geq 0\right\},$$

and

$$\gamma^{(2)}(t)=\left\{\sum_{j=1}^{i}\left[\sum_{k=1}^{n}b_j\theta_k\right]+2b_i,\quad i\geq 0\right\}.$$

Assuming that the searchers speeds β are very larger than the target's speed v. In addition, the target motion is independent of the search, and that is not specifically directed toward escape. In this problem, let the target speed is $v=1$. Then, the search plan $\gamma^{(\epsilon)}(t),\epsilon=1,2$ is finite, if:

$$\gamma^{(1)}(t)\geq E(\left|X_1^0X_2^0\right|)+t,$$

and

$$\gamma^{(2)}(t)\geq E(\left|X_1^0X_2^0\right|)+t,$$

where $E(\left|X_1^0X_2^0\right|)$ is the second order moment of the target initial position and it is finite.

Remark 1. *The repeat of the searching process on the searching area is the minimization of the probability of escaping the target.*

After searching, if S_1 returns to $(0,0)$ before S_2 then S_1 will wait at $(0,0)$ until S_2 returns also to $(0,0)$ and told if the target be found or not. If S_1 and S_2 didn't find the target they search to right and left and again they return to $(0,0)$ and so on until one of them find the target.

The first meeting time $\tau(\gamma)$ is a random variable valued in $\mathbb{R}^+$ which is defined as:

$$\tau(\gamma)=\begin{cases}\inf\left\{t:\text{either }\tau(\gamma^{(1)})=\eta(x,t)\text{ or }\tau(\gamma^{(2)})=\eta(x,t)\;,\;x=(x_1,x_2)\in X\right\},\\ \infty,\text{ if the set is empty.}\end{cases}$$

The target's motion is taken place in Euclidean $2-$space and is determined by a 2-dimensional stochastic parameter ξ such as the target's position at time 0. There is a target motion function η such that if $\xi=x$, then $\eta(x,t)$ gives the target's position at time t for $t\geq 0$. Thus, the target motion is deterministic when conditioned on the value of ξ. The distribution of ξ is assumed to be known to the searchers and is given by the probability density function p.

A moving target search plan is a function ψ such that $\psi(x,t)$ gives the rate at which effort density accumulates at point x at time t. There is a detection function Λ such that if $\xi=x$

and the search plan ψ is followed, then $\Lambda\left(\int_0^t \psi(\eta_s(x),s)ds\right)$ is the probability of detecting the target by time t.

Therefore, (2) becomes,

$$\int_{x_2}\int_{x_1} p(x_1,x_2)\Lambda\left(x_1,x_2;\left(\int_0^t \psi(\eta_s(x_1,x_2),s)ds\right)\right)dx_1dx_2 \tag{3}$$

Since each sector is divided into an equal small sectors l_k, $k=1,2,\dots,n$, where these sectors make a set of an equal cones have the same vertex $(0,0)$ as in Figure 2. However, the searchers can cover a tracks with width a_i-a_{i-1} and b_i-b_{i-1}, respectively, each one can cover an equal small areas from cones in the track number i. The cones is determined by a set of lines with equations $x_1=\mathscr{F}_k x_2=\tan\theta x_2$, where $\mathscr{F}_k$ is the slope of the line l_k and $\theta=\theta_k-\theta_{k-1}$, $k=1,2,\dots,n$. This set of equations make a set of an equal small areas, by which we are meaning for the moment that the searcher searches for every thing from his position, and nothing beyond that. Thus, to evaluate the expected value of the first meeting time between one of the searchers and the target, we use the polar coordinates with $x_1=r\cos\theta$ and $x_2=r\sin\theta$, $r:a_{i-1}\longrightarrow a_i$, $i=1,2,3,\dots$. in the right part and $r:b_{i-1}\longrightarrow b_i$, $i=1,2,3,\dots$, in the left part, $\theta:\theta_{k-1}\longrightarrow\theta_k$, $k=1,2,3,\dots,n$, where $a_0=b_0=r_0=0$, $\theta_0=0$. Hence, (3) becomes:

$$\int_{\theta}\int_{r} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta \tag{4}$$

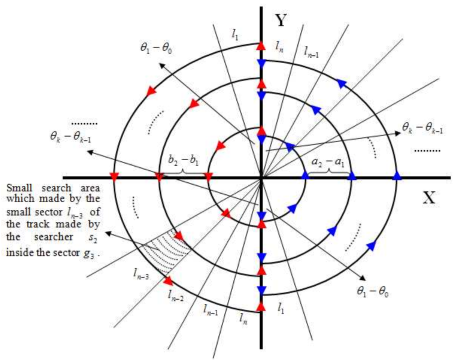

Figure 2: The small search area that made by small sectors l_k, $k=1,2,\dots,n$ made by the searchers inside the circles with radiuses a_i and b_i, $i=2,3,\dots$

Let $t_q, q=1,2$ be the time that S_1 and S_2 take them in the search paths $\{e_i, i\geq 0 \text{ and } i \text{ is an integer number}\}$ and $\{f_i, i\geq 0 \text{ and } i \text{ is an integer number}\}$ in the right and the left parts respectively to $(0,0)$. They go on x_2-axis from the origin before searching the sectors. They return after finishing on the sectors to the origin with equal speeds $(u_1 = u_2 = 1)$. In this case, the time of going through x_2-axis is equal to the distances which done. They are searching on the sectors h_i, g_i, $i = 1, 2, \ldots$. and its tracks (searching areas of the sectors) with "*regular speed*" β. Let $\tau(\gamma)$ be the time of the first meeting between one of the searchers and the target.

