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Equivalence between A Harmonic Form and A Closed Co-Closed Form in Both L^q and Non-L^q Spaces

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Abstract. For a differential k-form ω on a complete non-compact manifold, we establish an equivalent relation between a harmonic form and a closed co-closed form. We extend this equivalence from ω in L^2 spaces to ω with 2-balanced growth including L^2 spaces and non-L^2 spaces. Especially for a simple differential k-form $\bar{\omega}$ on a complete non-compact manifold, we generalize this equivalence from $\bar{\omega}$ in L^q spaces to $\bar{\omega}$ with 2-balanced growth including L^q spaces and non-L^q spaces for $2 \le q < 3$. Our research findings recapture the work of Andreotti and Vesentini. Our ideas and calculation methods in this paper could provide a new way of broadening L^q spaces to non-L^q spaces in a variety of energy for differential forms.

2010 Mathematics Subject Classifications: 53C23, 53C21

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1. Introduction

The study of a differential form ξ on a manifold has been one of the most active research topics in mathematical history. ξ can be classified according to different properties in the following directions:

(i) We can classify ξ by analyzing its vanishing properties in regard to an exterior differential operator d and/or an exterior co-differential operator d^* on a manifold M. For example, we define ξ as a closed form (i.e. $d\xi = 0$), a co-closed form (i.e. $d^*\xi = 0$), a harmonic form (i.e. $\triangle\xi = -(dd^* + d^*d)\xi = 0$), a p-pseudo-co-closed form (i.e. $d^*(|\xi|^{p-2}\xi) = 0$ for $p > 1$) and others.

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(ii) We can classify ξ by estimating its energy on a manifold M. Basically we define ξ with finite q-energy to be in L^q space (i.e. $\int_M |\xi|^q dv < \infty$) or ξ with infinite q-energy to be in non-L^q space (i.e. $\int_M |\xi|^q dv = \infty$) for some real number $q > 0$. Furthermore, various energy growths for ξ can be defined in broad spaces which consist of both L^q and non-L^q spaces. For example, definitions of p-balanced growth (including p-finite growth, p-mild growth, p-obtuse growth, p-moderate growth, and p-small growth) and its counter-part p-imbalanced growth (including respectively p-infinite growth, p-severe growth, p-acute growth, p-immoderate growth, and p-large growth) can be found in Wei-Li-Wu's[7] work. Spaces defined by p-balanced growth contain L^q spaces and extend from L^q spaces to Non-L^q spaces. The Liouville-type theorems for ξ (i.e. $\xi \equiv$ constant) and the vanishing theorems for ξ (i.e. $\xi \equiv 0$) were studied in L^q spaces by Zhang[12] in 2001. Recently, Wu and Li[10] extended the Liouville-type results for ξ from L^q spaces to broader spaces with p-balanced growth.

(iii) We can classify ξ by studying its manifold on either a compact manifold $\bar{M}$ or a complete non-compact manifold M. The theory of ξ on $\bar{M}$ has been almost completed and studied intensively by mathematicians in history. Compared with the theory of ξ on $\bar{M}$, the theory of ξ on M has been actively studied. Currently, mathematicians are spending more time on studying this challenging research theory of ξ on M.

In particular, the theory of a harmonic form ξ (i.e. $\triangle\xi = 0$) has been a very popular and challenging research topic for mathematicians. On a compact manifold $\bar{M}$, the equivalence between a harmonic form and a closed co-closed form is a well-known finding from the Stokes' Theorem. However, on a complete non-compact manifold M, the equivalence is not always true. In 1965, Andreotti and Vesentini[2] proved this equivalent relation when a harmonic form ξ has finite q-energy growth in L^q space for $q = 2$ on M. After that, many mathematicians have expressed interests in studying a harmonic form with finite q-energy in L^q spaces. In 1981, Greene and Wu[3] explored the vanishing theorem for a harmonic 1-form in L^q space on M with non-negative Ricci curvature. In 2015, Wei and Wu[8] generalized the vanishing theorem for a harmonic form from L^q space to a broader space with p-balanced growth for $p = 2$, which includes both L^q and non-L^q spaces.

In this paper, for a differential k-form ω on a complete non-compact manifold, we establish an equivalent relation between a harmonic form (i.e. $\triangle\omega = 0$) and a closed co-closed form (i.e. $d\omega = d^*\omega = 0$), that is:

$$\triangle\omega = 0 \Longleftrightarrow d\omega = d^*\omega = 0 \tag{1}$$

where ω has 2-balanced growth of

$$\liminf_{r\to\infty} \frac{\int_{B(x;r)} |\omega|^q dv}{r^2} < \infty \qquad \text{(i.e. 2-finite growth)} \tag{2}$$

and q denotes a real number specified in the context when it is used. First, we extend this equivalence from ω in L^2 spaces to ω with 2-balanced growth (2) for $q = 2$ including both L^2 spaces and non-L^2 spaces. Furthermore, we verify this equivalence from ω in L^q spaces to ω with 2-balanced growth (2) including both L^q spaces and non-L^q spaces for $2 < q < 3$ when ω satisfies

$$\langle d|\omega|^2 \wedge \omega, d\omega\rangle \leq 2|\omega|^2|d\omega|^2. \tag{3}$$

In addition, especially for a simple differential k-form $\bar{\omega}$, we broaden this equivalence from $\bar{\omega}$ in L^q spaces to $\bar{\omega}$ with 2-balanced (2) including both L^q spaces and non-L^q spaces for $2 \leq q < 3$. In particular, we recapture the work of Andreotti and Vesentini. Our research ideas and calculation methods in this paper can provide a new way of working on a differential form with a variety of energy in broader spaces.

