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Partial Sums for Certain Subclasses of Meromorphically Univalent Functions

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Abstract. In this paper we introduce and study a subclass $\mathscr{M}_p^{\lambda,k}(\alpha,\beta)$ of meromorphically univalent functions defined by generalized Liu-Srivastava operator. We obtain coefficient estimates, extreme points, growth and distortion bounds, radii of meromorphic starlikeness and meromorphic convexity for the class $\mathscr{M}_p^{\lambda,k}(\alpha,\beta)$ by fixing the second coefficient. Further, it is shown that the class $\mathscr{M}_p^{\lambda,k}(\alpha,\beta)$ is closed under convex linear combination.
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1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n \tag{1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}.$$

Let $\Sigma_{\mathscr{S}}$, $\Sigma^*(\gamma)$ and $\Sigma_K(\gamma)$, $(0 \le \gamma < 1)$ denote the subclasses of Σ that are meromorphic univalent, meromorphically starlike functions of order γ and meromophically convex functions of order γ respectively. Analytically, $f \in \Sigma^*(\gamma)$ if and only if, f is of the form (1) and satisfies

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in \mathbb{U},$$

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similarly, $f \in \Sigma_K(\gamma)$, if and only if, f is of the form (1) and satisfies

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \gamma, \quad z \in \mathbb{U},$$

and similar other classes of meromorphically univalent functions have been extensively studied by Altintas et al., [2], Aouf et al. [3, 4, 5, 6], Mogra et al. [18], Uralegadi [20] and others.

Let Σ_P be the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geq 0, \tag{2}$$

that are analytic and univalent in $\mathbb{U}^*$. For functions $f \in \Sigma$ given by (1) and $g \in \Sigma$ given

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n, \tag{3}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) := z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{4}$$

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m$) the generalized hypergeometric function ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \quad &:= \quad \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1;\ l, m \in \mathbb{N}_0 \quad &:= \quad \mathbb{N} \cup \{0\}; z \in U) \end{aligned} \tag{5}$$

where $\mathbb{N}$ denotes the set of all positive integers and $(\theta)_n$ is the Pochhammer symbol defined by

$$(\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1 & n = 0; \theta \in \mathbb{C} \setminus \{0\} \\ \theta(\theta+1)(\theta+2)\ldots(\theta+n-1), & n \in N; \theta \in \mathbb{C} \end{cases} \tag{6}$$

Corresponding to a function ${}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z)$ defined by

$$\mathcal{F}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) := z^{-1}{}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z),$$

Liu and Srivastava [16] (see also [17]) considered a linear operator $\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)$: $\Sigma \to \Sigma$ defined by the following Hadamard product (or convolution):

$$\begin{aligned} \mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)f(z) \quad &= \quad \mathcal{F}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= \quad z^{-1} + \sum_{n=1}^{\infty} \left| \frac{(\alpha_1)_{n+1} \ldots (\alpha_l)_{n+1}}{(\beta_1)_{n+1} \ldots (\beta_m)_{n+1}} \right| \frac{a_n\, z^n}{(n+1)!}, \end{aligned} \tag{7}$$

where, $\alpha_i > 0, (i = 1, 2, \ldots l), \beta_j > 0, (j = 1, 2, \ldots m), l \leq m+1$; $l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For notational simplicity, we use a shorter notations $\mathscr{H}_m^l[\alpha_1]$ for $\mathscr{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)$, in the sequel. We note that the linear operator $\mathscr{H}_m^l[\alpha_1]$ was motivated essentially by Dziok and Srivastava [9].