Theorem 1. *The expected value of the first meeting time for the searchers to return to the point* $(0,0)$ *after one of them has met the Conditionally Deterministic moving target is finite if:*

$$\sum_{i=1}^{\infty}\left\{\sum_{j=1}^{i}\left[\sum_{k=1}^{n}\left[\int_{\theta_{k-1}}^{\theta_k}\int_{b_{j-1}}^{b_j} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right.\right.\right.$$
$$\left.+\int_{\theta_{k-1}}^{\theta_k}\int_{a_{j-1}}^{a_j} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right]$$
$$\left.\left.\left.\times\left[2(a_j+b_j)+\sum_{\alpha=1}^{j}\left((a_\alpha+b_\alpha)\sum_{k=1}^{n}\theta_k\right)\right]\right]\right]\right\} \tag{5}$$

is finite.

The proof of Theorem 1 is in the Appendix.

Corollary 1. *In the case of a located target, the searchers are not needing to repeat the searching process in the searching area again. More interesting results have been got by Mohamed et al. [9], that gives the expected value of the meeting time for the searchers to return the origin after one of them has detected the target. Therefore,* (5) *becomes,*

$$E(t(\psi)) = \sum_{i=1}^{\infty}\left[\left(2a_i+\frac{\pi}{\omega_i}\right)\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{b_{s-1}}^{b_s}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right.$$
$$\left.+\left(2b_i+\frac{\pi}{\Gamma_i}\right)\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{a_{s-1}}^{a_s}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right], \tag{6}$$

where ω_i *and* Γ_i *are called the angular velocity in the right and the left parts, respectively. Furthermore, if* $a_i = b_i$, $i = 1, 2, \ldots$ *then* $\omega_i = \Gamma_i$ *and the expected value of the time for the searchers to return the origin after one of them has detected the target is [Mohamed et al. 7],*

$$E(t(\psi)) = \sum_{i=1}^{\infty}\left[\left(4a_i+\frac{2\pi}{\omega_i}\right)\left(\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{b_{s-1}}^{b_s}\int_{\theta_{k-1}}^{\theta_k} g(r,\theta)rdrd\theta\right)\right]. \tag{7}$$

3.1. Special Cases

Case 1

If the width in the right part is fixed (i.e. $a_i - a_{i-1} = a$), then $a_1 = a$, $a_2 = 2a$, $a_3 = 3a, \ldots$, in (5). Therefore, the expected value of the first meeting time is finite if:

$$\sum_{i=1}^{\infty}\left\{\sum_{j=1}^{i}\left[\sum_{k=1}^{n}\left[\int_{\theta_{k-1}}^{\theta_k}\int_{b_{j-1}}^{b_j} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) r dr d\theta \right.\right.\right.$$
$$\left. + \int_{\theta_{k-1}}^{\theta_k}\int_{(j-1)a}^{ja} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) r dr d\theta\right]$$
$$\left.\left.\left. \times \left[2(ja+b_j)+\sum_{\alpha=1}^{j}\left(\left(\alpha a+b_\alpha\right)\sum_{k=1}^{n}\theta_k\right)\right]\right]\right\}, \tag{8}$$

is finite.

Case 2

If the width in the left part is fixed (i.e. $b_i - b_{i-1} = b$), then $b_1 = b$, $b_2 = 2b$, $b_3 = 3b, \ldots$, in (5). Thus, the expected value of the first meeting time is finite if:

$$\sum_{i=1}^{\infty}\left\{\sum_{j=1}^{i}\left[\sum_{k=1}^{n}\left[\int_{\theta_{k-1}}^{\theta_k}\int_{(j-1)b}^{jb} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) r dr d\theta \right.\right.\right.$$
$$\left. + \int_{\theta_{k-1}}^{\theta_k}\int_{a_{j-1}}^{a_j} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) r dr d\theta\right]$$
$$\left.\left.\left. \times \left[2(a_j+jb)+\sum_{\alpha=1}^{j}\left(\left(a_\alpha+\alpha b\right)\sum_{k=1}^{n}\theta_k\right)\right]\right]\right\}, \tag{9}$$

finite.

Case 3

If the width in the two parts are fixed (i.e. $a_i - a_{i-1} = a$ and $b_i - b_{i-1} = b$), then the expected value of the first meeting time is finite if:

$$\sum_{i=1}^{\infty}\left\{\sum_{j=1}^{i}\left[\sum_{k=1}^{n}\left[\int_{\theta_{k-1}}^{\theta_k}\int_{(j-1)b}^{jb} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right.\right.\right.$$
$$\left.+\int_{\theta_{k-1}}^{\theta_k}\int_{(j-1)a}^{ja} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right]$$
$$\left.\left.\left.\times\left[2j(a+b)+\sum_{\alpha=1}^{j}\left(\alpha(a+b)\sum_{k=1}^{n}\theta_k\right)\right]\right]\right\}, \tag{10}$$

is finite.

4. Existence of an Optimal Search Plan

The goal of the searching strategy could be minimized the expected value of the first meeting time between one of the searchers and the target. Then, the main problem is to find a search paths $\gamma^{(\epsilon)}(t)$, $\epsilon = 1,2$. If such a search paths exists, we call it O.S.P.