2. Materials and Methods

In this section, we will give definitions of a harmonic form, a closed form, and a co-closed form on a complete non-compact manifold as the basic knowledge. In our study, an assumption of energy growth (2) satisfies the definition of p-finite growth for $p = 2$ as one of the five cases of p-balanced growth for a differential form ξ. In addition, we will give Lemma 1 as well as its proof. We have applied this lemma for proving the main theorems in Section 3.

2.1. Preliminary

Throughout this paper, we assume that M is a complete non-compact n-manifold with volume element dv. We denote the geodesic ball of radius r centered at x_0 in M by $B(x_0; r)$ or $B(r)$, and its boundary by $\partial B(x_0; r)$ or $\partial B(r)$. Let $\xi : V \to M$ be a vector bundle over M. Set $\mathcal{A}^k(\xi) = \Gamma(\wedge^k T^*M \otimes V)$ the space of smooth k-forms ξ on M with values in the vector bundle and let $d : \mathcal{A}^k(\xi) \to \mathcal{A}^{k+1}(\xi)$ be the exterior differential operator and $d^* : \mathcal{A}^k(\xi) \to \mathcal{A}^{k-1}(\xi)$ be its co-differential operator given by

$$d^* = (-1)^{nk+n+1} \star d\star$$

where $\star : \mathcal{A}^k \to \mathcal{A}^{n-k}$ is linear with respect to multiplication by functions. In particular, if $\nu \in \mathcal{A}^1(\xi)$, d^* is defined by $d^*\nu = -\text{trace}\nabla\nu = -\text{div}\nu$. The Hodge Laplacian Δ is defined on the V-valued differential forms by

$$\Delta = -(dd^* + d^*d) : \mathcal{A}^k(V) \to \mathcal{A}^k(V)$$

Thus, by our convention, in the space of smooth real-valued functions f on M, the Hodge Laplacian agrees with the connection Laplacian, or the Laplace-Beltrami operator; that is,

$$\Delta f = -d^*df = \text{trace}\nabla df = \text{div}(\nabla f).$$

More details can be found in [9].

Definition 1. *A differential k-form ξ is said to be harmonic if* $\Delta\xi = -(dd^* + d^*d)\xi = 0$, *closed if* $d\xi = 0$, *and co-closed if* $d^*\xi = 0$.

Definition 2. *A differential form ξ of degree k is called a simple differential k-form if there are differential 1-forms $\xi_1, \cdots, \xi_k$ such that $\xi = \xi_1 \wedge \cdots \wedge \xi_k$.*

We recall the definition of p-balanced growth as follows (cf.[7, 6]):

Definition 3. *A function or a differential form f has* p-finite growth (*or,* is p-finite) *if there exists $x_0 \in M$ such that*

$$\liminf_{r\to\infty} \frac{1}{r^p} \int_{B(x_0;r)} |f|^q dv < \infty$$

and has p-infinite growth (*or,* is p-infinite) *otherwise.*

A function or a differential form f has p-mild growth (*or,* is p-mild) *if there exist $x_0 \in M$, and a strictly increasing sequence of $\{r_j\}_0^\infty$ going to infinity, such that for every $l_0 > 0$, we have*

$$\sum_{j=\ell_0}^{\infty} \left(\frac{(r_{j+1} - r_j)^p}{\int_{B(x_0;r_{j+1})\setminus B(x_0;r_j)} |f|^q dv} \right)^{\frac{1}{p-1}} = \infty\,,$$

and has p-severe growth (*or,* is p-severe) *otherwise.*

A function or a differential form f has p-obtuse growth (*or,* is p-obtuse) *if there exists $x_0 \in M$ such that for every $a > 0$, we have*

$$\int_a^\infty \left(\frac{1}{\int_{\partial B(x_0;r)} |f|^q ds} \right)^{\frac{1}{p-1}} dr = \infty\,,$$

and has p-acute growth (*or,* is p-acute) *otherwise.*

A function or a differential form f has p-moderate growth (*or,* is p-moderate) *if there exist $x_0 \in M$, and $F(r) \in \mathcal{F}$, such that*

$$\limsup_{r\to\infty} \frac{1}{r^p F^{p-1}(r)} \int_{B(x_0;r)} |f|^q dv < \infty\,,$$

and has p-immoderate growth (*or,* is p-immoderate) *otherwise, where*

$$\mathcal{F} = \{F : [a,\infty) \longrightarrow (0,\infty) | \int_a^\infty \frac{dr}{rF(r)} = \infty \quad for \quad some \quad a \geq 0\}\,.$$

(*Notice that the functions in $\mathcal{F}$ are not necessarily monotone.*)

A function or a differential form f has p-small growth (*or,* is p-small) *if there exists $x_0 \in M$, such that for every $a > 0$, we have*

$$\int_a^\infty \left(\frac{r}{\int_{B(x_0;r)} |f|^q dv} \right)^{\frac{1}{p-1}} dr = \infty\,,$$

and has p-large growth (*or,* is p-large) *otherwise.*

In Definition 3, q denotes a real number whose value will be specified in the context in which Definition 3 is used.

