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General Integral Operators of p-valent Functions

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Abstract. In this paper we study new general integral operators of p-valent functions, which have to cover several integral operators from literature. We give sufficient conditions for these operators to be p-valently starlike, p-valently close-to-convex, uniformly p-valent close-to-convex and strongly starlike of order τ $(0 < \tau \leq 1)$ in U (open unit disk).

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1. Introduction

Let $\mathscr{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j,\ a_j \geq 0, p \in \{1, 2, \ldots\}, \quad z \in U \tag{1}$$

or

$$f(z) = z^p - \sum_{j=p+1}^{\infty} a_j z^j,\ a_j \geq 0, p \in \{1, 2, \ldots\}, \quad z \in U \tag{2}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. We note that $\mathscr{A}_1 = \mathscr{A}$.

A function $f \in \mathscr{A}_p$ is said to be p-valently starlike of order β $(0 \leq \beta < p)$ iff

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (z \in U).$$

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We denote by $\mathscr{S}_p^{\star}(\beta)$ the class of all such of functions. On the other hand, a function $f \in \mathscr{A}_p$ is said to be p-valently convex of order β $(0 \leq \beta < p)$ if and only if

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta \quad (z \in U).$$

Let $\mathscr{K}_p$ be the class of all these functions which are p-valently convex of order β in U. Furthermore, a function $f(z) \in \mathscr{A}_p$ is said to be in the class $\mathscr{C}_p(\beta)$ of p-valently close-to-convex functions of order β $(0 \leq \beta < p)$ if U iff

$$Re\left(\frac{f'(z)}{z^{p-1}}\right) > \beta \quad (z \in U).$$

It is easy to be seen that $\mathscr{S}_p^{\star}(0) = \mathscr{S}_p^{\star}$ and $\mathscr{C}_p(0) = \mathscr{C}_p$ are, respectively, the classes of p-valently starlike, p-valently convex and p-valently close-to-convex functions in U. We also note that $S_1^{\star} = S^{\star}$, $\mathscr{K}_1 = \mathscr{K}$ and $\mathscr{C}_1 = \mathscr{C}$ are, respectively, the well known classes of starlike, convex and close-to-convex functions in U.

A function $f \in \mathscr{A}_p$ is said to be in the class $\mathscr{UC}_p(\beta)$ of uniformly p-valent close-to-convex functions of order β $(0 \leq \beta < p)$ in U iff

$$Re\left(\frac{zf'(z)}{g(z)} - \beta\right) \geq \left|\frac{zf'(z)}{g(z)} - p\right| \quad (z \in U)$$

for some $g(z) \in \mathscr{US}_p(\beta)$ where $\mathscr{US}_p(\beta)$is the class of uniformly p-valent starlike functions of order β $(-1 \leq \beta < p)$ in U that satisfies

$$Re\left(\frac{zf'(z)}{f(z)} - \beta\right) \geq \left|\frac{zf'(z)}{f(z)} - p\right| \, (z \in U). \tag{3}$$

The uniformly starlike functions are firstly introduced in [8].

The purpose of this paper is to find the sufficient conditions for two generalized operators in order to be p-valently starlike, p-valently close-to-convex, uniformly p-valent close-to-convex and strongly starlike of order τ $(0 < \tau \leq 1)$ in U (open unit disk).

2. Preliminary Results

Definition 1 ([2]). *Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. We denote by D_λ^β the linear operator defined by*

$$D_\lambda^\beta : A \to A, \quad D_\lambda^\beta f(z) = z + \sum_{j=n+1}^{\infty} [1+(j-1)\lambda]^\beta a_j z^j. \tag{4}$$

Remark 1. *In [1] we have introduced the following operator concerning the complex functions with negative coefficients:*

$$D_{\lambda}^{\beta}:A\to A,\quad D_{\lambda}^{\beta}f(z)=z-\sum_{j=n+1}^{\infty}[1+(j-1)\lambda]^{\beta}a_j z^j. \tag{5}$$

The neighborhoods concerning the class of functions defined using the operator (5) *is studied in [5].*

Remark 2. *We have introduced and studied the following operator concerning the functions $f\in S$, $S=\{f\in\mathscr{A}: f$ is univalent in $U\}$:*

$$D_{\lambda_1,\lambda_2}^{n,\beta}f(z)=(h*\psi_1*f)(z)=z\pm\sum_{k\geq 2}\frac{[1-\lambda_1(k-1))]^{\beta-1}}{[1-\lambda_2(k-1))]^{\beta}}\cdot\frac{1+c}{k+c}\cdot C(n,k)\cdot a_k\cdot z^k \tag{6}$$

where $C(n,k)=\frac{(n+1)_{k-1}}{(1)_{k-1}}$; $(n)_k$ is the Pochammer symbol; $k\geq 2$, $c\geq 0$ and $Re\{c\}\geq 0$; $z\in U$.

