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Some Classes of Analytic Functions With Respect To Symmetric Conjugate Points

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Abstract. In this paper, the author introduces the notion of $(2j, k)$-symmetric conjugate functions. Several new classes of analytic functions with respect to $(2j, k)$-symmetric conjugate points are introduced. Inclusion relations, integral representation and conditions for starlikeness are the main results.

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1. Introduction, Definitions And Preliminaries

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad a_n \geq 0, \tag{1}$$

which are analytic in the open disc $\mathscr{U} = \{z \in \mathbb{C} \setminus \mid z \mid < 1\}$ and $\mathscr{S}$ be the class of functions $f \in \mathscr{A}$ which are univalent in $\mathscr{U}$.

We denote by $\mathscr{S}^*$, $\mathscr{C}$, $\mathscr{K}$ and $\mathscr{C}^*$ the familiar subclasses of $\mathscr{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathscr{U}$. Our favorite references of the field are [3, 4] which covers most of the topics in a lucid and economical style.

Let $f(z)$ and $g(z)$ be analytic in $\mathscr{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathscr{U}$, if there exists an analytic function $w(z)$ in $\mathscr{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathscr{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathscr{U}) \subset g(\mathscr{U})$.

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A function $f(z) \in \mathscr{A}$ is in the class $\mathscr{S}_s^k(\phi)$ if and only if it satisfies the condition

$$\frac{zf'(z)}{f_k(z)} \prec \phi(z), \quad (z \in \mathscr{U}),$$

where $\phi \in \mathscr{P}$, k is a fixed positive integer and $f_k(z)$ is defined by the following equality

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{-\nu} z) \qquad (\varepsilon = exp(2\pi/k); z \in \mathscr{U}). \tag{2}$$

The class $\mathscr{S}_s^k(\phi)$ is called the class of functions starlike with respect to k-symmetric points. Similarly, a function $f \in \mathscr{A}$ is said to be in $\mathscr{C}_s^k(\phi)$ of functions convex with respect to k-symmetric points if and only if

$$\frac{\left(zf'(z)\right)'}{f_k'(z)} \prec \phi(z), \quad (z \in \mathscr{U}),$$

where $\phi \in \mathscr{P}$, k is a fixed positive integer and $f_k(z)$ is defined by the following equality (2). The classes $\mathscr{S}_s^k(\phi)$ and $\mathscr{C}_s^k(\phi)$ were introduced recently by Wang, Gao and Yuan in [7].

Let k be a positive integer and $j = 0, 1, 2, \ldots (k-1)$. A function $f \in \mathscr{A}$ is said to be (j, k)-symmetrical if for each $z \in \mathscr{U}$

$$f(\varepsilon z) = \varepsilon^j f(z), \tag{3}$$

where $\varepsilon = \exp(2\pi i/k)$. The family of (j, k)-symmetrical functions will be denoted by $\mathscr{F}_k^j$. We observe that $\mathscr{F}_2^1, \mathscr{F}_2^0$ and $\mathscr{F}_k^1$ are well-known families of odd functions, even functions and k-symmetrical functions respectively.

Also let $f_{j,k}(z)$ be defined by the following equality

$$f_{j,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \frac{f(\varepsilon^\nu z)}{\varepsilon^{\nu j}}, \quad (f \in \mathscr{A}; k = 1, 2, \ldots; j = 0, 1, 2, \ldots (k-1)), \tag{4}$$

where ν is an integer.

It is obvious that $f_{j,k}(z)$ is a linear operator from $\mathscr{U}$ into $\mathscr{U}$. The notion of (j, k)-symmetric functions was introduced and studied by P. Liczberski and J. Połubiński in [5].

Al-Amiri, Coman and Mocanu in [1] introduced and investigated a class of functions starlike with respect to 2k-symmetric conjugate points, which satisfy the following inequality

$$Re\left\{\frac{zf'(z)}{f_{2k}(z)}\right\} > 0, \quad (z \in \mathscr{U}),$$

where k is a fixed positive integer and $f_{2k}(z)$ is defined by the following equality

$$f_{2k}(z) = \frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu} f(\varepsilon^{-\nu} z) + \varepsilon^{\nu}\overline{f(\varepsilon^{-\nu}\bar{z})}\right]. \tag{5}$$

The class of such functions is denoted by $\mathcal{S}_{sc}^{k}$.