ω satisfying the energy assumption of

$$\liminf_{r\to\infty} \frac{\int_{B(x;r)} |\omega|^q dv}{r^2} < \infty$$

is p-finite for $p = 2$. In particular, we see that ω in L^q space (i.e. $\int_M |\omega|^q dv < \infty$) must satisfy $\liminf_{r\to\infty} \frac{\int_{B(x;r)} |\omega|^q dv}{r^2} = 0 < \infty$.

2.2. Lemma

We begin with the following lemma:

Lemma 1. *Suppose ω is a differential k-form on an n-dimensional manifold M. Then, for any f differentiable on M, we have*

$$d^*(f\omega) = fd^*\omega + (-1)^{nk+n+1} \star (df \wedge \star\omega) \tag{4}$$

Proof.

$$\begin{aligned} d^*(f\omega) &= (-1)^{nk+n+1} \star d \star (f\omega) \\ &= (-1)^{nk+n+1} \star d(f \star \omega) \\ &= (-1)^{nk+n+1} \star (fd \star \omega + df \wedge \star\omega) \\ &= f(-1)^{nk+n+1} \star d \star \omega \\ &\quad +(-1)^{nk+n+1} \star (df \wedge \star\omega) \\ &= fd^*\omega + (-1)^{nk+n+1} \star (df \wedge \star\omega) \end{aligned}$$

where we apply the definition of d^* given by $d^* = (-1)^{nk+n+1} \star d\star$.

3. Results of Main Theorems

In this section, we will provide proofs of our main results in details. Three theorems will be stated.

Theorem 1. *On a complete non-compact manifold M, a differential k-form ω with 2-finite growth of*

$$\liminf_{r\to\infty} \frac{\int_{B(x;r)} |\omega|^q dv}{r^2} < \infty$$

for $q = 2$, is harmonic if and only if ω is closed and co-closed. In particular, a differential k-form in L^2 space is harmonic if and only if it is closed and co-closed.

Proof. ($\Leftarrow$) It is obvious that ω is harmonic if ω is both closed and co-closed.
($\Rightarrow$) Now we need to prove that ω is both closed and co-closed if ω is harmonic. Choose a smooth cut-off function $\psi(x)$ as in [5, (3.1)], i.e. for any $x_0 \in M$ and any pair of

positive numbers s, t with $0 < s < t$, a rotationally symmetric Lipschitz continuous non-negative function $\psi(x) = \psi(x; s, t)$ satisfies $\psi \equiv 1$ on $B(s)$, $\psi \equiv 0$ off $B(t)$, and $|\nabla\psi| \le \frac{C_1}{t-s}$, *a.e. on* $B(t)\backslash B(s)$, where $C_1 > 0$ is a constant (independent of x_0, s, t). By the harmonicity of ω, and the adjoint relationship of d and d^* on differential forms with compact support, and applying Lemma 1 for $d^*(\psi^2\omega)$ in which $f = \psi^2$, we have:

$$\begin{array}{rcl} 0 & = & \int_{B(t)} \langle \psi^2\omega, \Delta\omega\rangle dv \\ & = & \int_{B(t)} -\langle \psi^2\omega, d^*d\omega\rangle - \langle \psi^2\omega, dd^*\omega\rangle dv \\ & = & \int_{B(t)} -\langle d(\psi^2\omega), d\omega\rangle dv - \int_{B(t)} \langle d^*(\psi^2\omega), d^*\omega\rangle dv \\ & = & \int_{B(t)} -\langle 2\psi d\psi\wedge\omega + \psi^2 d\omega, d\omega\rangle dv \\ & & -\int_{B(t)} \langle (-1)^{nk+n+1} \star (d\psi^2 \wedge \star\omega), d^*\omega\rangle dv \\ & & -\int_{B(t)} \psi^2 |d^*\omega|^2 dv \\ & = & I + II + III. \end{array} \tag{5}$$

$$\begin{array}{rcl} I & \le & \int_{B(t)\backslash B(s)} 2\psi|d\psi||\omega||d\omega| dv - \int_{B(t)} \psi^2|d\omega|^2 dv \\ & \le & 2(\int_{B(t)\backslash B(s)} |d\psi|^2|\omega|^2 dv)^{\frac{1}{2}} (\int_{B(t)\backslash B(s)} \psi^2|d\omega|^2 dv)^{\frac{1}{2}} - \int_{B(t)} \psi^2|d\omega|^2 dv \end{array} \tag{6}$$

where we have used a generalized Hadmard Theorem $|d\psi \wedge \omega| \le |d\psi||\omega|$, and Cauchy-Schwarz inequality (applied to $\int_{B(t)\backslash B(s)} \psi|d\psi||\omega||d\omega|dv$). Similarly, we have the following estimates involved with the codifferential operator d^* :