Remark 3. *If we denote by $(x)_k$ the Pochhammer symbol, we define it as follows:*

$$(x)_k=\begin{cases}1 & \text{for } k=0, x\in\mathbb{C}\setminus\{0\}\\ x(x+1)(x+2)\cdot\ldots\cdot(x+k-1) & \text{for } k\in\mathbb{N}-\{0\} \text{ and } x\in\mathbb{C}\end{cases}.$$

The following generalised integral operators are univalent for all $n\in\mathbb{N}-\{0\}$ under given conditions:

$$I^1(z)=\left\{\beta\int_0^z t^{\beta\delta-1}\cdot\prod_{j=1}^{p}\left[\frac{((D_{\lambda_1,\lambda_2}^{n,\kappa}f_j(t^n)')^{2\gamma_1-1}}{t^{\sigma}}\right]^{\delta_j^1}\cdot\left[\frac{(D_{\lambda_1,\lambda_2}^{n,\kappa}f_j(t^n))^{2\gamma_2-1}}{t^{\sigma}}\right]^{\delta_j^2}dt\right\}^{\frac{1}{\beta}}, \tag{7}$$

where $\alpha,\gamma_1,\gamma_2,\beta\in\mathbb{C}$, $Re\alpha=a>0$ and $D_{\lambda_1,\lambda_2}^{n,\kappa}f_j(z)\in\mathscr{A}$, $\lambda_1,\lambda_2,\kappa\geq 0$, $\sigma\in\mathbb{R}$, $j=\overline{1,p}$, $p\in\mathbb{N}$, $D_{\lambda_1,\lambda_2}^{n,\kappa}f_j(z^n)$ of form (6) and

$$I^2(z)=\left\{\chi\int_0^z t^{\chi\delta-1}\prod_{j=1}^{p}\left[\frac{((D_{\lambda}^{\beta}f_j(t^n)')^{2\gamma_1-1}}{t^{\sigma}}\right]^{\delta_j^1}\left[\frac{(D_{\lambda}^{\beta}f_j(t^n))^{2\gamma_2-1}}{t^{\sigma}}\right]^{\delta_j^2}dt\right\}^{\frac{1}{\chi}}, \tag{8}$$

where $\alpha,\gamma_1,\gamma_2,\chi\in\mathbb{C}$, $Re\alpha=a>0$ and $D_{\lambda}^{\beta}f_j(z)\in\mathscr{A}$, $\beta\geq 0$, $\lambda\geq 0$, $\sigma\in\mathbb{R}$, $D_{\lambda}^{\beta}f(z^n)$ of form (5).

We will make use of the following Lemmas in order to derive our main results.

Lemma 1 ([10]). *If $f \in \mathscr{A}_p$ satisfies*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} < p+\frac{1}{4} \quad (z \in U),$$

then f is p-valently starlike in U.

Lemma 2 ([6]). *If $f \in \mathscr{A}_p$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right| < p+1 \quad (z \in U),$$

then f is p-valently starlike in U.

Lemma 3 ([14]). *If $f \in \mathscr{A}_p$ satisfies*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} < p+\frac{a+b}{(1+a)(1-b)} \quad (z \in U),$$

where $a > 0$, $b \geq 0$ and $a + 2b \leq 1$, then f is p-valently close-to-convex in U.

Lemma 4 ([3]). *If $f \in \mathscr{A}_p$ satisfies*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} < p+\frac{1}{3} \quad (z \in U),$$

then f is uniformly p-valent close-to-convex in U.

Lemma 5 ([12]). *If $f \in \mathscr{A}_p$ satisfies*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{p}{4}-1 \quad (z \in U),$$

then

$$Re\sqrt{1+\frac{zf'(z)}{f(z)}} > \frac{\sqrt{p}}{2} \quad (z \in U).$$

Lemma 6 ([11]). *If $f \in \mathscr{A}_p$ satisfies*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > p-\frac{\tau}{2} \quad (z \in U),$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| > \frac{\pi}{2}\tau \quad (z \in U).$$

We take into consideration the general integral operator of form (7) (or (8)), for which we study the sufficient conditions to be p-valently starlike, p-valently close-to-convex, uniformly p-valent close-to-convex and strongly starlike of order τ $(0 < \tau \leq 1)$ in U, giving also several examples which prove its relevance.

3. Main Results

If we consider that $\beta\delta - 1 = \sigma = p$ (or $\chi\delta - 1 = \sigma = p$) and the function $f(z)$ of form (1) or (2), we derive the following:

$$I_p^1(z) = \left\{ \beta \int_0^z p t^{p-1} \cdot \prod_{j=1}^{m} \left[\frac{((D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(t^n)')^{2\gamma_1 - 1}}{t^p} \right]^{\delta_j^1} \cdot \left[\frac{(D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(t^n))^{2\gamma_2 - 1}}{t^p} \right]^{\delta_j^2} dt \right\}^{\frac{1}{\beta}} \tag{9}$$

with respect to the general integral operator $I^1(z)$ of form (7) and

$$I_p^2(z) = \left\{ \chi \int_0^z p t^{p-1} \prod_{j=1}^{m} \left[\frac{((D_{\lambda}^{\beta} f_j(t^n)')^{2\gamma_1 - 1}}{t^p} \right]^{\delta_j^1} \left[\frac{(D_{\lambda}^{\beta} f_j(t^n))^{2\gamma_2 - 1}}{t^p} \right]^{\delta_j^2} dt \right\}^{\frac{1}{\chi}} \tag{10}$$

from the general integral operator $I^2(z)$ of form (8).

