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# Fixed Point in Semi-linear Uniform Space

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**Abstract.** In this paper, more properties of semi-linear uniform spaces are given. We also define Lipschitz condition for functions and contractions functions on such spaces. Finally we ask the natural question: “Does every contraction on a complete semi-linear uniform space have a unique fixed point”.

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**Key Words and Phrases**: Lipschitz Condition, Contractions, Metrizable Spaces, Uniform Spaces, Semi-linear Uniform Spaces

## 1. Introduction

The notion of uniformity has been investigated by several mathematician in the years 1936 to 1939, as Weil [8, 9], and [10]. L.W.Cohen [2, 3], and Graves [5].

The first systematic exposition of the theory of uniform spaces was given by Burbaki in [1]. Wiel’s booklet [10] contains the definition of uniformly continuous mapping.

In 2009, Tallafha, A. and Khalil, R. [6], defined a new type of uniform spaces, namely Semi-linear uniform space. Also they defined a set valued map $\rho$ on $X \times X$, by which they studied some cases of best approximation in such spaces.

In [7], Tallafha, A. defined another set valued map $\delta$ on $X \times X$, and gave more properties of semi-linear uniform spaces using the maps $\rho$ and $\delta$.

The object of this paper is to define Lipschitz condition, and contraction mapping on semi-linear uniform spaces, which enables us to study fixed point for such functions. Since Lipschitz condition, and contractions are usually discussed in metric and normed spaces, and never been studied in other weaker spaces. We believe that the structure of semi-linear uniform spaces is very rich, and all the known results on fixed point theory can be generalized.

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## 2. Semi-linear Uniform Space

Let $X$ be a set and $D_X$ be a collection of subsets of $X \times X$, such that each element $V$ of $D_X$ contains the diagonal $\Delta = \{(x,x) : x \in X\}$, and $V = V^{-1} = \{(y,x) : (x,y) \in V\}$ for all $V \in D_X$ (symmetric). $D_X$ is called the family of all entourages of the diagonal.

**Definition 1** ([4]). *Let $\Gamma$ be a subcollection of $D_X$, the pair $(X,\Gamma)$ is called a uniform space if,*

*(i)* $V_1 \cap V_2 \in \Gamma$ *For all* $V_1$, $V_2$ *in* $\Gamma$.

*(ii) For every* $V \in \Gamma$, *there exists* $U \in \Gamma$ *such that* $U \circ U \subseteq V$.

*(iii)* $\underset{V\in\Gamma}{\cap} V = \Delta$.

*(iv) If* $V \in \Gamma$ *and* $V \subseteq W \in D_X$, *then* $W \in \Gamma$.

Let $(X,\Gamma)$ be a uniform space. By a chain in $X \times X$, we mean a totally (or linearly) ordered collection of subsets of $X \times X$, where $V_1 \leq V_2$ means $V_1 \subseteq V_2$.

A uniform space $(X,\Gamma)$ is called a semi-linear uniform space, if $\Gamma$ is a chain and condition $(iv)$ in Definition 1 is replaced by $\bigcup\{V : V \in \Gamma\} = X \times X$ [6]. Clearly, if $\Gamma$ is a chain then condition $(i)$, is satisfied.

**Definition 2** ([4]). *For $x \in X$ and $V \in \Gamma$. The open ball of center $x$ and radius $V$ is defined by $B(x,V) = \{y : (x,y) \in V\}$, equivalently $B(x,V) = \{y : \rho(x,y) \subseteq V\}$. Clearly if $y \in B(x,V)$, then there is a $W \in \Gamma$ such that $B(y,W) \subseteq B(x,V)$.*

The family $\tau = \{G \subseteq X :$ for every $x \in G$ there is a $V \in \Gamma$ such that $B(x,V) \subseteq G\}$, is a topology on $X$. That is a set $G$ is open if for every point $x$ in $G$, there exist $V \in \Gamma$, such that $B(x,V) \subseteq G$. In [6] it is shown that open balls separate points, so if $X$ is finite then we have the discreet topology, therefore interesting examples are when $X$ is infinite. Also, if $X$ is infinite then $\Gamma$ should be infinite, otherwise $\Delta \in \Gamma$, which implies that, the topology is the discrete one. The elements of $\Gamma$ may be assumed to be open in the topology on $X \times X$, See [4].

**Definition 3** ([6]). . *Let $(X,\Gamma)$ be a semi-linear uniform space, for $(x,y) \in X \times X$, and let $\Gamma_{(x,y)} = \{V \in \Gamma : (x,y) \in V\}$. Then, the set valued map $\rho$ on $X \times X$, is defined by*

$$\rho(x,y) = \bigcap\left\{V : V \in \Gamma_{(x,y)}\right\}$$

Clearly for all $(x,y) \in X \times X$, we have $\rho(x,y) = \rho(y,x)$ and $\Delta \subseteq \rho(x,y)$. Let $\Gamma \backslash \Gamma_{(x,y)} = \left(\Gamma_{(x,y)}\right)^c = \{V \in \Gamma : (x,y) \notin V\}$, from now on, we shall denote $\Gamma \backslash \Gamma_{(x,y)}$ by $\Gamma^c_{(x,y)}$.

**Definition 4** ([7]). *Let $(X,\Gamma)$ be a semi-linear uniform space. Then, the set valued map $\delta$ on $X \times X$, is defined by,*

$$\delta(x,y) = \begin{cases} \bigcup\{V : V \in \Gamma^c_{(x,y)}\} & \text{if } x \neq y \\ \phi & \text{if } x = y \end{cases}.$$

In [7], Tallafha gave some important properties of semi-linear uniform spaces, using the set valued map $\rho$ and $\delta$. Now we shall give more properties of semi-linear uniform spaces.