Next, we define the linear operator $\mathscr{D}_{\lambda,k}^{l,m} : \Sigma \to \Sigma$ by

$$\mathscr{D}_{\lambda,0}^{l,m} f(z) = f(z),$$

$$\mathscr{D}_{\lambda,1}^{l,m} f(z) = (1-\lambda)\mathscr{H}_m^l[\alpha_1]f(z) + \frac{\lambda}{z}(z^2 \mathscr{H}_m^l[\alpha_1]f(z))' = \mathscr{D}_{\lambda}^{l,m} f(z), (\lambda \geq 0).$$

and (in general),

$$\begin{aligned} \mathscr{D}_{\lambda,k}^{l,m} f(z) &= \mathscr{D}_{\lambda}^{l,m}(\mathscr{D}_{\lambda,k-1}^{l,m} f(z)) \\ \mathscr{D}_{\lambda,k}^{l,m} f(z) &:= \frac{1}{z} + \sum_{n=1}^{\infty} \Gamma_n(\alpha_1, k, \lambda)\, a_n\, z^n, \end{aligned} \tag{8}$$

where,

$$\Gamma_n(\alpha_1, k, \lambda) = \frac{(\alpha_1)_{n+1} \ldots (\alpha_l)_{n+1}}{(\beta_1)_{n+1} \ldots (\beta_m)_{n+1}}\ \frac{[1 + \lambda(n-1)]^k}{(n+1)!}, \quad (k \in \mathbb{N}_0,\ \lambda > 0). \tag{9}$$

We note that, for $k = 1$ and $\lambda = 0$ the operator $\mathscr{D}_{0,1}^{l,m} f(z) = \mathscr{H}_m^l[\alpha_1]f(z)$ which was investigated by Liu and Srivastava [16], (see also [8]), for $l = 2$, $m = 1$, $\alpha_2 = 1$, $\lambda = 0$ and $k = 1$ the operator $\mathscr{D}_{0,1}^{2,1}[\alpha_1, 1; \beta_1]f(z) = \mathscr{L}[\alpha_1; \beta_1]f(z)$ was introduced and studied by Liu and Srivastava [15] (see also [1], [12] and [22]). Further, we remark in passing that this operator $\mathscr{L}[\alpha_1; \beta_1]$ is closely related to the Carlson-Shaffer operator $\mathscr{L}[\alpha_1; \beta_1]$ defined on the space of analytic and univalent functions in $\mathbb{U}$. For $l = 2$, $m = 1$, $\alpha_1 = \delta + 1$, $\beta_1 = \alpha_2 = 1$, $\lambda = 0$ and $k = 1$, the operator $\mathscr{D}_{0,1}^{2,1}[\delta + 1, 1; 1]f(z) = \mathscr{D}^{\delta} f(z) = \frac{1}{z(1-z)^{\delta+1}} * f(z) (\delta > -1)$, where $\mathscr{D}^{\delta}$ is the differential operator which was introduced by Ganigi and Uralegadi [10] (see also [8]) and then it was generalized by Yang [21].

Now by making use of the operator $\mathscr{D}_{\lambda,k}^{l,m}$, we define a new subclass of functions in Σ_P as follows.

Definition 1. *For $\alpha > 1$ and $0 < \beta \leq 1$, let $\mathscr{M}^{\lambda,k}(\alpha, \beta)$ denote a subclass of Σ consisting functions of the form* (1) *satisfying the condition that*

$$\Re\left\{z\mathscr{D}_{\lambda,k}^{l,m} f(z) - \alpha z^2 (\mathscr{D}_{\lambda,k}^{l,m} f(z))'\right\} > \beta, \quad z \in \mathbb{U}^*, \tag{10}$$

where $\mathscr{D}_{\lambda,k}^{l,m} f(z)$ is given by (8). *Furthermore, we say that a function $f \in \mathscr{M}_P^{\lambda,k}(\alpha, \beta, \gamma)$, whenever $f(z)$ is of the form* (2).

In this paper, we assume that $\alpha > 1$, $0 < \beta \leq 1$ and $\Gamma_n(\alpha_1, k, \lambda)$ is given by (9) one or otherwise stated in sequel. We observe that, by specializing the parameters l, m, $\alpha_1, \ldots \alpha_l$, $\beta_1, \ldots \beta_m$, k, γ, λ and k the class leads to various subclasses. As for illustrations, we present some examples for the cases.