Further we give sufficient conditions for the operator $I_p^1(z)$ of form (9) to be p-valently starlike, p-valently close-to-convex, uniformly p-valent close-to-convex and strongly starlike of order τ $(0 < \tau \leq 1)$ in U.

3.1. Sufficient Conditions for the Operator $I_p^1(z)$

For further simplification, we note the integral operator $I_p^1(z)$ of form (9) as follows:

$$I_p^1(z) = \left\{ \beta \int_0^z p t^{p-1} \cdot \prod_{\substack{j=1 \\ a \in \{1,2\}}}^{m} \left[\frac{((D_{\lambda_1,\lambda_2}^{n,\kappa,a} f_j(t^n))^{2\gamma_a - 1}}{t^p} \right]^{\delta_j^a} dt \right\}^{\frac{1}{\beta}}, \tag{11}$$

where $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f_j(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))'$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f_j(z^n) = D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n)$, $D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n)$ is of form (6), $z \in U$.

We firstly study the sufficient conditions for the operator I_p^1 to be in $\mathscr{S}_p^\star$.

Theorem 1. *Let $\delta_j^a, \gamma_a \in \mathbb{C}$, $a \in \{1,2\}$, $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$. If $f_j \in \mathscr{A}_p$ for all $j = \overline{1,m}$ satisfies*

$$Re\left[\frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f_j(z^n))^{2\gamma_a - 1}]'}{(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f_j(z^n))^{2\gamma_a - 1}} \right] < p + \frac{1}{4 \cdot \sum\limits_{\substack{j=1 \\ a \in \{1,2\}}}^{m} \delta_j^a} \quad (z \in U), \tag{12}$$

where $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f_j(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))'$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f_j(z^n) = D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n)$, $z \in U$, $D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n)$ is of form (6), then $I_p^1(z)$ is p-valently starlike in U.

Proof. From (9) (or (11)) we see that $I_p^1(z) \in \mathscr{A}_p$. Moreover, by differentiating (11) logarithmically, multiplying by z and adding 1, we have

$$1+\frac{z[I_p^1(z)]''}{[I_p^1(z)]'}=\frac{1-\beta}{\beta}\cdot\frac{\left[\int\limits_0^z pt^{p-1}\cdot\prod_{\substack{j=1\\ a\in\{1,2\}}}^m\left[\frac{((D_{\lambda_1,\lambda_2}^{n,\kappa,a}f_j(t^n))^{2\gamma_a-1}}{t^p}\right]^{\delta_j^a}dt\right]'}{\int\limits_0^z pt^{p-1}\cdot\prod_{\substack{j=1\\ a\in\{1,2\}}}^m\left[\frac{((D_{\lambda_1,\lambda_2}^{n,\kappa,a}f_j(t^n))^{2\gamma_a-1}}{t^p}\right]^{\delta_j^a}dt} \tag{13}$$

$$+p\left(1-\sum_{\substack{j=1\\ a\in\{1,2\}}}^m\delta_j^a\right)+\sum_{\substack{j=1\\ a\in\{1,2\}}}^m\delta_j^a(2\gamma_a-1)\cdot nz^{n-1}\cdot\frac{z(D_{\lambda_1,\lambda_2}^{n,\kappa,a}f_j(z^n))'}{D_{\lambda_1,\lambda_2}^{n,\kappa,a}f_j(z^n)}.$$

Next we take the real part of (13) and consider the conditions of the integral operator (7), from where we obtain the following

$$Re\left(1+\frac{z[I_p^1(z)]''}{[I_p^1(z)]'}\right)\le\frac{1-\beta}{\beta}+p\left(1-\sum_{\substack{j=1\\ a\in\{1,2\}}}^m\delta_j^a\right)+\sum_{\substack{j=1\\ a\in\{1,2\}}}^m\delta_j^a\cdot Re\left[\frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,a}f_j(z^n))^{2\gamma_a-1}]'}{(D_{\lambda_1,\lambda_2}^{n,\kappa,a}f_j(z^n))^{2\gamma_a-1}}\right]. \tag{14}$$

From (14) and (12), we obtain that

$$Re\left(1+\frac{z[I_p^1(z)]''}{[I_p^1(z)]'}\right)<\frac{1-\beta}{\beta}+p\left(1-\sum_{\substack{j=1\\ a\in\{1,2\}}}^m\delta_j^a\right)+\sum_{\substack{j=1\\ a\in\{1,2\}}}^m\delta_j^a\cdot\left(p+\frac{1}{4\cdot\sum\limits_{\substack{j=1\\ a\in\{1,2\}}}^m\delta_j^a}\right) \tag{15}$$

$$=\frac{1-\beta}{\beta}+p+\frac{1}{4}. \tag{16}$$

Applying Lemma 1 to an integral operator of order $\frac{1}{\beta}$, we immediately obtain that $I_p^1(z)\in\mathscr{S}_p^\star$.