Definition 2. *Let $\left\{\gamma_m^{(\epsilon)}(t)\right\}_{m\geq 1} \in \Gamma_1(t)$, $\epsilon = 1,2$ be sequences of search plans, we say that $\gamma_m^{(\epsilon)}(t)$ converges to $\gamma^{(\epsilon)}(t)$ as m tends to ∞ if and only if any $t \in \mathbb{R}^+$, $\gamma_m^{(\epsilon)}(t)$ converges to $\gamma^{(\epsilon)}(t)$ uniformly on every compact space [5, El-Rayes et. al.].*

Theorem 2. *Let for any $t \in \mathbb{R}^+$ and $D(t)$ be a Conditionally Deterministic process with continuous sample paths. Then, the mapping*

$$(\gamma^{(1)}(t),\gamma^{(2)}(t)) \longrightarrow E(\tau_\gamma) \in \mathbb{R}^+,$$

is lower semi-continuous on $\Gamma_1(t)$.

Proof. Let $I(\gamma^{(\epsilon)},t)$ be the indicator function of the set $\left\{\tau_{\gamma^{(\epsilon)}} > t,\ \epsilon = 1,2\right\}$, by the Fatou-Lesbesque theorem [Stone 12] we obtain,

$$E(\tau_\gamma) = E\left[\sum_{t=1}^{\infty} I(\gamma^{(\epsilon)},t)\right] = E\left[\sum_{t=1}^{\infty}\lim_{m\to\infty}\inf I(\gamma_m^{(\epsilon)},t)\right] \leq \sum_{t=1}^{\infty}\lim_{m\to\infty}\inf E(\tau_{\gamma_m}),$$

for any sequences $\gamma_m^{(\epsilon)} \longrightarrow \gamma^{(\epsilon)}$ in $\Gamma_1(t)$, where $\Gamma_1(t)$ is sequentially compact [5, El-Rayes et. al.]. Thus, the mapping $\gamma^{(\epsilon)} \longrightarrow E(\tau_\gamma)$ is lower semi continuous mapping on $\Gamma_1(t)$, then this mapping attains its minimum. □

5. Necessary Conditions for Optimal Search Plan

Definition 3. *Let $\gamma^* \in \Omega$ be a search plan, then γ^* is an optimal search plan, if*

$$E(t(\gamma^*)) = \inf\{E(t(\gamma)), \gamma \in \Omega\}.$$

$\gamma^{(\epsilon)*}(t), \epsilon = 1,2$ are optimal search paths if the sequences $a = \{a_i i \geq 0\}$ and $b = \{b_i; i \geq 0\}$ are optimal sequences, i.e., $a^* = \{a_i^*; i \geq 0\}$ and $b^* = \{b_i^*; i \geq 0\}$. The problem is to find which values of the turning points a_i and b_i are optimal for a given distribution function of stochastic parameter ξ. There is obviously some similarity between this problem and the well known Linear Search Problem which had been studied before by Balkhi [1, 2]. In that problem, the two searchers are starting at zero and moving with speed one, aims to minimize the expected value of some function of the time taken to meet an object moving according to a Conditionally Deterministic motion in the plane. An optimizing searcher goes to successively increasing distances in alternating directions until the object had met. In our problem, any searcher wishes to find the optimal search paths to search the sectors and its tracks.

The search paths $\gamma^{(\epsilon)}(t)$, $\epsilon = 1,2$ are optimal search paths if the sequences $a = \{a_i; i \geq 0\}$ and $b = \{b_i; i \geq 0\}$ on x_2-axis are optimal. Therefore, we can assumed the certain conditions (necessary) on underlying distribution under which, there exists a search path $\gamma^{(\epsilon)*}(t)$, $\epsilon = 1,2$.

As it can be seen that the search path depends on two unknown factors. Those are the stochastic parameter ξ distribution, and the search path $\gamma^{(\epsilon)}(t)$, $\epsilon = 1,2$ that depend on $a = \{a_i; i \geq 0\}$ and $b = \{b_i; i \geq 0\}$ used by the searchers in the right and the left parts respectively. Let us assume that the stochastic parameter ξ distribution is known. Nevertheless we are still facing a difficult optimization problem. Because this problem has an infinite number of variables; there are $a = \{a_i; i \geq 0\}$ and $b = \{b_i; i \geq 0\}$.

The following recursions gives a necessary conditions for a strategy to be optimal with respect to *Circular Normal Distribution*.

5.1. The Case of a Circular Normal Distribution

If we assume that the stochastic parameter ξ has a bivariate normal distribution with parameters σ_1 and σ_2 at time 0. And, (X_1, X_2) give the target's actual position at time t. Then X_1 is normally distributed with mean 0 and standard deviation σ_1. In addition, X_1 is independent of X_2, which is normally distributed with mean 0 and standard deviation σ_2. Let

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left[-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right)\right], \text{ for } (X_1, X_2) \in E, \tag{11}$$

be the probability density function of the bivariate normal distribution. Thus, the distribution of error in the navigation system yields f as given in (12) for the density of the target distribution. If $\sigma_1 = \sigma_2 = \sigma$ then (11) becomes,

$$f(x_1, x_2) = \frac{1}{2\pi\sigma^2} \exp[-(x_1^2 + x_2^2)/2\sigma^2], \text{ for } (X_1, X_2) \in E, \tag{12}$$

and the target distribution is called *Circular Normal*.

Then the structure of the search path becomes easy and even simple as we shall see below.

Let ξ be the target's position at time 0. It is assumed that ξ has a *Circular Normal* distribution with parameter $\sigma > 0$ so, the density of the target distribution is given by (10). Assume that $T = [0, \infty)$ and the target's velocity is assumed to be constant with its speed proportional to its distance from the origin to its position at time 0. Let q be the proportionality constant for the target speed. Then, assume that the detection function Λ is given by,

$$\Lambda\left(x_1, x_2; \left(\int_0^t \psi(\eta_s(x_1, x_2), s)ds\right)\right) = 1 - \exp\left(-\int_0^t \psi(\eta_s(x_1, x_2), s)ds\right) = 1 - e^{-\lambda}$$

for $\lambda \geq 0$, [Stone 12].