$$\begin{array}{rcl} II & \le & 2\int_{B(t)\backslash B(s)} \psi|\omega||d\psi||d^*\omega| dv \\ & \le & 2\big(\int_{B(t)\backslash B(s)} |d\psi|^2|\omega|^2 dv\big)^{\frac{1}{2}} \big(\int_{B(t)\backslash B(s)} \psi^2|d^*\omega|^2 dv\big)^{\frac{1}{2}} \end{array} \tag{7}$$

where in the first step we have used

$$|(-1)^{nk+n+1} \star (d\psi^2 \wedge \star\omega)| = |d\psi^2 \wedge \star\omega| \le |d\psi^2|| \star \omega| = 2\psi|d\psi||\omega| \tag{8}$$

and in the second step we have applied Cauchy-Schwarz inequality to $\int_{B(t)\backslash B(s)} \psi|\omega||d\psi||d^*\omega|dv$. Substituting (6) and (7) into (5), we obtain the following inequality:

$$\begin{array}{rl} & \displaystyle\int_{B(t)} \psi^2|d\omega|^2 dv + \int_{B(t)} \psi^2|d^*\omega|^2 dv \\ \le & 2\big(\displaystyle\int_{B(t)\backslash B(s)} |d\psi|^2|\omega|^2 dv\big)^{\frac{1}{2}} \cdot \{\big(\int_{B(t)\backslash B(s)} \psi^2|d\omega|^2 dv\big)^{\frac{1}{2}} + \big(\int_{B(t)\backslash B(s)} \psi^2|d^*\omega|^2 dv\big)^{\frac{1}{2}}\} \\ \le & 2\sqrt{2}\big(\displaystyle\int_{B(t)\backslash B(s)} |d\psi|^2|\omega|^2 dv\big)^{\frac{1}{2}} \cdot \big(\int_{B(t)\backslash B(s)} \psi^2(|d\omega|^2 + |d^*\omega|^2) dv\big)^{\frac{1}{2}} \end{array} \tag{9}$$

where in the last step we have used

$$a^{\frac{1}{2}} + b^{\frac{1}{2}} \le \sqrt{2}(a+b)^{\frac{1}{2}} \qquad \text{for } a,\, b \ge 0$$

in which

$$a = \int_{B(t)\backslash B(s)} \psi^2|d\omega|^2 dv \quad \text{and}$$

$$b = \int_{B(t)\backslash B(s)} \psi^2 |d^*\omega|^2 dv.$$

Hence,

$$\begin{array}{rl} & \int_{B(t)} \psi^2(|d\omega|^2 + |d^*\omega|^2)dv \\ \leq & 2\sqrt{2}\big(\int_{B(t)\backslash B(s)} |d\psi|^2|\omega|^2 dv\big)^{\frac{1}{2}} \cdot \big(\int_{B(t)\backslash B(s)} \psi^2(|d\omega|^2 + |d^*\omega|^2)dv\big)^{\frac{1}{2}} \\ \leq & \frac{2\sqrt{2}C_1}{t-s}\big(\int_{B(t)\backslash B(s)} |\omega|^2 dv\big)^{\frac{1}{2}} \cdot \big(\int_{B(t)\backslash B(s)} \psi^2(|d\omega|^2 + |d^*\omega|^2)dv\big)^{\frac{1}{2}} \end{array} \tag{10}$$

Let $\{r_j\}$ be a strictly increasing sequence of positive real numbers going to infinity and define:

$$\begin{array}{rcl} A_j & = & \frac{1}{r_j^2}\int_{B(r_j)} |\omega|^2 dv \\ \varphi_j(x) & = & \psi(x; r_j, r_{j+1}) \\ Q_{j+1} & = & \int_{B(r_{j+1})} \varphi_j^2(|d\omega|^2 + |d^*\omega|^2)dv \\ C & = & (2\sqrt{2}C_1)^2. \end{array} \tag{11}$$

Using the above notations, substituting $s = r_j$, $t = r_{j+1}$ and $\psi = \varphi_j$ into (10), and squaring both sides of (10), via $0 \leq \varphi_{j-1} \leq \varphi_j$, we have,

$$Q_{j+1}^2 \leq C\left(\frac{r_{j+1}^2 A_{j+1} - r_j^2 A_j}{(r_{j+1} - r_j)^2}\right)\left(Q_{j+1} - Q_j\right) \tag{12}$$

We claim that $Q_j \to 0$ as $j \to \infty$. Choosing $\{r_j\}$ such that $r_{j+1} \geq 2r_j$ (that is, $r_{j+1} - r_j \geq \frac{1}{2}r_{j+1}$), then:

$$\frac{r_{j+1}^2 A_{j+1} - r_j^2 A_j}{(r_{j+1} - r_j)^2} \leq 4A_{j+1} \tag{13}$$

It follows from (12) and (13) that

$$0 \leq Q_{j+1} \leq 4CA_{j+1}. \tag{14}$$

Now we use the assumption that ω satisfies (2) for $q = 2$, there exists a constant $K > 0$, and a sequence $\{r_j\}$ with $r_{j+1} \geq 2r_j$, such that $A_{j+1} \leq K$. It follows from (14) that $\{Q_{j+1}(\leq 4CK)\}$ is an increasing sequence. In (12), summing over j, we have for $\forall N > 1$,

$$\sum_{j=1}^{N} Q_{j+1}^2 \leq 4CK(Q_{N+1} - Q_1) \leq 16C^2K^2.$$

Therefore, $Q_j \to 0$ as $j \to \infty$, which indicates $\int_M |d\omega|^2 dv = \int_M |d^*\omega|^2 dv = 0$. This shows that $d\omega = 0$ and $d^*\omega = 0$ almost everywhere. By the continuity, $d\omega = d^*\omega = 0$ on M. In particular, a differential k-form in L^2 space is harmonic if and only if it is closed and co-closed.