Remark 4. *If $\beta=1$ we can just use Lemma 1 to* (15)*, which directly follows to $I_p^1(z)\in\mathscr{S}_p^\star$ $(\beta=1)$, where* (15) *becomes $Re\left(1+\frac{z[I_p^1(z)]''}{[I_p^1(z)]'}\right)<p+\frac{1}{4}$.*

Remark 5. *Let $\delta_j^1=0$. Then, for all $j=\overline{1,m}$, $m\in\mathbb{N}-\{0\}$, we obtain the following integral operator:*

$$I_p^{11}(z)=\left\{\beta\int\limits_0^z pt^{p-1}\cdot\prod_{j=1}^m\left[\frac{(D_{\lambda_1,\lambda_2}^{n,\kappa}f_j(t^n))^{2\gamma_2-1}}{t^p}\right]^{\delta_j^2}dt\right\}^{\frac{1}{\beta}}. \tag{17}$$

On the other hand, if $\delta_j^2 = 0$, we obtain the following integral operator:

$$I_p^{12}(z) = \left\{ \beta \int_0^z p t^{p-1} \cdot \prod_{j=1}^{m} \left[\frac{(D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(t^n))^{2\gamma_1 - 1}}{t^p} \right]^{\delta_j^1} dt \right\}^{\frac{1}{\beta}}. \tag{18}$$

Corollary 1. *a) Let $\delta_j^1 = 0$, $\delta_j^2, \gamma_2 \in \mathbb{C}$, $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$. If $f_j \in \mathscr{A}_p$ for all $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, satisfies*

$$Re \left[\frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))^{2\gamma_2 - 1}]'}{(D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))^{2\gamma_2 - 1}} \right] < p + \frac{1}{4 \cdot \sum_{j=1}^{m} \delta_j^2} \quad (z \in U),$$

where $D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n)$ is of form (6), then $I_p^{11}(z)$ of form (17) is p-valently starlike in U.

b) Let $\delta_j^2 = 0$, $\delta_j^1, \gamma_1 \in \mathbb{C}$, $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$. If $f_j \in \mathscr{A}_p$ for all $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, satisfies

$$Re \left[\frac{z[((D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))')^{2\gamma_1 - 1}]'}{((D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))')^{2\gamma_1 - 1}} \right] < p + \frac{1}{4 \cdot \sum_{j=1}^{m} \delta_j^1} \quad (z \in U),$$

where $D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n)$ is of form (6), then $I_p^{12}(z)$ of form (18) is p-valently starlike in U.

Furthermore, if $j = p = 1$, $\delta_1^1 = \delta^1 \in \mathbb{C}$, $\delta_1^2 = \delta^2 \in \mathbb{C}$ and $D_{\lambda_1,\lambda_2}^{n,\kappa} f_1(z^n) = D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ in Theorem 1, we have:

Corollary 2. *If $f \in \mathscr{A}$, $a \in \{1,2\}$, satisfies the following condition*

$$Re \left[\frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(z^n))^{2\gamma_a - 1}]'}{(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(z^n))^{2\gamma_a - 1}} \right] < 1 + \frac{1}{4 \prod_{a \in \{1,2\}} \delta^a} \quad (z \in U),$$

then $\left\{ \beta \int_0^z p t^{p-1} \cdot \prod_{a \in \{1,2\}} \left[\frac{(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(t^n))^{2\gamma_a - 1}}{t^p} \right]^{\delta^a} dt \right\}^{\frac{1}{\beta}}$ is starlike in U, for any $\gamma_1, \gamma_2 \in \mathbb{C}$, $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))'$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f(z^n) = D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$, where $D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ is of form (6), $j = 1, z \in U$.

Moreover, if we take $\delta_1^1 = \delta_1^2 = \delta \in \mathbb{C}$ in Corollary 2, we have the following result:

Corollary 3. *If $f \in \mathscr{A}$, $\delta \in \mathbb{C}$, $a \in \{1,2\}$, satisfies the condition*

$$Re \left[\frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(z^n))^{2\gamma_a - 1}]'}{(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(z^n))^{2\gamma_a - 1}} \right] < 1 + \frac{1}{4\delta} \quad (z \in U),$$

then $\int_0^z \prod_{a\in\{1,2\}} \left[\frac{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(t^n))^{2\gamma_a-1}}{t}\right]^{\delta} dt$ *is starlike in U, for any* $\gamma_1, \gamma_2 \in \mathbb{C}$,
$D^{n,\kappa,1}_{\lambda_1,\lambda_2} f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$, *where* $D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$ *is of form* (6), $j = 1, z \in U$.

Theorem 2. *Let* δ^a_j, $\gamma_a \in \mathbb{C}$, $a \in \{1,2\}$, $j = \overline{1,m}$. *If* $f_j \in \mathscr{A}_p$ *for all* $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, *satisfies*

$$\left|\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a-1}} - p\right| < \frac{p+1}{4\cdot \sum\limits_{\substack{j=1\\ a\in\{1,2\}}}^{m} \delta^a_j} \quad (z \in U), \tag{19}$$

where $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f_j(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$, $z \in U$, $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$ *is of form* (6), *then* $I^1_p(z)$ *is p-valently starlike in U.*