Example 1. *If $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $f(z)$ of the form (2), then we obtain the new subclass $\mathscr{M}_P(\alpha, \beta, \gamma)$ defined by*

$$\Re\left\{zf(z) - \alpha z^2 f'(z)\right\} > \beta.$$

The class were introduced and studied by Aouf [3], Kulkarni and Joshi [13].

Example 2. *For $l = 2$, $m = 1$, $\alpha_1 = \delta + 1$ $\beta_1 = \alpha_2 = 1$ and $f(z)$ of the form (2), then we get the new subclass $\mathscr{D}_P^{\delta}(\alpha, \beta, \gamma)$ defined by*

$$\Re\left\{z\mathscr{D}^{\delta} f(z) - \alpha z^2(\mathscr{D}^{\delta} f(z))'\right\} > \beta.$$

where $\mathscr{D}^{\delta} f(z) = \frac{1}{z(1-z)^{\delta+1}} * f(z)(\delta > -1)$, is the differential operator which was introduced by Ganigi and Uralegadi [10].

Example 3. *For $l = 2$, $m = 1$, $\alpha_2 = 1$ and $f(z)$ of the form (2), then we obtain the new subclass $\mathscr{L}_P(\alpha, \beta, \gamma)$ defined by*

$$\Re\left\{z\mathscr{L}[\alpha_1, \beta_1]f(z) - \alpha z^2(\mathscr{L}[\alpha_1, \beta_1]f(z))'\right\} > \beta.$$

where the operator $\mathscr{L}[\alpha_1; \beta_1]$ was introduced and studied by Liu and Srivastava [15] (see also [11] and [22]).

Example 4. *For $\lambda = 0$, $k = 1$ and $f(z)$ of the form (2), then we obtain the new subclass $\mathscr{H}_P(\alpha, \beta, \gamma)$ defined by*

$$\Re\left\{z\mathscr{H}[\alpha_1]f(z) - \alpha z^2(\mathscr{H}[\alpha_1]f(z))'\right\} > \beta.$$

where the operator $\mathscr{H}[\alpha_1]$ was introduced and studied by Liu and Srivastava [16] for multivalent functions.

2. Coefficients Inequalities

Our first theorem gives a necessary and sufficient condition for a function f to be in the class $\Sigma_P(\alpha, \beta, \gamma, \lambda, k)$.

Theorem 1. *Let $f \in \Sigma_P$ be given by (2). Then $f \in \Sigma_P(\alpha, \beta, \gamma, \lambda, k)$ if and only if*

$$\sum_{n=1}^{\infty}(n\alpha - 1)\Gamma_n(\alpha_1, k, \lambda)a_n \leq 1 + \alpha - \beta. \tag{11}$$

Proof. Suppose that $f \in \Sigma_P(\alpha, \beta, \gamma, \lambda, k)$. Then

$$\Re\left\{z\left(\frac{1}{z} + \sum_{n=1}^{\infty}\Gamma_n(\alpha_1, k, \lambda)a_n z^n\right) - \alpha z^2\left(\frac{-1}{z^2} + \sum_{n=1}^{\infty}n\Gamma_n(\alpha_1, k, \lambda)a_n z^{n-1}\right)\right\}$$

$$= \Re\left\{1+\alpha-\sum_{n=1}^{\infty}(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)a_n z^{n+1})\right\} > \beta.$$

If we choose z to be real, let $z \to 1-$, we get

$$1+\alpha-\sum_{n=1}^{\infty}(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)a_n \leq \beta$$

which is equivalent to (11). Conversely, let us suppose that the inequality (11) holds true. Then we have

$$\begin{aligned}\left|z\mathscr{D}^{l,m}_{\lambda,k}f(z)-\alpha z^2(\mathscr{D}^{l,m}_{\lambda,k}f(z))'\right| &= \left|-\sum_{n=1}^{\infty}(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)a_n z^{n+1}\right|\\ &\leq \sum_{n=1}^{\infty}(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)|a_n||z|^{n+1}\\ &\leq 1+\alpha-\beta\end{aligned}$$

which implies that $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$. Finally, we note that the assertion (11) of Theorem 1 is sharp, the extremal function being

$$f(z) = \frac{1}{z}+\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)}z$$

The coefficient estimate for functions in the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ is sated in the following corollary.