Theorem 2. *On a complete non-compact manifold M, a differential k-form ω satisfying:*

$$\langle d|\omega|^2 \wedge \omega, d\omega\rangle \leq 2|\omega|^2|d\omega|^2$$

and

$$\liminf_{r\to\infty} \frac{\int_{B(x;r)} |\omega|^q dv}{r^2} < \infty,$$

(i.e. 2-finite growth) for $2 < q < 3$, is harmonic if and only if ω is closed and co-closed. In particular, a differential k-form in L^q for $2 < q < 3$ satisfying $\langle d|\omega|^2 \wedge \omega, d\omega\rangle \leq 2|\omega|^2|d\omega|^2$ is harmonic if and only if it is closed and co-closed.

Proof. ($\Leftarrow$) It is obvious that ω is harmonic if ω is both closed and co-closed.
($\Rightarrow$) Now we need to prove that ω is both closed and co-closed if ω is harmonic. Choose a smooth cut-off function $\psi(x)$ as in [5, (3.1)], i.e. for any $x_0 \in M$ and any pair of positive numbers s, t with $0 < s < t$, a rotationally symmetric Lipschitz continuous non-negative function $\psi(x) = \psi(x; s, t)$ satisfies $\psi \equiv 1$ on $B(s)$, $\psi \equiv 0$ off $B(t)$, and $|\nabla\psi| \leq \frac{C_1}{t-s}$, *a.e. on* $B(t)\backslash B(s)$, where $C_1 > 0$ is a constant (independent of x_0, s, t). For $m > 2$, we have $\int_{B(t)}\langle\psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}\omega, \Delta\omega\rangle dv = 0$ due to the harmonicity of ω. Meanwhile, by the adjoint relationship of d and d^* on differential forms with compact support, and applying Lemma 1 for $d^*(\psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}\omega)$ in which $f = \psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}$, we have for any constant $\epsilon > 0$,

$$\begin{aligned}
0 &= \int_{B(t)}\langle\psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}\omega, \Delta\omega\rangle dv \\
&= \int_{B(t)} -\langle\psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}\omega, d^*d\omega\rangle - \langle\psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}\omega, dd^*\omega\rangle dv \\
&= \int_{B(t)} -\langle d(\psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}\omega), d\omega\rangle dv - \int_{B(t)}\langle d^*(\psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}\omega), d^*\omega\rangle dv \\
&= \int_{B(t)} -\langle 2\psi(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}d\psi \wedge \omega + \psi^2 d(|\omega|^2 + \epsilon)^{\frac{m}{2}-1} \wedge \omega + \psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}d\omega, d\omega\rangle dv \\
&\quad - \int_{B(t)} \psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}|d^*\omega|^2 dv \\
&\quad - \int_{B(t)}\langle(-1)^{nk+n+1} \star (d(\psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}) \wedge \star\omega), d^*\omega\rangle dv \\
&= I + II + III
\end{aligned} \tag{15}$$

Since $m > 2$,

$$\begin{aligned}
I &\leq \int_{B(t)\backslash B(s)} 2\psi(|\omega|^2 + \epsilon)^{\frac{m-1}{2}}|d\psi||d\omega|dv \\
&\quad + \int_{B(t)} |m-2|\psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}|d\omega|^2 dv \\
&\quad - \int_{B(t)} \psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}|d\omega|^2 dv \\
&\leq 2\Big(\int_{B(t)\backslash B(s)} |d\psi|^2(|\omega|^2 + \epsilon)^{\frac{m}{2}} dv\Big)^{\frac{1}{2}} \\
&\quad \cdot\Big(\int_{B(t)\backslash B(s)} \psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}|d\omega|^2 dv\Big)^{\frac{1}{2}} \\
&\quad +(|m-2| - 1)\int_{B(t)} \psi^2(|\omega|^2 + \epsilon)^{\frac{m}{2}-1}|d\omega|^2 dv
\end{aligned} \tag{16}$$

where we have used a generalized Hadmard Theorem $|d\psi \wedge \omega| \leq |d\psi||\omega|$, Cauchy-Schwarz inequality (applied to $\int_{B(t)\backslash B(s)} \psi(|\omega|^2+\epsilon)^{\frac{m-1}{2}}|d\psi||d\omega|dv$), and (3) (applied to $\int_{B(t)}\langle\psi^2 d(|\omega|^2+\epsilon)^{\frac{m}{2}-1} \wedge \omega, d\omega\rangle dv$)

$$|\langle d(|\omega|^2 + \epsilon) \wedge \omega, d\omega\rangle| = |\langle d|\omega|^2 \wedge \omega, d\omega\rangle| \leq 2|\omega|^2|d\omega|^2. \tag{17}$$