Proof. We make use of (13) and the hypothesis (19) and obtain

$$\begin{aligned}\left|1 + \frac{z[I^1_p(z)]''}{[I^1_p(z)]'} - p\right| &\leq \frac{1-\beta}{\beta} + \left|\sum_{\substack{j=1\\ a\in\{1,2\}}}^{m} \delta^a_j \left(\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a-1}} - p\right)\right| \\ &< \frac{1-\beta}{\beta} + \sum_{\substack{j=1\\ a\in\{1,2\}}}^{m} \delta^a_j \left|\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a-1}} - p\right| \\ &< \frac{1-\beta}{\beta} + \sum_{\substack{j=1\\ a\in\{1,2\}}}^{m} \delta^a_j \left(\frac{p+1}{4\cdot \sum\limits_{\substack{j=1\\ a\in\{1,2\}}}^{m} \delta^a_j}\right) = \frac{1-\beta}{\beta} + p + 1.\end{aligned}$$

Using Lemma 2 for an integral operator of order $\frac{1}{\beta}$, we get $I^1_p(z) \in \mathscr{S}^\star_p$ immediately.

Remark 6. *If* $\beta = 1$ *we can just use Lemma 2 to the relation above, which directly follows to* $I^1_p(z) \in \mathscr{S}^\star_p$ $(\beta = 1)$, *under the condition* (19), *from where we obtain*
$\left|1 + \frac{z[I^1_p(z)]''}{[I^1_p(z)]'} - p\right| < p + 1.$

Letting $j = p = 1, \forall j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, $\delta^1_1 = \delta^1 \in \mathbb{C}$, $\delta^2_1 = \delta^2 \in \mathbb{C}$ and $D^{n,\kappa}_{\lambda_1,\lambda_2} f_1(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$ in Theorem 2, we have:

Corollary 4. *If* $f \in \mathscr{A}$, $a \in \{1,2\}$, *satisfies the following condition*

$$\left|\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_a-1}} - 1\right| < \frac{2}{\prod\limits_{a\in\{1,2\}} \delta^a} \quad (z \in U),$$

then $\left\{\beta\int\limits_0^z \prod\limits_{a\in\{1,2\}} \left[\frac{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(t^n))^{2\gamma_a-1}}{t}\right]^{\delta^a} dt\right\}^{\frac{1}{\beta}}$ *is starlike in U, for any* $\gamma_1,\gamma_2\in\mathbb{C}$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$*, where* $D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$ *is of form* (6), $(j=1)$, $z\in U$.

Moreover, if we take $\delta_1^1=\delta_1^2=\delta\in\mathbb{C}$ in Corollary 4, we have the following result:

Corollary 5. *If* $f\in\mathscr{A}$, $\delta\in\mathbb{C}$, $a\in\{1,2\}$*, satisfies the condition*

$$\left|\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_a-1}}-1\right| < \frac{2}{\delta} \quad (z\in U),$$

then $\left\{\beta\int\limits_0^z \prod\limits_{a\in\{1,2\}} \left[\frac{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(t^n))^{2\gamma_a-1}}{t}\right]^{\delta} dt\right\}^{\frac{1}{\beta}}$ *is starlike in U, for any* $\gamma_1,\gamma_2\in\mathbb{C}$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$*, where* $D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$ *is of form* (6), $(j=1)$, $z\in U$.

Next we apply Lemma 3 and Lemma 4 and we obtain sufficient conditions for I_p^1 to be p-valently close-to-convex and uniformly p-valent close to convex in U.

Theorem 3. *Let* $\delta_j^\upsilon,\gamma_\upsilon\in\mathbb{C}$, $\upsilon\in\{1,2\}$, $j=\overline{1,m}$*. If* $f_j\in\mathscr{A}_p$ *for all* $j=\overline{1,m}$, $m\in\mathbb{N}-\{0\}$*, satisfies*

$$Re\left[\frac{z[(D^{n,\kappa,\upsilon}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_\upsilon-1}]'}{(D^{n,\kappa,\upsilon}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_\upsilon-1}}\right] < p+\frac{a+b}{(1+a)(1-b)\cdot\sum\limits_{\substack{j=1\\ \upsilon\in\{1,2\}}}^{m}\delta_j^\upsilon} \quad (z\in U), \tag{20}$$

where $a>0$, $b\geq 0$, $a+2b\leq 1$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f_j(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$, $z\in U$, $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$ *is of form* (6)*, then* $I_p^1(z)$ *is p-valently close-to-convex in U.*

Proof. From (14), together with (20), and making use of Lemma 3 for an integral operator of order $\frac{1}{\beta}$, we have $I_p^1(z)\in\mathscr{C}_p(\alpha)$ $(0\leq\alpha<p)$.