Now we introduce the concept of analytic functions with respect to $2jk$-symmetric conjugate points. For fixed positive integers j and k, let $f_{2j,k}(z)$ be defined by the following equality

$$f_{2j,k}(z)=\frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu j}f(\varepsilon^{\nu}z)+\varepsilon^{\nu j}\overline{f(\varepsilon^{\nu}\bar{z})}\right], \quad (f\in\mathcal{A}). \tag{6}$$

If ν is an integer, then the following identities follow directly from (6):

$$\begin{aligned} f_{2j,k}^{'}(z)&=\frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu j+\nu}f^{'}(\varepsilon^{\nu}z)+\varepsilon^{\nu j-\nu}\overline{f^{'}(\varepsilon^{\nu}\bar{z})}\right]\\ f_{2j,k}^{''}(z)&=\frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu j+2\nu}f^{''}(\varepsilon^{\nu}z)+\varepsilon^{\nu j-2\nu}\overline{f^{''}(\varepsilon^{\nu}\bar{z})}\right]. \end{aligned} \tag{7}$$

and

$$\begin{aligned} f_{2j,k}(\varepsilon^{\nu}z)=\varepsilon^{\nu j}f_{2j,k}(z), \quad f_{2j,k}(z)=\overline{f_{2j,k}(\overline{z})}\\ f_{2j,k}^{'}(\varepsilon^{\nu}z)=\varepsilon^{\nu j-\nu}f_{2j,k}^{'}(z), \quad f_{2j,k}^{'}(\overline{z})=\overline{f_{2j,k}^{'}(z)} \end{aligned} \tag{8}$$

Motivated by $\mathcal{S}_{sc}^{k}$, we now introduce the following.
A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{S}_{sc}^{(j,k)}(\phi)$ if and only if it satisfies the condition

$$\frac{zf^{'}(z)}{f_{2j,k}(z)}\prec\phi(z) \quad (z\in\mathcal{U}), \tag{9}$$

where $\phi(z)\in\mathcal{P}$, the class of functions with positive real part and $f_{2j,k}(z)$ is defined by the equality (6). We call the functions $f\in\mathcal{A}$ that satisfies the condition (9) to be starlike with respect to $2jk$-symmetric points.

Similarly, let $\mathcal{C}_{sc}^{(j,k)}(\phi)$ denote the class of functions in $\mathcal{S}$ satisfying the condition

$$\frac{\left(zf^{'}(z)\right)^{'}}{f_{2j,k}^{'}(z)}\prec\phi(z), \tag{10}$$

$$(z\in\mathcal{U};\, k=1,2,\ldots;\, j=0,1,2,\ldots(k-1)),$$

where $\phi\in\mathcal{P}$.

Remark 1. *The notion of $(2j,k)$-symmetric conjugate function is a generalization of the notion of even, odd and $2k$-symmetric conjugate functions. For different choices of the parameters j,k and the function $\phi(z)$, the classes $\mathcal{S}_{sc}^{(j,k)}(\phi)$ and $\mathcal{C}_{sc}^{(j,k)}(\phi)$ reduces to various other well-known and new subclasses of analytic functions.*

2. Main Results

Theorem 1. *If* $f \in \mathscr{S}_{sc}^{(j,k)}(\phi)$*, then* f *is univalent in* $\mathscr{U}$.

Proof. From the definition of $\mathscr{S}_{sc}^{(j,k)}(\phi)$,

$$Re\left(\frac{zf'(z)}{f_{2j,k}(z)}\right) > 0, \tag{11}$$

since $Re\{\phi(z)\} > 0$. If we replace z by $\varepsilon^{\nu}z$ in (11), then (11) will be of the form

$$Re\left\{\frac{\varepsilon^{\nu}zf'(\varepsilon^{\nu}z)}{f_{2j,k}(\varepsilon^{\nu}z)}\right\} > 0, \quad (z \in \mathscr{U};\ \nu = 0, 1, 2, \ldots, k-1). \tag{12}$$

From inequality (12), we have

$$Re\left\{\frac{\overline{\varepsilon^{\nu}\overline{z}}\,\overline{f'(\varepsilon^{\nu}\overline{z})}}{\overline{f_{2j,k}(\varepsilon^{\nu}\overline{z})}}\right\} > 0, \quad (z \in \mathscr{U};\ \nu = 0, 1, 2, \ldots, k-1). \tag{13}$$