Definition 4. *If $a = \{a_i; i \geq 0$ and i is an integer number$\}$ and $b = \{b_i; i \geq 0$ and i is an integer number$\}$ are search paths such that the derivative of $E(\tau(\gamma))$ with respect to a and b exists and all partial derivatives of $E(\tau(\gamma))$ with respect to the a_i's and b_i's vanish, then a and b are said to be critical search paths (C.S.P).*

Remark 2. *We infer that if $E(\tau(\gamma))$ is differentiable then the set of critical search paths will contain all of the relative minimal and relative maximal search paths. Of course this set may also contain search paths at which $E(\tau(\gamma))$ does not have relative minimal or maximal search paths. In addition the function $E(\tau(\gamma))$ may have relative extremum at a search path at which the derivative of $E(\tau(\gamma))$ with respect to a and b do not exist or $E(\tau(\gamma))$ may have a relative extremum at a search path which is not an interior point.*

If $a = \{a_i; i \geq 0\}$ and $b = \{b_i; i \geq 0\}$ are C.S.P, then $\frac{\partial E(\tau(\gamma))}{\partial a_i}$ and $\frac{\partial E(\tau(\gamma))}{\partial b_i}$ are exist for all pertinent values of i, and then

$$\frac{\partial E(\tau(\gamma))}{\partial a_i} = 0, \qquad \frac{\partial E(\tau(\gamma))}{\partial b_i} = 0, \qquad i \geq 0. \tag{13}$$

Theorem 3. *If the stochastic parameter ξ has a Circular Normal distribution with joint density function $f(x_1, x_2)$ as in (12), and if the condition (5) holds then a_ι's and b_ι's of a C.S.P $a = \{a_\iota; \iota \geq 0\}$ and $b = \{b_\iota; \iota \geq 0\}$ are given by the following relations (with $a_0 = b_0 = 0$):*

$$a_\iota \exp\left(-\frac{a_\iota^2}{2\sigma^2}\right) = \sigma^2 \sum_{i=1}^{\infty} \left[\frac{\left(2 + \frac{i\pi}{2}\right) \left[\sum_{k=1}^{n} \int_{\theta_{k-1}}^{\theta_k} \int_{b_{i-1}}^{b_{i+\iota-1}} \exp\left(-\frac{r^2}{2\sigma^2}\right) d\left(-\frac{r^2}{2\sigma^2}\right) d\theta\right]}{\left[\sum_{j=1}^{i+\iota-1} \left(2b_j + (i - j + \iota) \sum_{k=1}^{n} b_j \theta_k\right)\right] \left(\sum_{k=1}^{n} (\theta_k - \theta_{k-1})\right)} \right], \tag{14}$$

and

$$b_\iota \exp\left(-\frac{b_\iota^2}{2\sigma^2}\right) = \sigma^2 \sum_{i=1}^{\infty} \left[\frac{\left(2+\frac{i\pi}{2}\right)\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{a_{i-1}}^{a_{i+\iota-1}} \exp\left(-\frac{r^2}{2\sigma^2}\right) d\left(-\frac{r^2}{2\sigma^2}\right) d\theta\right]}{\left[\sum_{j=1}^{i+\iota-1}\left(2a_j + (i-j+\iota)\sum_{k=1}^{n} a_j\theta_k\right)\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)} \right]. \tag{15}$$

The proof of Theorem 3 is in the Appendix.

The searchers search inside the tracks with width $a_\iota - a_{\iota-1}$ and $b_\iota - b_{\iota-1}$ in the right and the left parts of x_2-axis, respectively. By choosing many values of a_ι, $\iota = 1,2,3,\ldots$, we can find b_ι, $\iota = 1,2,3,\ldots$, where the above theorem is true for all values of a_ι, $\iota = 2,3,\ldots$, and vice versa. Therefore, we need to satisfy the conditions $a_\iota \geq a_{\iota-1}$ and $b_\iota \geq b_{\iota-1}$ along the searching process and this is called the optimal case otherwise we stop the process and reject the values of a_ι or b_ι, $\iota = 1,2,3,\ldots$.

6. Conclusion and Future Work

A coordinated search technique for a moving target has been presented. The condition which make the expected value of the first meeting time be finite has been given. The necessary conditions have been given to make the search plan be optimal.

The proposed model will be extendible to the multiple searchers case by considering the combinations of movement of multiple targets in the plane.

ACKNOWLEDGEMENTS The authors are very grateful to the Editor-in-Chief and the anonymous referees for their insights.

References

[1] Z. Balkhi. The generalized linear search problem, existence of optimal search paths. *J. Oper. Res. Soc. Japan*, (30) 4:399-420, 1987.

[2] Z. Balkhi. Generalized optimal search paths for continuous univariate random variable. *Oper. Res.*, (23)1:67-96, 1987.

[3] F. Bourgault, T. Furukawa and H. Durrant-Whyte. Coordinated decentralized search for a lost target in a bayesian world. *Proceedings IEEE/RSJ International Conference, Intelligent Robots and Systems*, 1:48-53, 2003.

[4] F. Bourgault, T. Furukawa and H. Durrant-Whyte. Optimal search for a lost target in a bayesian world. *S. Yaut et al. (EDS.):Field and Service Robotics*, 24:209-222, 2006.