Corollary 1. *If $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$, then*

$$a_n \leq \frac{1+\alpha-\beta}{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)},\ n\geq 1. \tag{12}$$

The result is sharp for the function

$$f_n(z) = \frac{1}{z}+\frac{1+\alpha-\beta}{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)},\ n\geq 1. \tag{13}$$

Next we obtain the growth theorem for the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$.

Theorem 2. *If $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$, then*

$$\frac{1}{r}-\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)}r \leq |f(z)| \leq \frac{1}{r}+\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)}r \quad (|z|=r)$$

and

$$\frac{1}{r^2}-\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)} \leq |f'(z)| \leq \frac{1}{r^2}+\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)} \quad (|z|=r).$$

The result is sharp for

$$f(z) = \frac{1}{z}+\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)}z. \tag{14}$$

Proof. Since $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n$, we have

$$|f(z)|\leq \frac{1}{r}+\sum_{n=1}^{\infty}a_n r^n \leq \frac{1}{r}+r\sum_{n=1}^{\infty}a_n. \tag{15}$$

Given that $f\in\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$, from the equation (11), we have

$$\begin{aligned}(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)\sum_{n=1}^{\infty}a_n &\leq \sum_{n=1}^{\infty}(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)\,a_n\\ &\leq 1+\alpha-\beta.\end{aligned}$$

That is,

$$\sum_{n=1}^{\infty}a_n\leq\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)}.$$

Using the above equation in (15), we have

$$|f(z)|\leq\frac{1}{r}+\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)}r$$

and

$$|f(z)|\geq\frac{1}{r}-\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)}r.$$

The result is sharp for $f(z)=\frac{1}{z}+\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)}z$. Similarly we have,

$$|f'(z)|\geq\frac{1}{r^2}-\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)}$$

and

$$|f'(z)|\leq\frac{1}{r^2}+\frac{1+\alpha-\beta}{(\alpha-1)\Gamma_1(\alpha_1,k,\lambda)}.$$

Let the functions $f_j(z)$ $(j=1,2,\ldots,m)$ be given by

$$f_j(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,j}z^n,\quad a_{n,j}\geq 0,\ n\in\mathbb{N},n\geq 1. \tag{16}$$

We state the following closure theorem for the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ without proof.

Theorem 3. *Let the function $f_j(z)$ defined by (16) be in the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ for every $j=1,2,\ldots,m$. Then the function $f(z)$ defined by*

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n,$$

belongs to the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$, where $a_n=\frac{1}{m}\sum_{j=1}^{m}a_{n,j}$, $(n=1,2,\ldots)$.

Our next result gives the extreme points for functions in the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$.

Theorem 4. *(Extreme Points) Let*

$$f_0(z)=\frac{1}{z} \text{ and } f_n(z)=\frac{1}{z}+\frac{1+\alpha-\beta}{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)}z^n, \quad (n\geq 1). \tag{17}$$

Then $f\in\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$, if and only if it can be represented in the form

$$f(z)=\sum_{n=0}^{\infty}\mu_n f_n(z), \quad (\mu_n\geq 0, \sum_{n=0}^{\infty}\mu_n=1). \tag{18}$$

Proof. Suppose $f(z)$ can be expressed as in (18). Then

$$\begin{aligned} f(z)&=\sum_{n=0}^{\infty}\mu_n f_n(z)\\ &=\mu_0 f_0(z)+\sum_{n=1}^{\infty}\mu_n f_n(z)\\ &=\frac{1}{z}+\sum_{n=1}^{\infty}\mu_n\frac{1+\alpha-\beta}{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)}z^n. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{n=1}^{\infty}\mu_n\frac{1+\alpha-\beta}{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)}\frac{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)}{1+\alpha-\beta}z^n\\ &=\sum_{n=1}^{\infty}\mu_n-1=1-\mu_0\leq 1. \end{aligned}$$

So by Theorem 1, $f\in\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$.