Using the equality (8), the inequalities (12) and (13) can be rewritten as

$$Re\left\{\frac{\varepsilon^{\nu-\nu j}zf'(\varepsilon^{\nu}z)}{f_{2j,k}(z)}\right\} > 0, \quad (z \in \mathscr{U};\ \nu = 0, 1, 2, \ldots, k-1), \tag{14}$$

and

$$Re\left\{\frac{\varepsilon^{\nu j-\nu}\, z\overline{f'(\varepsilon^{\nu}\overline{z})}}{f_{2j,k}(z)}\right\} > 0, \quad (z \in \mathscr{U};\ \nu = 0, 1, 2, \ldots, k-1). \tag{15}$$

Adding the inequalities (14) and (15), we get

$$Re\left\{\frac{z\left[\varepsilon^{\nu-\nu j}f'(\varepsilon^{\nu}z) + \varepsilon^{\nu j-\nu}\,\overline{f'(\varepsilon^{\nu}\overline{z})}\right]}{f_{2j,k}(z)}\right\} > 0, \quad (z \in \mathscr{U};\ \nu = 0, 1, 2, \ldots, k-1). \tag{16}$$

Let $\nu = 0, 1, 2, \ldots, k-1$ in (13) respectively and summing them, we get

$$Re\left\{\frac{z\left[\frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu j+\nu}f'(\varepsilon^{\nu}z) + \varepsilon^{\nu j-\nu}\overline{f'(\varepsilon^{\nu}\overline{z})}\right]\right]}{f_{2j,k}(z)}\right\} > 0, \quad (z \in \mathscr{U}),$$

or equivalently,

$$Re\left(\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right) > 0, \quad (z \in \mathscr{U}),$$

that is $f_{2j,k}(z) \in \mathscr{S}^*$. Using this together with the condition (11) shows the functions in $\mathscr{S}_{sc}^{(j,k)}(\phi)$ are close-to-convex in $\mathscr{U}$. It is well-known that the class of close-to-convex functions are univalent, hence functions which are starlike with respect to $(2j, k)$-symmetric points are univalent.

Using arguments similar to those detailed in Theorem 1, we can prove next theorem.

Theorem 2. *If $f \in \mathscr{C}_{sc}^{(j,k)}(\phi)$, then $f_{2j,k}(z) \in \mathscr{C}$.*

Remark 2. *Using the condition (10) together with Theorem 2 shows the functions in $\mathscr{C}_{sc}^{(j,k)}$ are quasi-convex. It is well-known that the class of quasi-convex functions are univalent, hence functions which are convex with respect to $(2j, k)$-symmetric points are univalent.*

Theorem 3. *Let $f \in \mathscr{S}_{sc}^{(j,k)}(\phi)$, then we have*

$$f_{2j,k}(z) = z\exp\left\{\frac{1}{2k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[\phi\left(w(\varepsilon^\nu\zeta)\right) + \overline{\phi\left(w(\varepsilon^\nu\overline{\zeta})\right)} - 2\right]d\zeta\right\} \tag{17}$$

where $f_{2j,k}(z)$ defined by equality (6), $w(z)$ is analytic in $\mathscr{U}$ and $w(0) = 0$, $|\, w(z)\,| < 1$.

Proof. Let $f \in \mathscr{S}_{sc}^{(j,k)}(\phi)$, from the definition of $\mathscr{S}_{sc}^{(j,k)}(\phi)$, we have

$$\frac{zf'(z)}{f_{2j,k}(z)} = \phi\left(w(z)\right), \tag{18}$$

where $w(z)$ is analytic in $\mathscr{U}$ and $w(0) = 0$, $|\, w(z)\,| < 1$. Substituting z by $\varepsilon^\nu z$ in the equality (18) respectively ($\nu = 0, 1, 2, \ldots, k-1$, $\varepsilon^k = 1$), we have

$$\frac{\varepsilon^\nu z f'(\varepsilon^\nu z)}{f_{2j,k}(\varepsilon^\nu z)} = \phi\left(w(\varepsilon^\nu z)\right). \tag{19}$$