[5] A. El-Rayes, A. Mohamed and H. Abou-Gabal. Linear search for a brownian target motion. *Acta Math. Sci. J.* 23(B), 321–327, 2003.

[6] T. Furukawa, F. Bourgault, B. Lavis and H. Durrant-Whyte. Recursive bayesian search-and-tracking using coordinated UAVs for lost targets. *Proceedings of IEEE international conference on robotics and automation, Orlando, Florida*, 2521–2526, 2006.

[7] A. Mohamed, H. Abou Gabal, and M. El-Hadidy. Coordinated search for a randomly located target on the plane. *Eur. J. Pure Appl. Math,*, (2)1:97-111, 2009.

[8] A. Mohamed, H. Abou Gabal and W. Afifi. On the coordinated search problem. *Inter. J. Appl. Math.*, 5: 627-636, 2007.

[9] A. Mohamed, H. Fergany and M. El-Hadidy. On the coordinated search problem on the plane. *J. Sch. Bus. Admin., Istan. Univer.* 41:80-102, 2012.

[10] D. Reyniers. Coordinated search for an object on the line. *Eur. J. Oper. Res.*, 95:663-670, 1996.

[11] D. Reyniers. Coordinated two searchers for an object hidden on an interval. *J. Oper. Res. Soc.*, 46:1386-1392, 1995.

[12] L. Stone. Theory of optimal search. volume 118 of Mathematics in science and Engineering. Academic Press, New York, 1975.

[13] L. Thomas. Finding your kids when they are lost. *J. Oper. Res. Soc.*, 43:637-639, 1992.

Appendix

Proof. [Theorem 1] If the target moves inside the track of g_1, then $t_1 = 2a_1 + \sum_{k=1}^{n} a_1\theta_k$.

If the target moves inside in any track of g_i, $i = 1, 2$, then

$$t_1 = 2a_1 + 2a_2 + 2\sum_{k=1}^{n} a_1\theta_k + \sum_{k=1}^{n} a_2\theta_k.$$

If the target moves inside in any track of g_i, $i = 1, 2, 3$, then

$$t_1 = 2a_1 + 2a_2 + 2a_3 + 3\sum_{k=1}^{n} a_1\theta_k + 2\sum_{k=1}^{n} a_2\theta_k + \sum_{k=1}^{n} a_3\theta_k,$$

and so on.

If the target moves inside the track of h_1, then $t_2 = 2b_1 + \sum_{k=1}^{n} b_1\theta_k$.

If the target moves inside in any track of h_i, $i = 1, 2$, then

$$t_2 = 2b_1 + 2b_2 + 2\sum_{k=1}^{n} b_1\theta_k + \sum_{k=1}^{n} b_2\theta_k.$$

If the target moves inside in any track of h_i, $i = 1,2,3$, then

$$t_1 = 2b_1 + 2b_2 + 2b_3 + 3\sum_{k=1}^{n} b_1\theta_k + 2\sum_{k=1}^{n} b_2\theta_k + \sum_{k=1}^{n} b_3\theta_k,$$

and so on.

Then,

$$\begin{aligned}
E(\tau(\gamma)) = &\left(\sum_{k=1}^{n} a_1\theta_k + 2a_1\right)\left[\int_0^{\theta_1}\int_0^{b_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta + \dots\right.\\
&\left.+\int_{\theta_{n-1}}^{\theta_n}\int_0^{b_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta\right]\\
&+\left(\sum_{k=1}^{n} a_1\theta_k + 2a_1 + \sum_{k=1}^{n} a_1\theta_k + \sum_{k=1}^{n} a_2\theta_k + 2a_2\right)\\
&\times\left[\int_0^{\theta_1}\int_0^{b_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta + \dots\right.\\
&+\int_{\theta_{n-1}}^{\theta_n}\int_0^{b_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta\\
&+\int_0^{\theta_1}\int_{b_1}^{b_2} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta + \dots\\
&\left.+\int_{\theta_{n-1}}^{\theta_n}\int_{b_1}^{b_2} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta\right]\\
&+\left(\sum_{k=1}^{n} a_1\theta_k + 2a_1 + \sum_{k=1}^{n} a_1\theta_k + \sum_{k=1}^{n} a_2\theta_k + 2a_2 + \sum_{k=1}^{n} a_1\theta_k + \sum_{k=1}^{n} a_2\theta_k + \sum_{k=1}^{n} a_3\theta_k + 2a_3\right)\\
&\times\left[\int_0^{\theta_1}\int_0^{b_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta + \dots\right.\\
&+\int_{\theta_{n-1}}^{\theta_n}\int_0^{b_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta
\end{aligned}$$

$$+\int_{0}^{\theta_1}\int_{b_1}^{b_2} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta+\ldots$$

$$+\int_{\theta_{n-1}}^{\theta_n}\int_{b_1}^{b_2} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta$$

$$+\int_{0}^{\theta_1}\int_{b_2}^{b_3} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta+\ldots$$

$$+\left.\int_{\theta_{n-1}}^{\theta_n}\int_{b_2}^{b_3} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right]$$

$$+\left(\sum_{k=1}^{n} b_1\theta_k+2b_1\right)\left[\int_{0}^{\theta_1}\int_{0}^{a_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta+\ldots\right.$$

$$+\left.\int_{\theta_{n-1}}^{\theta_n}\int_{0}^{a_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right]+\ldots$$

$$+\left(\sum_{k=1}^{n} b_1\theta_k+2b_1+\sum_{k=1}^{n} b_1\theta_k+\sum_{k=1}^{n} b_2\theta_k+2b_2\right)$$