Similarly, we have the following estimates involved with the co-differential operator d^* :

$$\begin{array}{rcl} II+III & \leq & -\int_{B(t)}\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d^*\omega|^2dv \\ & & +|\int_{B(t)}\langle d(\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1})\wedge\star\omega,\star d^*\omega\rangle dv| \\ & \leq & -\int_{B(t)}\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d^*\omega|^2dv \\ & & +\int_{B(t)\backslash B(s)}|\langle 2\psi(|\omega|^2+\epsilon)^{\frac{m}{2}-1}d\psi\wedge\star\omega,\star d^*\omega\rangle|dv \\ & & +\int_{B(t)}|\frac{m}{2}-1|\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-2} \\ & & \cdot|\langle d(|\omega|^2+\epsilon)\wedge\star\omega,\star d^*\omega\rangle|dv \\ & \leq & -\int_{B(t)}\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d^*\omega|^2dv \\ & & +2\int_{B(t)\backslash B(s)}\psi(|\omega|^2+\epsilon)^{\frac{m-1}{2}}|d\psi||d^*\omega|dv \\ & & +\int_{B(t)}|m-2|\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d^*\omega|^2dv \\ & \leq & (|m-2|-1)\int_{B(t)}\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d^*\omega|^2dv \\ & & +2\big(\int_{B(t)\backslash B(s)}|d\psi|^2(|\omega|^2+\epsilon)^{\frac{m}{2}}dv\big)^{\frac{1}{2}} \\ & & \cdot\big(\int_{B(t)\backslash B(s)}\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d^*\omega|^2dv\big)^{\frac{1}{2}} \end{array} \tag{18}$$

where we have used $|\star\omega|=|\omega|$, $|\star d^*\omega|=|d^*\omega|$, and in the last step we apply Cauchy-Schwarz inequality to $\int_{B(t)\backslash B(s)}\psi(|\omega|^2+\epsilon)^{\frac{m-1}{2}}|d\psi||d^*\omega|dv$, and in the last second step we apply (3) for a differential form $\star\omega$ in $\int_{B(t)}|\frac{m}{2}-1|\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-2}\cdot|\langle d(|\omega|^2+\epsilon)\wedge\star\omega,\star d^*\omega\rangle|dv$ to obtain:

$$\begin{array}{rcl} |\langle d(|\omega|^2+\epsilon)\wedge\star\omega,\star d^*\omega\rangle| & = & |\langle d|\star\omega|^2\wedge\star\omega,\star((-1)^{nk+n+1}\star d\star\omega)\rangle| \\ & = & |\langle d|\star\omega|^2\wedge\star\omega, d\star\omega\rangle| \\ & \leq & 2|\star\omega|^2|d\star\omega|^2 \\ & = & 2|\omega|^2|\star d\star\omega|^2 \\ & = & 2|\omega|^2|d^*\omega|^2. \end{array} \tag{19}$$

Substituting (16) and (18) into (15), we obtain the following inequality:

$$\begin{array}{rl} & (1-|m-2|)\int_{B(t)}\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d\omega|^2dv \\ & +(1-|m-2|)\int_{B(t)}\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d^*\omega|^2dv \\ \leq & 2\big(\int_{B(t)\backslash B(s)}|d\psi|^2(|\omega|^2+\epsilon)^{\frac{m}{2}}dv\big)^{\frac{1}{2}} \\ & \cdot\{\big(\int_{B(t)\backslash B(s)}\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d\omega|^2dv\big)^{\frac{1}{2}} \\ & \quad+\big(\int_{B(t)\backslash B(s)}\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d^*\omega|^2dv\big)^{\frac{1}{2}}\} \\ \leq & 2\sqrt{2}\big(\int_{B(t)\backslash B(s)}|d\psi|^2(|\omega|^2+\epsilon)^{\frac{m}{2}}dv\big)^{\frac{1}{2}} \\ & \cdot\big(\int_{B(t)\backslash B(s)}\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}(|d\omega|^2+|d^*\omega|^2)dv\big)^{\frac{1}{2}} \end{array} \tag{20}$$

where in the last step we have applied the inequality

$$a^{\frac{1}{2}}+b^{\frac{1}{2}}\leq\sqrt{2}(a+b)^{\frac{1}{2}} \qquad \text{for } a,\, b\geq 0$$

in which

$$a = \int_{B(t)\backslash B(s)} \psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d\omega|^2 dv \quad \text{and}$$

$$b = \int_{B(t)\backslash B(s)} \psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}|d^*\omega|^2 dv\,.$$

We note that if $2 < m < 3$

$$\begin{array}{rl} & (1-|m-2|)\int_{B(t)} \psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}(|d\omega|^2+|d^*\omega|^2)dv \\ \leq & 2\sqrt{2}\big(\int_{B(t)\backslash B(s)} |d\psi|^2(|\omega|^2+\epsilon)^{\frac{m}{2}}dv\big)^{\frac{1}{2}} \\ & \cdot\big(\int_{B(t)\backslash B(s)} \psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}(|d\omega|^2+|d^*\omega|^2)dv\big)^{\frac{1}{2}} \\ \leq & \frac{2\sqrt{2}C_1}{t-s}\big(\int_{B(t)\backslash B(s)}(|\omega|^2+\epsilon)^{\frac{m}{2}}dv\big)^{\frac{1}{2}} \\ & \cdot\big(\int_{B(t)\backslash B(s)} \psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}(|d\omega|^2+|d^*\omega|^2)dv\big)^{\frac{1}{2}} \end{array} \tag{21}$$