On simple computation, we get

$$\frac{\overline{\varepsilon^\nu\overline{z}}\,\overline{f'(\varepsilon^\nu\overline{z})}}{\overline{f_{2j,k}(\varepsilon^\nu\overline{z})}} = \overline{\phi\left(w(\varepsilon^\nu\overline{z})\right)}. \tag{20}$$

Following the steps as in Theorem 1, we get

$$\frac{zf'_{2jk}(z)}{f_{2j,k}(z)} = \frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\phi\left(w(\varepsilon^\nu z)\right) + \overline{\phi\left(w(\varepsilon^\nu\overline{z})\right)}\right], \tag{21}$$

which can be rewritten as

$$\frac{f_{2jk}^{'}(z)}{f_{2j,k}(z)}-\frac{1}{z}=\frac{1}{2k}\sum_{\nu=0}^{k-1}\frac{1}{z}\left[\phi\left(w(\varepsilon^{\nu}z)\right)+\overline{\phi\left(w(\varepsilon^{\nu}\overline{z})\right)}-2\right]. \tag{22}$$

Integrating the equality (22), we have

$$\log\left\{\frac{f_{2j,k}(z)}{z}\right\}=\frac{1}{2k}\sum_{\nu=0}^{k-1}\int_0^z\frac{1}{\zeta}\left[\phi\left(w(\varepsilon^{\nu}\zeta)\right)+\overline{\phi\left(w(\varepsilon^{\nu}\overline{\zeta})\right)}-2\right]d\zeta, \tag{23}$$

or equivalently

$$f_{2j,k}(z)=z\exp\left\{\frac{1}{2k}\sum_{\nu=0}^{k-1}\int_0^z\frac{1}{\zeta}\left[\phi\left(w(\varepsilon^{\nu}\zeta)\right)+\overline{\phi\left(w(\varepsilon^{\nu}\overline{\zeta})\right)}-2\right]d\zeta\right\}.$$

This completes the proof of Theorem 3.

Theorem 4. *Let $f\in\mathscr{C}_{sc}^{(j,k)}(\phi)$, then we have*

$$f_{2j,k}(z)=\int_0^z\exp\left\{\frac{1}{2k}\sum_{\nu=0}^{k-1}\int_0^{\xi}\frac{1}{\zeta}\left[\phi\left(w(\varepsilon^{\nu}\zeta)\right)+\overline{\phi\left(w(\varepsilon^{\nu}\overline{\zeta})\right)}-2\right]d\zeta\right\}d\xi \tag{24}$$

where $f_{2j,k}(z)$ defined by equality (6), $w(z)$ is analytic in $\mathscr{U}$ and $w(0)=0$, $\mid w(z)\mid<1$.

3. Conditions for starlikeness with respect to Symmetric points

We now state the following result which will be used in the sequel.

Lemma 1. *[6, 2] Let the function q be univalent in the open unit disc $\mathscr{U}$ and θ and ϕ be analytic in a domain D containing $q(\mathscr{U})$ with $\phi(w)\neq 0$ when $w\in q(\mathscr{U})$. set $Q(z)=zq^{'}(z)\phi(q(z))$, $h(z)=\theta(q(z))+Q(z)$. Suppose that*

1. *Q is starlike univalent in $\mathscr{U}$, and*
2. *$Re\left(\frac{zh^{'}(z)}{Q(z)}\right)>0$ for $z\in\mathscr{U}$.*

If

$$\theta(p(z))+zp^{'}(z)\phi(p(z))\prec\theta(q(z))+zq^{'}(z)\phi(q(z)), \tag{25}$$

then $p(z)\prec q(z)$ and q is the best dominant.

Theorem 5. *Let the function $g(z)$ be convex univalent in $\mathscr{U}$ and also let*

$$Re\left\{\alpha\left(\frac{g(z)}{zg^{'}(z)}(g(z)-1)+1\right)+\beta\frac{g(z)}{zg^{'}(z)}\right\}>0 \tag{26}$$

and

$$h(z) = \alpha z g'(z) + \alpha g^2(z) + (\beta - \alpha) g(z),$$

where $\alpha > 0$, $\alpha + \beta > 0$. *If* $f \in \mathscr{A}$ *with* $\frac{f_{2j,k}(z)}{z} \neq 0$ *satisfies the condition*