$$\times\left[\int_{0}^{\theta_1}\int_{0}^{a_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta+\ldots\right.$$

$$+\int_{\theta_{n-1}}^{\theta_n}\int_{0}^{a_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta$$

$$+\int_{0}^{\theta_1}\int_{a_1}^{a_2} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta+\ldots$$

$$+\left.\int_{\theta_{n-1}}^{\theta_n}\int_{a_1}^{a_2} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right]$$

$$+\left(\sum_{k=1}^{n} b_1\theta_k + 2b_1 + \sum_{k=1}^{n} b_1\theta_k + \sum_{k=1}^{n} b_2\theta_k + 2b_2 + \sum_{k=1}^{n} b_1\theta_k + \sum_{k=1}^{n} b_2\theta_k + \sum_{k=1}^{n} b_3\theta_k + 2b_3\right)$$

$$\times\left[\int_0^{\theta_1}\int_0^{a_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta + \dots\right.$$

$$+\int_{\theta_{n-1}}^{\theta_n}\int_0^{a_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta$$

$$+\int_0^{\theta_1}\int_{a_1}^{a_2} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta + \dots$$

$$+\int_{\theta_{n-1}}^{\theta_n}\int_{a_1}^{a_2} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta + \dots$$

$$+\int_0^{\theta_1}\int_{a_2}^{a_3} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta + \dots$$

$$\left.+\int_{\theta_{n-1}}^{\theta_n}\int_{a_2}^{a_3} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta\right]$$

$$+\dots$$

and so on, then

$$E(\tau(\gamma)) = \left[2a_1 + \sum_{k=1}^{n} a_1\theta_k\right]\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_0^{b_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta\right]$$

$$+\left[2a_1 + 2a_2 + 2\sum_{k=1}^{n} a_1\theta_k + \sum_{k=1}^{n} a_2\theta_k\right]$$

$$\times\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_0^{b_1} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta\right.$$

$$\left.+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_1}^{b_2} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta\right]$$

$$+\left[2a_1+2a_2+2a_3+3\sum_{k=1}^{n}a_1\theta_k+2\sum_{k=1}^{n}a_2\theta_k+\sum_{k=1}^{n}a_3\theta_k\right]$$

$$\times\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{b_1}g(r,\theta)z\left(r,\theta;\left(\int_0^t\psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right.$$

$$+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_1}^{b_2}g(r,\theta)z\left(r,\theta;\left(\int_0^t\psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta$$

$$\left.+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_2}^{b_3}g(r,\theta)z\left(r,\theta;\left(\int_0^t\psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right]$$

$$+\dots$$

$$+\left[2b_1+\sum_{k=1}^{n}b_1\theta_k\right]\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{a_1}g(r,\theta)z\left(r,\theta;\left(\int_0^t\psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right]$$

$$+\left[2b_1+2b_2+2\sum_{k=1}^{n}b_1\theta_k+\sum_{k=1}^{n}b_2\theta_k\right]$$

$$\times\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{a_1}g(r,\theta)z\left(r,\theta;\left(\int_0^t\psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right.$$

$$\left.+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{a_1}^{a_2}g(r,\theta)z\left(r,\theta;\left(\int_0^t\psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right]$$

$$+\left[2b_1+2b_2+2b_3+3\sum_{k=1}^{n}b_1\theta_k+2\sum_{k=1}^{n}b_2\theta_k+\sum_{k=1}^{n}b_3\theta_k\right]$$

$$\times\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{a_1}g(r,\theta)z\left(r,\theta;\left(\int_0^t\psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta\right.$$

$$+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{a_1}^{a_2}g(r,\theta)z\left(r,\theta;\left(\int_0^t\psi(\eta_s(r,\theta),s)ds\right)\right)rdrd\theta$$

$$+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{a_2}^{a_3} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta\Bigg]$$

$$+\ldots$$

$$=\sum_{i=1}^{\infty}\Bigg\{\sum_{j=1}^{i}\Bigg[\sum_{k=1}^{n}\Bigg[\int_{\theta_{k-1}}^{\theta_k}\int_{b_{j-1}}^{b_j} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta$$

$$+\int_{\theta_{k-1}}^{\theta_k}\int_{a_{j-1}}^{a_j} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta\Bigg]$$

$$\times\left[2(a_j+b_j)+\sum_{\alpha=1}^{j}\left((a_\alpha+b_\alpha)\sum_{k=1}^{n}\theta_k\right)\right]\Bigg]\Bigg\},$$

therefore, the expected value of the first meeting time between one of the searchers and the target is finite if:

$$\sum_{i=1}^{\infty}\Bigg\{\sum_{j=1}^{i}\Bigg[\sum_{k=1}^{n}\Bigg[\int_{\theta_{k-1}}^{\theta_k}\int_{b_{j-1}}^{b_j} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta$$

$$+\int_{\theta_{k-1}}^{\theta_k}\int_{a_{j-1}}^{a_j} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta\Bigg]$$

$$\times\left[2(a_j+b_j)+\sum_{\alpha=1}^{j}\left((a_\alpha+b_\alpha)\sum_{k=1}^{n}\theta_k\right)\right]\Bigg]\Bigg\},$$

is finite. □

Proof. [Theorem 3] From (5) we obtain,

$$E(\tau(\gamma))=\sum_{i=1}^{\infty}\Bigg\{\sum_{j=1}^{i}\Bigg[\sum_{k=1}^{n}\Bigg[\int_{\theta_{k-1}}^{\theta_k}\int_{b_{j-1}}^{b_j} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta$$

$$+\int_{\theta_{k-1}}^{\theta_k}\int_{a_{j-1}}^{a_j} g(r,\theta)z\left(r,\theta;\left(\int_0^t \psi(\eta_s(r,\theta),s)ds\right)\right) rdrd\theta\Bigg]$$

$$\times \left[2(a_j + b_j) + \sum_{\alpha=1}^{j} \left((a_\alpha + b_\alpha) \sum_{k=1}^{n} \theta_k \right) \right] \Bigg] \Bigg\}$$