Conversely, we suppose $f\in\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$. Since

$$a_n\leq\frac{1+\alpha-\beta}{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)}, \quad n\geq 1.$$

We set,

$$\mu_n=\frac{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)}{1+\alpha-\beta}a_n, \quad n\geq 1$$

and $\mu_0=1-\sum\limits_{n=1}^{\infty}\mu_n$. Then we have,

$$\begin{aligned} f(z)&=\sum_{n=0}^{\infty}\mu_n f_n(z)\\ &=\mu_0 f_0(z)+\sum_{n=1}^{\infty}\mu_n f_n(z). \end{aligned}$$

Hence the results follows.

3. Radii of Meromorphically Starlikeness and Meromorphically Convexity

In this section, we obtain the radii of starlikeness and convexity of order δ for functions in the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$.

Theorem 5. *Let $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$. Then f is meromorphically starlike of order $\delta (0 \le \delta < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf_n \left[\left(\frac{1-\delta}{n+2-\delta} \right) \frac{(n\alpha - 1)\Gamma_n(\alpha_1,k,\lambda)}{1+\alpha-\beta} \right]^{\frac{1}{n+1}} \qquad (n \ge 1),$$

The result is sharp for the extremal function $f(z)$ given by (17).

Proof. The function $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ of the form (1) is meromorphically starlike of order δ in the disc $|z| < r_1$, if and only if it satisfies the condition

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| < 1 - \delta. \qquad (19)$$

Since

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \le \left| \frac{\sum_{n=1}^{\infty}(n+1)a_n z^{n+1}}{1+\sum_{n=1}^{\infty} a_n z^{n+1}} \right| \le \frac{\sum_{n=1}^{\infty}(n+1)|a_n||z|^{n+1}}{1-\sum_{n=1}^{\infty} |a_n||z|^{n+1}}.$$

The above expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty} \frac{n+2-\delta}{1-\delta} |a_n| \; |z|^{n-1} < 1.$$

Using the fact, that $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ if and only if

$$\sum_{n=2}^{\infty} \frac{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)}{1+\alpha-\beta} a_n \; < 1.$$

We say (19) is true if

$$\frac{n+2-\delta}{1-\delta} |z|^{n+1} < \frac{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)}{1+\alpha-\beta}.$$

Or, equivalently,

$$|z|^{n+1} < \frac{(1-\delta)}{(n+2-\delta)} \frac{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)}{1+\alpha-\beta}$$

which yields the starlikeness of the family.

Theorem 6. *Let $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$. Then f is meromorphically convex of order δ $(0 \le \delta < 1)$ in the unit disc $|z| < r_2$, where*

$$r_2 = \inf_n \left[\left(\frac{1-\delta}{n+2-\delta} \right) \frac{(\alpha - 1)\Gamma_n(\alpha_1,k,\lambda)}{1+\alpha-\beta} \right]^{\frac{1}{n+1}} \qquad (n \ge 1),$$

The result is sharp for the extremal function $f(z)$ given by (14).

Proof. The proof is analogous to that of Theorem 5, and we omit the details.

4. Partial Sums

Let $f \in \Sigma$ be a function of the form (1). Motivated by Silverman [19], Cho and Owa [7], Latha and Shivarudrappa [14], we define the partial sums f_m defined by

$$f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n z^n \quad (m \in \mathbb{N}). \tag{20}$$

In this section, we consider partial sums of functions from the class $\Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ and obtain sharp lower bounds for the real part of the ratios of f to f_m and f' to f'_m.