$$\alpha \left\{ \frac{z^2 f''(z)}{f_{2j,k}(z)} - \frac{z^2 f'(z) f'_{j,k}(z)}{\left(f_{2j,k}(z)\right)^2} + \frac{z^2 \left(f'(z)\right)^2}{\left(f_{2j,k}(z)\right)^2} \right\} + \beta \frac{z f'(z)}{f_{2j,k}(z)} \prec h(z), \tag{27}$$

then $f \in \mathscr{S}_{sc}^{(j,k)}(g)$ *and* g *is the best dominant.*

Proof. Let the function p be defined by

$$p(z) = \frac{z f'(z)}{f_{2j,k}(z)} \qquad (z \in \mathscr{U};\, z \neq 0;\, f \in \mathscr{A}), \tag{28}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathscr{P}$. By a straight forward computation, we have

$$z p'(z) = \frac{z f'(z)}{f_{2j,k}(z)} + \frac{z^2 f''(z)}{f_{2j,k}(z)} - \frac{z^2 f'(z) f'_{j,k}(z)}{\left(f_{2j,k}(z)\right)^2}.$$

Thus by (27), we have

$$\alpha z p'(z) + \alpha p^2(z) + (\beta - \alpha) p(z) \prec h(z). \tag{29}$$

By setting

$$\theta(w) := \alpha w^2 + (\beta - \alpha) w \qquad \text{and} \qquad \phi(w) := \alpha, \tag{30}$$

it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}$ with $\phi(0) \neq 0$ in the w-plane. Also, by letting

$$Q(z) = z g'(z) \phi(g(z)) = \alpha z g'(z) \tag{31}$$

and

$$h(z) = \theta(g(z)) + Q(z) = \alpha (g(z))^2 + (\beta - \alpha) g(z) + \alpha z g'(z). \tag{32}$$

Since $g(z)$ is convex univalent in $\mathscr{U}$ it implies that $Q(z)$ is starlike univalent in $\mathscr{U}$. Further, we have

$$Re \frac{z h'(z)}{Q(z)} = Re \left\{ \alpha \left(\frac{g(z)}{z g'(z)} (g(z) - 1) + 1 \right) + \beta \frac{g(z)}{z g'(z)} \right\} > 0.$$

The assertion of the Theorem 5 now follows by application of Lemma 1.

Corollary 1. *If* $f \in \mathscr{A}$ *with* $\frac{f_{2j,k}(z)}{z} \neq 0$ *satisfies the condition*

$$\alpha \left\{ \frac{z^2 f^{''}(z)}{f_{2j,k}(z)} - \frac{z^2 f^{'}(z) f^{'}_{j,k}(z)}{\left(f_{2j,k}(z)\right)^2} + \frac{z^2 \left(f^{'}(z)\right)^2}{\left(f_{2j,k}(z)\right)^2} \right\} + \beta \frac{z f^{'}(z)}{f_{2j,k}(z)} \prec h(z),$$

where

$$h(z) = \frac{a[\alpha(a-b) + \beta b]z^2 + [2\alpha(a-b) + \beta(a+b)]z + \beta}{(1+bz)^2},$$
$$-1 \leq b < a \leq 1$$

and

$$\beta \geq 2\alpha^2 \left(\frac{\mid b \mid}{1 + \mid b \mid} - \frac{1-a}{1-b} \right)$$

then $f \in \mathscr{S}_{sc}^{(j,k)}\left(\frac{1+az}{1+bz}\right)$.

Proof. We let $g(z) = \frac{1+az}{1+bz}$, in Theorem 5. Clearly $g(z)$ is convex univalent in $\mathscr{U}$. Hence the proof of the Corollary follows from Theorem 5.

Corollary 2. *If* $f \in \mathscr{A}$ *with* $\frac{f_{2j,k}(z)}{z} \neq 0$, $z \in \mathscr{U}$ *and*

$$D = \mathbb{C} \setminus \left\{ z \in \mathbb{C} : Re\, z \leq -\frac{1}{2}, Im\, z = 0 \right\},$$

then

$$\frac{z^2 f^{''}(z)}{f_{2j,k}(z)} - \frac{z^2 f^{'}(z) f^{'}_{j,k}(z)}{\left(f_{2j,k}(z)\right)^2} + \frac{z^2 \left(f^{'}(z)\right)^2}{\left(f_{2j,k}(z)\right)^2} + \frac{z f^{'}(z)}{f_{2j,k}(z)} \in D \quad \Longrightarrow f \in \mathscr{S}_{sc}^{(j,k)}.$$

Proof. If we let $\alpha = 1$, $\beta = 1$ and $g(z) = \frac{1+z}{1-z}$, in Theorem 5. It follows that $h(z)$ is convex with respect to the point $u = -1/2$. Hence the proof of the Corollary.