Since, $\Lambda \left(x_1, x_2; \left(\int_0^t \psi(\eta_s(x_1, x_2), s) ds \right) \right) = 1 - e^{-\lambda}$, then

$$E(\tau(\gamma)) = \sum_{i=1}^{\infty} \left\{ \sum_{j=1}^{i} \left[\sum_{k=1}^{n} \left[\int_{\theta_{k-1}}^{\theta_k} \int_{b_{j-1}}^{b_j} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2} \right) \left(1 - e^{-\lambda} \right) r dr d\theta \right.\right.\right.$$

$$\left. + \int_{\theta_{k-1}}^{\theta_k} \int_{a_{j-1}}^{a_j} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2} \right) \left(1 - e^{-\lambda} \right) r dr d\theta \right]$$

$$\left.\left. \times \left[2(a_j + b_j) + \sum_{\alpha=1}^{j} \left((a_\alpha + b_\alpha) \sum_{k=1}^{n} \theta_k \right) \right] \right] \right\}$$

$$= \left\{ \frac{e^{-\lambda} - 1}{2\pi} \right\} \left\{ \left[2a_1 + \sum_{k=1}^{n} a_1 \theta_k \right] \left[\sum_{k=1}^{n} \int_0^{b_1} \int_{\theta_{k-1}}^{\theta_k} \exp\left(-\frac{r^2}{2\sigma^2} \right) d\left(-\frac{r^2}{2\sigma^2} \right) d\theta \right] \right.$$

$$+ \left[2a_1 + 2a_2 + 2\sum_{k=1}^{n} a_1 \theta_k + \sum_{k=1}^{n} a_2 \theta_k \right] \left[\sum_{k=1}^{n} \int_{\theta_{k-1}}^{\theta_k} \int_0^{b_1} \exp\left(-\frac{r^2}{2\sigma^2} \right) d\left(-\frac{r^2}{2\sigma^2} \right) d\theta \right.$$

$$\left. + \sum_{k=1}^{n} \int_{\theta_{k-1}}^{\theta_k} \int_{b_1}^{b_2} \exp\left(-\frac{r^2}{2\sigma^2} \right) d\left(-\frac{r^2}{2\sigma^2} \right) d\theta \right]$$

$$+ \left[2a_1 + 2a_2 + 2a_3 + 3\sum_{k=1}^{n} a_1 \theta_k + 2\sum_{k=1}^{n} a_2 \theta_k + \sum_{k=1}^{n} a_3 \theta_k \right]$$

$$\times \left[\sum_{k=1}^{n} \int_{\theta_{k-1}}^{\theta_k} \int_0^{b_1} \exp\left(-\frac{r^2}{2\sigma^2} \right) d\left(-\frac{r^2}{2\sigma^2} \right) d\theta \right.$$

$$+ \sum_{k=1}^{n} \int_{\theta_{k-1}}^{\theta_k} \int_{b_1}^{b_2} \exp\left(-\frac{r^2}{2\sigma^2} \right) d\left(-\frac{r^2}{2\sigma^2} \right) d\theta$$

$$\left. + \sum_{k=1}^{n} \int_{\theta_{k-1}}^{\theta_k} \int_{b_2}^{b_3} \exp\left(-\frac{r^2}{2\sigma^2} \right) d\left(-\frac{r^2}{2\sigma^2} \right) d\theta \right]$$

$$+\dots$$

$$+\left[2b_1+\sum_{k=1}^{n}b_1\theta_k\right]\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{a_1}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right]$$

$$+\left[2b_1+2b_2+2\sum_{k=1}^{n}b_1\theta_k+\sum_{k=1}^{n}b_2\theta_k\right]\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{a_1}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right.$$

$$\left.+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{a_1}^{a_2}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right]$$

$$+\left[2b_1+2b_2+2b_3+3\sum_{k=1}^{n}b_1\theta_k+2\sum_{k=1}^{n}b_2\theta_k+\sum_{k=1}^{n}b_3\theta_k\right]$$

$$\times\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{a_1}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{a_1}^{a_2}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right.$$

$$\left.\left.+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{a_2}^{a_3}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right]+\dots\right\}$$

By differentiation with respect to a_1, then we get,

$$\frac{\partial E(\tau(\gamma))}{\partial a_1}=\left\{\frac{e^{-\lambda}-1}{2\pi}\right\}.\left\{\left[2+\sum_{k=1}^{n}\theta_k\right]\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{b_1}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right]\right.$$

$$+\left[2+2\sum_{k=1}^{n}\theta_k\right]\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{b_1}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right.$$

$$\left.+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_1}^{b_2}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right]$$