Letting $j=p=1$, $\delta_1^1=\delta^1\in\mathbb{C}$, $\delta_1^2=\delta^2\in\mathbb{C}$ and $D^{n,\kappa}_{\lambda_1,\lambda_2} f_1(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$ in Theorem 3, we have:

Corollary 6. *If* $f\in\mathscr{A}$, $\upsilon\in\{1,2\}$*, satisfies the following condition*

$$Re\left[\frac{z[(D^{n,\kappa,\upsilon}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_\upsilon-1}]'}{(D^{n,\kappa,\upsilon}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_\upsilon-1}}\right] < p+\frac{a+b}{(1+a)(1-b)\cdot\sum\limits_{\upsilon\in\{1,2\}}^{m}\delta^\upsilon} \quad (z\in U),$$

where $a > 0,\ b \geq 0,\ a + 2b \leq 1$, *then* $\left\{\beta \int_0^z \prod_{\upsilon\in\{1,2\}} \left[\frac{(D^{n,\kappa,\upsilon}_{\lambda_1,\lambda_2} f(t^n))^{2\gamma_\upsilon-1}}{t}\right]^{\delta^\upsilon} dt\right\}^{\frac{1}{\beta}}$ *is close-to-convex in* U*, for any* $\gamma_1, \gamma_2 \in \mathbb{C}$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$*, where* $D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$ *is of form* (6)*,* $(j = 1)$*,* $z \in U$*.*

Moreover, if we take $\delta_1^1 = \delta_1^2 = \delta \in \mathbb{C}$ in Corollary 6, we have the following result:

Corollary 7. *If* $f \in \mathscr{A}$*,* $\delta \in \mathbb{C}$*,* $\upsilon \in \{1,2\}$*, satisfies the condition*

$$Re\left[\frac{z[(D^{n,\kappa,\upsilon}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_\upsilon-1}]'}{(D^{n,\kappa,\upsilon}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_\upsilon-1}}\right] < p + \frac{a+b}{(1+a)(1-b)\cdot\delta} \quad (z \in U),$$

where $a > 0,\ b \geq 0,\ a + 2b \leq 1$, *then* $\left\{\beta \int_0^z \prod_{\upsilon\in\{1,2\}} \left[\frac{(D^{n,\kappa,\upsilon}_{\lambda_1,\lambda_2} f(t^n))^{2\gamma_\upsilon-1}}{t}\right]^{\delta} dt\right\}^{\frac{1}{\beta}}$ *is close-to-convex in* U*, for any* $\gamma_1,\ \gamma_2 \in \mathbb{C}$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$*,* $D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$ *is of form* (6)*,* $(j = 1)$*,* $z \in U$*.*

Theorem 4. *Let* $\delta_j^a, \gamma_a \in \mathbb{C}$*,* $a \in \{1,2\}$*,* $j = \overline{1,m}$*. If* $f_j \in \mathscr{A}_p$ *for all* $j = \overline{1,m}$ *satisfies*

$$Re\left[\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a-1}}\right] < p + \frac{1}{3\cdot \sum\limits_{\substack{j=1\\ a\in\{1,2\}}}^{m} \delta_j^a} \quad (z \in U), \tag{21}$$

where $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f_j(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$*,* $z \in U$*,* $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$ *is of form* (6)*, then* $I_p^1(z)$ *is uniformly p-valent close-to-convex in* U*.*

Proof. From (14), together with (21), and making use of Lemma 3 for an integral operator of order $\frac{1}{\beta}$, we get $I_p^1(z) \in \mathscr{U}\mathscr{C}_p(\alpha)$ $(0 \leq \alpha > p)$.

Letting $j = p = 1$, $\delta_1^1 = \delta^1 \in \mathbb{C}$, $\delta_1^2 = \delta^2 \in \mathbb{C}$ and $D^{n,\kappa}_{\lambda_1,\lambda_2} f_1(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$ in Theorem 4, we have:

Corollary 8. *If* $f \in \mathscr{A}$*,* $a \in \{1,2\}$*, satisfies the following condition*

$$Re\left[\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_a-1}}\right] < p + \frac{1}{3\cdot \sum\limits_{a\in\{1,2\}}^{m} \delta^a} \quad (z \in U),$$

then $\left\{\beta \int_0^z \prod_{a\in\{1,2\}} \left[\frac{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(t^n))^{2\gamma_a-1}}{t}\right]^{\delta^a} dt\right\}^{\frac{1}{\beta}}$ *is uniformly close-to-convex in* U*, for any* $\gamma_1, \gamma_2 \in \mathbb{C}$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$*, where* $D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$ *is of form* (6)*,* $(j = 1)$*,* $z \in U$*.*

Moreover, if we take $\delta_1^1 = \delta_1^2 = \delta \in \mathbb{C}$ in Corollary 8, we have the following result:

Corollary 9. *If $f \in \mathscr{A}$, $\delta \in \mathbb{C}$, $a \in \{1,2\}$, satisfies the condition*

$$Re\left[\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_a-1}}\right] < p + \frac{1}{3\delta} \quad (z \in U),$$

then $\left\{\beta \int_0^z \prod_{a\in\{1,2\}} \left[\frac{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(t^n))^{2\gamma_a-1}}{t}\right]^\delta dt\right\}^{\frac{1}{\beta}}$ *is uniformly close-to-convex in U, for any $\gamma_1, \gamma_2 \in \mathbb{C}$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n))'$ and $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$, $D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$ is of form (6), $(j=1)$, $z \in U$.*

Theorem 5. *Let $\delta_j^a, \gamma_a \in \mathbb{C}$, $a \in \{1,2\}$, $j = \overline{1,m}$. If $f_j \in \mathscr{A}_p$ for all $j = \overline{1,m}$ satisfies*

$$Re\left[\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a-1}}\right] > p - \frac{3p+4}{4\cdot \sum\limits_{\substack{j=1\\ a\in\{1,2\}}}^{m} \delta_j^a} \quad (z \in U), \tag{22}$$

then

$$Re\sqrt{\frac{z[I_p^1(z)]'}{I_p^1(z)}} > \frac{\sqrt{p}}{2} \quad (z \in U),$$

where $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))'$ and $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f_j(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$, $z \in U$, $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$ is of form (6).