Theorem 7. *Let $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$ be given by (1) and define the partial sums $f_1(z)$ and $f_m(z)$, by*

$$f_1(z) = \frac{1}{z} \text{ and } f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} |a_n| z^n, \ (m \in \mathbb{N}/\{1\}). \tag{21}$$

Suppose also that

$$\sum_{n=1}^{\infty} d_n |a_n| \leq 1,$$

where

$$d_n \geq \begin{cases} 1 & \text{for } n = 1,2,3,\ldots,m \\ \frac{(n\alpha-1)\Gamma_n(\alpha_1,k,\lambda)}{1+\alpha-\beta} & \text{for } n = m+1, m+2, m+3 \cdots \end{cases}. \tag{22}$$

Then $f \in \Sigma_P(\alpha,\beta,\gamma,\lambda,k)$. Furthermore,

$$Re\left(\frac{f(z)}{f_m(z)}\right) > 1 - \frac{1}{d_{m+1}} \tag{23}$$

and

$$Re\left(\frac{f_m(z)}{f(z)}\right) > \frac{d_{m+1}}{1+d_{m+1}}. \tag{24}$$

Proof. For the coefficients d_n given by (22) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \tag{25}$$

Therefore we have

$$\sum_{n=1}^{m} |a_n| + d_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} d_n |a_n| \leq 1 \tag{26}$$

by using the hypothesis (22). By setting

$$\Phi_1(z) \quad = \quad d_{m+1}\left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right)$$

$$= \quad 1+\frac{d_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1+\sum\limits_{n=1}^{m} a_n z^{n-1}},$$

then it suffices to show that

$$\Re\left(\Phi_1(z)\right) \geq 0 \quad (z \in \mathbb{U}^*)$$

or,

$$\left|\frac{\Phi_1(z)-1}{\Phi_1(z)+1}\right| \leq 1 \quad (z \in \mathbb{U}^*)$$

and applying (26), we find that

$$\begin{aligned}\left|\frac{\Phi_1(z)-1}{\Phi_1(z)+1}\right| &\leq \frac{d_{m+1}\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{m}|a_n|-d_{m+1}\sum\limits_{n=m+1}^{\infty}|a_n|}\\ &\leq 1, \quad z\in\mathbb{U}^*,\end{aligned}$$

which readily yields the assertion (23) of Theorem 7. In order to see that

$$f(z)=\frac{1}{z}+\frac{z^{m+1}}{d_{m+1}} \tag{27}$$

gives sharp result, we observe that for $z = re^{i\pi/m}$ that $\frac{f(z)}{f_m(z)} = 1-\frac{r^{m+2}}{d_{m+1}} \to 1-\frac{1}{d_{m+1}}$ as $r \to 1^-$.

Similarly, if we take

$$\Phi_2(z) \quad = \quad (1+d_{m+1})\left(\frac{f_m(z)}{f(z)}-\frac{d_{m+1}}{1+d_{m+1}}\right)$$

and making use of (26), we deduce that

$$\left|\frac{\Phi_2(z)-1}{\Phi_2(z)+1}\right| \leq \frac{(1+d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{m}|a_n|-(1-d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}$$

which leads us immediately to the assertion (24) of Theorem 7. The bound in (24) is sharp for each $m \in \mathbb{N}$ with the extremal function $f(z)$ given by (27).

Theorem 8. *If $f(z)$ of the form* (1) *satisfies the condition* (11). *Then*

$$\Re\left(\frac{f'(z)}{f'_m(z)}\right) \geq 1-\frac{m+1}{d_{m+1}}$$

and

$$\Re\left(\frac{f_m'(z)}{f'(z)}\right) \geq \frac{d_{m+1}}{m+1+d_{m+1}},$$

where

$$d_n \geq \begin{cases} n & \textit{for } n = 2, 3, \ldots, m \\ \frac{(\alpha-1)\Gamma_n(\alpha_1,k,\lambda)}{1+\alpha-\beta} & \textit{for } n = m+1, m+2, m+3\cdots \end{cases}.$$

The bounds are sharp, with the extremal function $f(z)$ of the form (14).

Proof. The proof is analogous to that of Theorem 7, and we omit the details.

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