$$+\left[2+3\sum_{k=1}^{n}\theta_k\right]\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{b_1}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right.$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_1}^{b_2}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\\
&+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_2}^{b_3}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\Bigg]\\
&+\ldots\\
&+\left[2b_1+\sum_{k=1}^{n}b_1\theta_k\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)\left(-\frac{a_1}{\sigma^2}\right)\exp\left(-\frac{a_1^2}{2\sigma^2}\right)\\
&+\left[2b_1+2b_2+2\sum_{k=1}^{n}b_1\theta_k+\sum_{k=1}^{n}b_2\theta_k\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)\left(-\frac{a_1}{\sigma^2}\right)\exp\left(-\frac{a_1^2}{2\sigma^2}\right)\\
&+\left[2b_1+2b_2+2b_3+3\sum_{k=1}^{n}b_1\theta_k+2\sum_{k=1}^{n}b_2\theta_k+\sum_{k=1}^{n}b_1\theta_k\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)\\
&\times\left(-\frac{a_1}{\sigma^2}\right)\exp\left(-\frac{a_1^2}{2\sigma^2}\right)+\ldots\Bigg\}\\
=&0,
\end{aligned}
$$

then,

$$
\begin{aligned}
&\sum_{i=1}^{\infty}\left[\left(2+\frac{i\pi}{2}\right)\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_{i-1}}^{b_i}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right]\right]\\
&=\sum_{i=1}^{\infty}\left[\sum_{j=1}^{i}\left(2b_j+(i-j+1)\sum_{k=1}^{n}b_j\theta_k\right)\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)\left(\frac{a_1}{\sigma^2}\right)\exp\left(-\frac{a_1^2}{2\sigma^2}\right).
\end{aligned}
$$

Consequently,

$$
a_1\exp\left(-\frac{a_1^2}{2\sigma^2}\right)=\sigma^2\sum_{i=1}^{\infty}\left[\frac{\left(2+\frac{i\pi}{2}\right)\cdot\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_{i-1}}^{b_i}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right]}{\left[\sum_{j=1}^{i}\left(2b_j+(i-j+1)\sum_{k=1}^{n}b_j\theta_k\right)\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)}\right].
$$

By differentiation with respect to a_2, then we get,

$$
\frac{\partial E(\tau(\gamma))}{\partial a_2}=\left\{\frac{e^{-\lambda}-1}{2\pi}\right\}\Bigg\{\left[2+\sum_{k=1}^{n}\theta_k\right]\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{b_1}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right.
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_1}^{b_2}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\Bigg]\\
&+\left[2+2\sum_{k=1}^{n}\theta_k\right]\Bigg[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{b_1}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\\
&+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_1}^{b_2}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\\
&+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_2}^{b_3}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\Bigg]\\
&+\left[2+3\sum_{k=1}^{n}\theta_k\right]\Bigg[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{0}^{b_1}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\\
&+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_1}^{b_2}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\\
&+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_2}^{b_3}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\\
&+\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_3}^{b_4}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\Bigg]\\
&+\dots\\
&+\left[2b_1+2b_2+2\sum_{k=1}^{n}b_1\theta_k+\sum_{k=1}^{n}b_2\theta_k\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)\left(-\frac{a_2}{\sigma^2}\right)\exp\left(-\frac{a_2^2}{2\sigma^2}\right)\\
&+\left[2b_1+2b_2+2b_3+3\sum_{k=1}^{n}b_1\theta_k+2\sum_{k=1}^{n}b_2\theta_k+\sum_{k=1}^{n}b_3\theta_k\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)\\
&\times\left(-\frac{a_2}{\sigma^2}\right)\exp\left(-\frac{a_2^2}{2\sigma^2}\right)+\dots\Bigg\}\\
=&0,
\end{aligned}
$$

thus,

$$\sum_{i=1}^{\infty}\left[\left(2+\frac{i\pi}{2}\right)\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_{i-1}}^{b_{i+1}}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right]\right]$$
$$=\sum_{i=1}^{\infty}\left[\sum_{j=1}^{i+1}\left(2b_j+(i-j+2)\sum_{k=1}^{n}b_j\theta_k\right)\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)\left(\frac{a_1}{\sigma^2}\right)\exp\left(-\frac{a_1^2}{2\sigma^2}\right)$$

leads to,

$$a_2\exp\left(-\frac{a_2^2}{2\sigma^2}\right)=\sigma^2\sum_{i=1}^{\infty}\left[\frac{\left(2+\frac{i\pi}{2}\right)\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_{i-1}}^{b_{i+1}}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right]}{\left[\sum_{j=1}^{i+1}\left(2b_j+(i-j+2)\sum_{k=1}^{n}b_j\theta_k\right)\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)}\right].$$

Similarly,

$$a_3\exp\left(-\frac{a_3^2}{2\sigma^2}\right)=\sigma^2\sum_{i=1}^{\infty}\left[\frac{\left(2+\frac{i\pi}{2}\right)\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_{i-1}}^{b_{i+2}}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right]}{\left[\sum_{j=1}^{i+2}\left(2b_j+(i-j+3)\sum_{k=1}^{n}b_j\theta_k\right)\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)}\right],$$

and etc... By induction from the above expressions we obtain,

$$a_\iota\exp\left(-\frac{a_\iota^2}{2\sigma^2}\right)=\sigma^2\sum_{i=1}^{\infty}\left[\frac{\left(2+\frac{i\pi}{2}\right)\left[\sum_{k=1}^{n}\int_{\theta_{k-1}}^{\theta_k}\int_{b_{i-1}}^{b_{i+\iota-1}}\exp\left(-\frac{r^2}{2\sigma^2}\right)d\left(-\frac{r^2}{2\sigma^2}\right)d\theta\right]}{\left[\sum_{j=1}^{i+\iota-1}\left(2b_j+(i-j+\iota)\sum_{k=1}^{n}b_j\theta_k\right)\right]\left(\sum_{k=1}^{n}(\theta_k-\theta_{k-1})\right)}\right].$$

Using the same method, we can prove (15). □