Proof. From (14) and (22) we get

$$Re\left(1 + \frac{z[I_p^1(z)]''}{[I_p^1(z)]'}\right) > p\left(1 - \sum_{\substack{j=1\\ a\in\{1,2\}}}^{m} \delta_j^a\right) + \sum_{\substack{j=1\\ a\in\{1,2\}}}^{m} \delta_j^a \cdot \left(p - \frac{3p+4}{4\cdot \sum\limits_{\substack{j=1\\ a\in\{1,2\}}}^{m} \delta_j^a}\right) = \frac{p}{4} - 1.$$

We apply Lemma 5 and we get

$$Re\sqrt{\frac{z[I_p^1(z)]'}{I_p^1(z)}} > \frac{\sqrt{p}}{2} \quad (z \in U).$$

Letting $j = p = 1$, $\delta_1^1 = \delta^1 \in \mathbb{C}$, $\delta_1^2 = \delta^2 \in \mathbb{C}$ and $D^{n,\kappa}_{\lambda_1,\lambda_2} f_1(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n)$ in Theorem 5, we have:

Corollary 10. *If $f \in \mathscr{A}$, $a \in \{1,2\}$, satisfies the following condition*

$$Re\left[\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}}\right] > -\frac{3}{4\cdot\sum\limits_{a\in\{1,2\}}^{m}\delta^a} \quad (z \in U),$$

then

$$Re\sqrt{\frac{z[I^1_1(z)]'}{I^1_p(z)}} > \frac{1}{2} \quad (z \in U),$$

Moreover, if we take $\delta_1^1 = \delta_1^2 = \delta \in \mathbb{C}$ in Corollary 10, we have the following result:

Corollary 11. *If $f \in \mathscr{A}$, $\delta \in \mathbb{C}$, $a \in \{1,2\}$, satisfies the condition*

$$Re\left[\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}}\right] > -\frac{3}{4\cdot\sum\limits_{a\in\{1,2\}}^{m}\delta} \quad (z \in U),$$

then

$$Re\sqrt{\frac{z[I^1_1(z)]'}{I^1_p(z)}} > \frac{1}{2} \quad (z \in U),$$

Furthermore, if we take $\delta = 1$ in Corollary 11, we have the following result:

Corollary 12. *If $f \in \mathscr{A}$, $\delta \in \mathbb{C}$, $a \in \{1,2\}$, satisfies the condition*

$$Re\left[\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}}\right] > -\frac{3}{4} \quad (z \in U),$$

then

$$Re\sqrt{\frac{z[I^1_1(z)]'}{I^1_p(z)}} > \frac{1}{2} \quad (z \in U),$$

Remark 7. *The sufficient conditions for the operator $I^2_p(z)$ of form* (10) *can be obtained in a similar way.*

3.2. Strong Starlikeness of the Integral Operator $I^1_p(z)$

Theorem 6. *Let $\delta^a_j, \gamma_a \in \mathbb{C}$, $a \in \{1,2\}$, $j = \overline{1,m}$. If $f_j \in \mathscr{A}_p$ for all $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, satisfies*

$$Re\left[\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(z^n))^{2\gamma_a-1}}\right] > p - \frac{\gamma}{2\cdot\sum\limits_{\substack{j=1\\ a\in\{1,2\}}}^{m}\delta^a_j} \quad (z \in U), \tag{23}$$

then I_p^1 *is strongly starlike of order* γ $(0<\gamma\le 1)$ *in* U*, where* $D^{n,\kappa,1}_{\lambda_1,\lambda_2}f_j(z^n)=(D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2}f_j(z^n)=D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n)$*,* $z\in U$*,* $D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n)$ *is of form* (6).

Proof. Applying Lemma 6 and making use of (14), it follows that I_p^1 is strongly starlike of order γ $(0<\gamma\le 1)$ in U.

Letting $j=p=1$, $\delta_1^1=\delta^1\in\mathbb{C}$, $\delta_1^2=\delta^2\in\mathbb{C}$ and $D^{n,\kappa}_{\lambda_1,\lambda_2}f_1(z^n)=D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n)$ in Theorem 5, we have:

Corollary 13. *If* $f\in\mathscr{A}$*,* $a\in\{1,2\}$*, satisfies the following condition*

$$Re\left[\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}}\right]>1-\frac{\gamma}{2\cdot\sum\limits_{a\in\{1,2\}}^{m}\delta^a}\quad(z\in U),$$

then $\left\{\beta\int\limits_0^z\prod\limits_{a\in\{1,2\}}\left[\frac{(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(t^n))^{2\gamma_a-1}}{t}\right]^{\delta^a}dt\right\}^{\frac{1}{\beta}}$ *is strongly starlike of order* γ $(0<\gamma\le 1)$ *in* U*, where* $D^{n,\kappa,1}_{\lambda_1,\lambda_2}f(z^n)=(D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2}f(z^n)=D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n)$*,* $z\in U$*,* $z\in U$*,* $z\in U$*,* $D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n)$ *is of form* (6).