Corollary 3. *If* $f \in \mathscr{A}$ *with* $\frac{f_{2j,k}(z)}{z} \neq 0$, $z \in \mathscr{U}$, *satisfy the condition*

$$\Phi_k^j(z) = \alpha \left\{ \frac{z^2 f^{''}(z)}{f_{2j,k}(z)} - \frac{z^2 f^{'}(z) f^{'}_{j,k}(z)}{\left(f_{2j,k}(z)\right)^2} + \frac{z^2 \left(f^{'}(z)\right)^2}{\left(f_{2j,k}(z)\right)^2} \right\} + \frac{z f^{'}(z)}{f_{2j,k}(z)} \prec 1 + \delta z,$$

where $\delta = \mu(2\alpha + 1 - \alpha\mu)$ *and* $0 < \mu \leq \left(1 + \frac{1}{2\alpha}\right)$. *Then*

$$\frac{z f^{'}(z)}{f_{2j,k}(z)} \prec 1 + \mu z.$$

Proof. If we let $\beta = 1$ and $g(z) = 1 + \mu z$ in Theorem 5. Then $h(z)$ will be of the form $h(z) = 1 + (2\alpha + 1)\mu z + \alpha\mu^2 z^2$. For $|z| = 1$,

$$|h(z) - 1| = \mu \left|2\alpha + 1 + \alpha\mu z\right| \geq \mu \left(2\alpha + 1 - \alpha\mu\right).$$

If we put $\delta = \left(2\alpha + 1 - \alpha\mu\right)$, then from the above inequality it follows that $h(z)$ is superordinate to $1 + \delta z$. Hence the proof of the Corollary.

If we let $\alpha = 1$ and $\mu = 1$ in the Corollary 3, then we have the following result.

Corollary 4. *If $f \in \mathscr{A}$ with $\frac{f_{2j,k}(z)}{z} \neq 0$, $z \in \mathscr{U}$, then*

$$\left|\frac{zf'(z)}{f_{2j,k}(z)}\left(1 + \frac{f''(z)}{f'(z)} - \frac{zf'_{j,k}(z)}{f_{2j,k}(z)} + \frac{zf'(z)}{f_{2j,k}(z)}\right) - 1\right| < 2 \quad (z \in \mathscr{U})$$

implies $\left|\frac{zf'(z)}{f_{2j,k}(z)} - 1\right| < 1$, *for all* $z \in \mathscr{U}$.

It is well-known that a function $f \in \mathscr{A}$ is called strongly-starlike of order λ, $0 < \lambda \leq 1$, if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \lambda\frac{\pi}{2}, \quad (z \in \mathscr{U}),$$

and we denote by $\mathscr{S}\mathscr{S}^*(\lambda)$ the class of such functions. Similarly, we denote the class of strongly-starlike functions of order λ with respect to $(2j, k)$-symmetric points by $\mathscr{S}\mathscr{S}_{sc}^{(j,k)}(\lambda)$.

Now, we give the sufficient conditions for strongly-starlike of order λ with respect to $(2j, k)$-symmetric points

Corollary 5. *Let $0 < \lambda < 1$, and let*

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\lambda}\left[\frac{2\lambda z}{1-z^2} + \left(\frac{1+z}{1-z}\right)^{\lambda}\right].$$

If $f \in \mathscr{A}$ with $\frac{f_{2j,k}(z)}{z} \neq 0$, $z \in \mathscr{U}$, satisfies the condition

$$\frac{z^2 f''(z)}{f_{2j,k}(z)} - \frac{z^2 f'(z) f'_{j,k}(z)}{\left(f_{2j,k}(z)\right)^2} + \frac{z^2\left(f'(z)\right)^2}{\left(f_{2j,k}(z)\right)^2} + \frac{zf'(z)}{f_{2j,k}(z)} \prec h(z),$$

then $f \in \mathscr{S}\mathscr{S}_{sc}^{(j,k)}(\lambda)$.

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