Let $\{r_j\}$ be a strictly increasing sequence of positive real numbers going to infinity and define:

$$\begin{array}{rcl} \widetilde{A}_j(\epsilon) & = & \frac{1}{r_j^2}\int_{B(r_j)}(|\omega|^2+\epsilon)^{\frac{m}{2}}dv \\ \varphi_j(x) & = & \psi(x; r_j, r_{j+1}) \\ \widetilde{Q}_{j+1}(\epsilon) & = & \int_{B(r_{j+1})} \varphi_j^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}(|d\omega|^2+|d^*\omega|^2)dv \\ \widetilde{C} & = & (\frac{2\sqrt{2}C_1}{1-|m-2|})^2 \end{array} \tag{22}$$

Using the above notations, substituting $s = r_j$, $t = r_{j+1}$ and $\psi = \varphi_j$ into (21), and squaring both sides of (21), via $\varphi_{j-1} \leq \varphi_j$ we have,

$$\widetilde{Q}^2_{j+1}(\epsilon) \leq \widetilde{C}\left(\frac{r_{j+1}^2\widetilde{A}_{j+1}(\epsilon)-r_j^2\widetilde{A}_j(\epsilon)}{(r_{j+1}-r_j)^2}\right)\left(\widetilde{Q}_{j+1}(\epsilon)-\widetilde{Q}_j(\epsilon)\right) \tag{23}$$

Now we let $\epsilon \to 0$, then we claim that $\widetilde{Q}_j \to 0$ as $j \to \infty$ where we define $\widetilde{Q}_j := \lim_{\epsilon\to 0}\widetilde{Q}_j(\epsilon)$ and $\widetilde{A}_j := \lim_{\epsilon\to 0}\widetilde{A}_j(\epsilon)$. It is clear that $\widetilde{A}_j = \widetilde{A}_j(0) < \infty$ for every fixed j since $\widetilde{A}_j(\epsilon)$ is monotonic as $\epsilon \to 0$ for $m > 2$ and the energy growth (2) for $q = m$ and $2 < m < 3$. $\widetilde{A}_j(\epsilon) \to \widetilde{A}_j(0) = \widetilde{A}_j$ as $\epsilon \to 0$ and $\widetilde{A}_j < \infty$ as $j \to \infty$ due to the 2-finite growth for ω.

We have via (23),

$$\begin{array}{rcl} \widetilde{Q}_{j+1}(\epsilon) & \leq & \dfrac{\widetilde{Q}^2_{j+1}(\epsilon)}{\widetilde{Q}_{j+1}(\epsilon)-\widetilde{Q}_j(\epsilon)} \\ & \leq & \widetilde{C}\dfrac{r_{j+1}^2\widetilde{A}_{j+1}(\epsilon)-r_j^2\widetilde{A}_j(\epsilon)}{(r_{j+1}-r_j)^2}. \end{array} \tag{24}$$

Choosing $\{r_j\}$ such that $r_{j+1} \geq 2r_j$ (that is, $r_{j+1} - r_j \geq \frac{1}{2}r_{j+1}$), then:

$$\frac{r_{j+1}^2\widetilde{A}_{j+1} - r_j^2\widetilde{A}_j}{(r_{j+1}-r_j)^2} \leq 4\widetilde{A}_{j+1}$$

Therefore,

$$\begin{aligned}\widetilde{Q}_{j+1}=\lim_{\epsilon\to 0}\widetilde{Q}_{j+1}(\epsilon) &\leq \lim_{\epsilon\to 0}\widetilde{C}\frac{r_{j+1}^2\widetilde{A}_{j+1}(\epsilon)-r_j^2\widetilde{A}_j(\epsilon)}{(r_{j+1}-r_j)^2} \\ &= \widetilde{C}\frac{r_{j+1}^2\widetilde{A}_{j+1}-r_j^2\widetilde{A}_j}{(r_{j+1}-r_j)^2} \\ &\leq 4\widetilde{C}\widetilde{A}_{j+1} \\ &< \infty.\end{aligned} \tag{25}$$

Hence, let $\epsilon \to 0$ in (23), we obtain

$$\widetilde{Q}_{j+1}^2 \leq \widetilde{C}\left(\frac{r_{j+1}^2\widetilde{A}_{j+1}-r_j^2\widetilde{A}_j}{(r_{j+1}-r_j)^2}\right)\left(\widetilde{Q}_{j+1}-\widetilde{Q}_j\right). \tag{26}$$

We have $\omega = 0$ or $d\omega = d^*\omega = 0$ on M , due to an argument similar to that in Theorem 1 via (2), (26), and $q = m$, $2 < q < 3$.

Theorem 3. *On a complete non-compact manifold M, a simple differential k-form $\bar{\omega}$ with 2-finite growth of*

$$\liminf_{r\to\infty}\frac{\int_{B(x;r)}|\bar{\omega}|^q dv}{r^2} < \infty,$$

for $2 \leq q < 3$ is harmonic if and only if $\bar{\omega}$ is closed and co-closed. In particular, a simple differential k-form $\bar{\omega}$ in L^q spaces for $2 \leq q < 3$ is harmonic if and only if $\bar{\omega}$ is closed and co-closed.