Moreover, if we take $\delta_1^1=\delta_1^2=\delta\in\mathbb{C}$ in Corollary 13, we have the following result:

Corollary 14. *If* $f\in\mathscr{A}$*,* $\delta\in\mathbb{C}$*,* $a\in\{1,2\}$*, satisfies the condition*

$$Re\left[\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}}\right]>1-\frac{\gamma}{2\delta}\quad(z\in U),$$

then $\left\{\beta\int\limits_0^z\prod\limits_{a\in\{1,2\}}\left[\frac{(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(t^n))^{2\gamma_a-1}}{t}\right]^{\delta}dt\right\}^{\frac{1}{\beta}}$ *is strongly starlike of order* γ $(0<\gamma\le 1)$ *in* U*, where* $D^{n,\kappa,1}_{\lambda_1,\lambda_2}f(z^n)=(D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n))'$ *and* $D^{n,\kappa,2}_{\lambda_1,\lambda_2}f(z^n)=D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n)$*,* $z\in U$*,* $D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n)$ *is of form* (6).

Remark 8. *The strong starlikeness of the integral operator* $I_p^2(z)$ *of form* (10) *can be obtained in a similar way.*

Example 1. *Let* $\beta=0$ *in* $D^{\beta}_{\lambda}f(z)$ *of form* (4) *(or* (5)*),* $n=1$*. So we have that* $D^0_\lambda f(z)=f(z)$*,* $\forall\lambda\ge 0, f(z)$ *of form* (1) *(or* (2)*). We will use this form of the integral operator, where the function* f *is of form* (1) *with respect to the operator* (10)*. For further simplification, we consider that* $\gamma_1=\gamma_2=1$*, and* $\delta=1$

If $\chi=1$*,* $\delta_j^{=}0$*,* $\forall j=\overline{1,m}$*,* $m\in\mathbb{N}-\{0\}$ *and we consider* $\delta_j^2=\alpha_j$*,* $\forall j=\overline{1,m}$*,* $m\in\mathbb{N}-\{0\}$*, we obtain the operator* $F_p(z)=\int\limits_0^z pt^{p-1}\prod\limits_{j=1}^{n}\left(\frac{f_j(t)}{t^p}\right)^{\alpha_j}dt$*, which is recently studied in [7].*

Remark 9. *There are other integral operators in literature for whom the sufficient conditions for these operators to be p-valently starlike, p-valently close-to-convex, uniformly p-valent close-to-convex and strongly starlike of order* τ $(0 < \tau \leq 1)$ *in* U *(open unit disk) is covered by this paper (e.g. see [4, 9, 13]).*

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References

[1] M. Acu, I. Dorca, and S. Owa. *On some starlike functions with negative coefficients*, Proceedings of the Interational Coference on Theory and Applications of Mathematics and Informatics, ICTAMI 2011, Alba Iulia. 2011.

[2] M. Acu and S. Owa. *Note on a class of starlike functions*, Proceeding Of the International Short Joint Work on Study on Calculus Operators in Univalent Function Theory - Kyoto, 1-10. 2006.

[3] H.A. Al-Karsani and S.S. Al-Hajiry. *A note on certain inequalities for p-valent functions*, J. Inequal. Pure Appl. Math., 9(3), Art. 90. 2008.

[4] D. Breaz. *Integral operators on univalent function spaces*, Ed. Acad. Române, Bucureşti, 159-203. 2004.

[5] I. Dorca, M. Acu, and D. Breaz. *Note on Neighborhoods of Some Classes of Analytic Functions with Negative Coefficients*, ISRN Mathematical Analysis, vol. 2011, Article ID 610549, 7 pages, 2011. doi:10.5402/2011/610549. 2011.

[6] J. Dziok. *Applications of the Jack lemma*, Acta Math. Hungar., 105(1-2), 93-102. 2004.

[7] B. A. Frasin. *Integral Operators of p-valent Functions*, J. Inequal. Pure Appl. Math., 10(4), Art. 109. 2009.

[8] A.W. Goodman. *On uniformly starlike functions*, J. Math. Anal. Appl., 55, 364-370. 1991.

[9] Y.J. Kim and E.P. Merkes. *On an integral of powers of a spirallike function*, Kyungpook Mathematical Journal, 12, 249-252. 1972.

[10] M. Nunokawa. *On the multivalent functions*, Indian J. Pure Appl. Math., 20, 577-582. 1989.

[11] M. Nunokawa, S. Owa and A. Ikeda. *On the strongly starlikeness of multivalently convex functions of order* α, IJMMS, 28(1), 51-55. 2001.

[12] S. Owa. *On the Nunokawas conjecture for multivalent functions*, Bull. Austral. Math. Soc., 41, 301-305. 1990.

[13] N.N. Pascu and V. Pescar. *On the integral operators of Kim-Merkes and Pfaltzgraff*, Mathematica, 32(55)(2), 185-192. 1990.

[14] R.K. Raina and I.B. Bapna. *Inequalities defining ceratin subclasses of analytic functions involving fractional calculus operators*, J. Inequal. Pure Appl. Math., 5(2), Art. 28. 2004.