Proof. We claim that inequality (3) is true for all simple differential k-forms. Indeed, Let $\bar{\omega} = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Then $|\bar{\omega}|^2 = f^2$, $d|\bar{\omega}|^2 = 2f df$, $d\bar{\omega} = df \wedge dx_{i_1} \wedge dx_{i_2} \cdots \wedge dx_{i_k}$ and

$$\begin{aligned}\langle d|\bar{\omega}|^2 \wedge \bar{\omega}, d\bar{\omega}\rangle &= \langle 2f df \wedge f dx_{i_1} \wedge \cdots \wedge dx_{i_k}, d\bar{\omega}\rangle \\ &= 2f^2\langle d\bar{\omega}, d\bar{\omega}\rangle \\ &= 2f^2|d\bar{\omega}|^2 \\ &= 2|\bar{\omega}|^2|d\bar{\omega}|^2.\end{aligned} \tag{27}$$

Therefore, the proof follows at once from Theorems 1 and 2.

4. Discussion

In this section, we will compare our results with results obtained by other mathematicians who have the same research interests. We discuss the significance of condition (3) for any differential k-form in the proof of Theorem 2 and give a counter-example of (3) (cf. Remark 1). Condition (3) as a necessary condition has been observed and mentioned by other mathematicians in their research work such as Yau[11], Pigola, Rigoli, Setti[4], Alexandru-Rugina[1] (cf. Remark 2). Our research findings have gone beyond the work of Andreotti and Vesentini from L^2 spaces to both L^q spaces and non-L^q spaces for $2 \leq q < 3$ (cf. Remark 3).

Remark 1. *(Counter-Example of* (3)*) Inequality* (3) *is not true in general. The following is a counter-example: Let the differential* 1*-form* $\omega = x_1 x_n dx_1 + x_n dx_n$ *in* R^n*. Then* $d\omega = -x_1 dx_1 \wedge dx_n$*,* $|\omega|^2 = (x_1^2+1)x_n^2$*,* $d|\omega|^2 = 2x_1x_n^2dx_1 + 2(x_1^2+1)x_ndx_n$*, hence*

$$\begin{aligned}\langle d|\omega|^2 \wedge \omega, d\omega\rangle &= \langle (2x_1x_n^3 - 2(x_1^2+1)x_1x_n^2)dx_1 \wedge dx_n, d\omega\rangle \\ &= \langle \big(-2x_n^3 + 2(x_1^2+1)x_n^2\big)d\omega, d\omega\rangle \\ &= -2x_n^3|d\omega|^2 + 2(x_1^2+1)x_n^2|d\omega|^2 \\ &= -2x_n^3|d\omega|^2 + 2|\omega|^2|d\omega|^2.\end{aligned}$$

This reverses the sign "$\leq$" in (3) *as* $x_n < 0$*.*

Remark 2. *Inequality* (3) *is necessary in the proof of the equivalence* (1) *in* L^q *spaces for* $q \neq 2$*. This point is observed by Pigola-Rigoli-Setti[4, p.262, Remark B.8] who quote Alexandru-Rugina[1] about* L^q*-harmonic* k*-forms on the hyperbolic space* H^m_{-1}*,* $m \geq 3$ *which are neither closed and nor co-closed.*

Remark 3. *Equation* (15) *is treated in [11, p.664, (2.25)], except the term* $III = -\int_{B(t)}\langle(-1)^{nk+n+1} \star (d(\psi^2(|\omega|^2+\epsilon)^{\frac{m}{2}-1}) \wedge \star\omega), d^*\omega\rangle dv$*. We provide a complete calculation in* (15) *and* (18) *involving the term* III *and the co-differential operator* $d^\star$ *based on Lemma 1. We extend the work of Andreotti and Vesentini[2] for* L^2 *differential forms* ω*.*

5. Conclusions

In this paper, we verify the equivalent relation between a harmonic form and a closed co-closed form on a complete non-compact manifold, that is,

$$\triangle\omega = 0 \iff d\omega = d^*\omega = 0.$$

For a differential k-form ω, we extend this equivalence from ω in L^q spaces to ω with 2-balanced growth of

$$\liminf_{r\to\infty} \frac{\int_{B(x;r)} |\omega|^q dv}{r^2} < \infty \qquad \text{(i.e. 2-finite growth)}$$

including L^q and non-L^q spaces, where either $q = 2$ (cf. Theorem 1) or $2 < q < 3$ with ω satisfying

$$\langle d|\omega|^2 \wedge \omega, d\omega\rangle \leq 2|\omega|^2|d\omega|^2$$

(cf. Theorem 2). For a simple differential k-form $\bar{\omega}$, we generalize this equivalence for $\bar{\omega}$ in L^q spaces for $2 \leq q < 3$ (cf. Theorem 3). Our research work recaptures prior work related with the equivalence in L^q spaces from other mathematicians such as Andreotti and Vesentini. Our ongoing research will be the study of this equivalent relationship in the other 4 cases of p-balanced growth. Information gathered in this study could lead to the future research on a differential form with all kinds of energy growth estimates.

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