**Proposition 1.** *Let $(X,\Gamma)$ be a semi-linear uniform space, If $\Lambda$ is a sub collection of $\Gamma$, such that $\bigcap_{V\in\Lambda} V \neq \Delta$, then there exist $U \in \Gamma$ such that $U \subsetneqq \bigcap_{V\in\Lambda} V$.*

*Proof.* Since $\bigcap_{V\in\Lambda} V \neq \Delta$, then there exists a point $(x,y) \in \bigcap_{V\in\Lambda} V$, such that $x \neq y$. Let $U \in \Gamma$, be such that $(x,y) \notin U$. Clearly since $\Gamma$ is a chain, $U$ is the required set. □

The following is an immediate consequence of Proposition 1, and Proposition 3.2 in [7].

**Corollary 1.** *Let $(X,\Gamma)$ be a semi-linear uniform space, If $\rho(x,y) \neq \Delta$, then,*

1. *There exist $U \in \Gamma$ such that $U \subsetneqq \rho(x,y)$.*
2. *$U \subseteq \delta(x,y)$.*

Let $(X,\Gamma)$ be a semi-linear uniform space. If $V \in \Gamma$, then for all $n \in \mathbb{N}$, by $nV$, we mean $V \circ V \circ \ldots \circ V$ n-times.

Let $\Lambda$ be a sub collection of $\Gamma$, $n \in \mathbb{N}$, and $U_1, U_2, \ldots U_n \in \Lambda$. Since $\Gamma$ is a chain, there exist $U \in \Lambda$, such that, $U_1 \circ U_2 \circ \ldots \circ U_n \subseteq U \circ \ldots \circ U = nU$. So

$$\begin{aligned}\bigcup_{W\in\Lambda} nW &\subseteq \bigcup_{U_1\in\Lambda} U_1 \circ \bigcup_{U_2\in\Lambda} U_2 \circ \ldots \circ \bigcup_{U_n\in\Lambda} U_n = n \bigcup_{V\in\Lambda} V \\ &= \bigcup_{U_1,U_2,\ldots U_n\in\Lambda} U_1 \circ U_2 \circ \ldots \circ U_n \subseteq \bigcup_{U\in\Lambda} U \circ \ldots \circ U \\ &= \bigcup_{W\in\Lambda} nW.\end{aligned}$$

This complete the proof of the following Proposition.

**Proposition 2.** *Let $(X,\Gamma)$ be a semi-linear uniform space, and $\Lambda$ a sub collection of $\Gamma$, then, for all $n \in \mathbb{N}$, we have, $\bigcup_{V\in\Lambda} nV = n \bigcup_{V\in\Lambda} V$.*

**Corollary 2.** *Let $(X,\Gamma)$ be a semi-linear uniform space. If $x \neq y$, then $n\delta(x,y) = \bigcup_{V\in\Gamma^c_{(x,y)}} nV$.*

The above corollary is one of the important properties of $\delta$. Now the question is wither $\rho$ satisfies a similar property. It is clear that $n\rho(x,y) \subseteq \bigcap_{V\in\Gamma_{(x,y)}} nV$, so we have.

**Question 1.** *Does $\bigcap_{V\in\Gamma_{(x,y)}} nV \subseteq n\rho(x,y)$?*

Using the definition of $\delta$, $\rho$, we have the following proposition.

**Proposition 3.** *Let $(X,\Gamma)$ be a semi-linear uniform space, and $(x,y) \in X \times X$, if $\rho(x,y) \neq \Delta$, then for all $r,n \in \mathbb{N}$, $r \leq n$, we have,*

1. $r\rho(x,y) \subseteq n\rho(x,y)$.
2. $r\delta(x,y) \subseteq n\delta(x,y)$.

**Remark 1.** *If $r < n$, does,*

1. $r\rho(x,y) \subsetneqq n\rho(x,y)$.
2. $r\delta(x,y) \subsetneqq n\delta(x,y)$.

## 3. Contractions

**Definition 5** ([6])**.** *Let $(X,\Gamma)$ be a semi-linear uniform space, and $(x_n)$ be a sequence in $X$. We say $x_n$ converges to $x$ in $X$, and we write $x_n \to x$, if or every $V \in \Gamma$ there exists $k$ such that $(x_n, x) \in V$ for every $n \geq k$.*

**Definition 6** ([6])**.** *Let $(X,\Gamma)$ be a semi-linear uniform space and $(x_n)$ be a sequence in $X$, $(x_n)$ is called Cauchy if for every $V \in \Gamma$ there exists $k$, such that $(x_n, x_m) \in V$ for all $n,m \geq k$.*

Now we shall discuss some topological properties of a semi-linear uniform spaces. Since semi-linear uniform spaces is a topological space then the continuity of a function is as in topology. The concept of uniform continuity is given by Wiel's [10], so we have.

**Definition 7** ([10])**.** *Let $f : (X,\Gamma_X) \to (Y,\Gamma_Y)$, then $f$ is uniformly continuous if $\forall U \in \Gamma_Y$, $\exists V \in \Gamma_X$, such that if $(x,y) \in V$, then $(f(x), f(y)) \in U$.*

Clearly using our notation we have:

**Proposition 4.** *Let $f : (X,\Gamma_X) \to (Y,\Gamma_Y)$. Then $f$ is uniformly continuous, if and only if $\forall U \in \Gamma_Y$, $\exists V \in \Gamma_X$ such that, for all $x,y \in X$, if $\rho_X(x,y) \subseteq V$, then $\rho_Y(f(x), f(y)) \subseteq U$.*

The following Proposition, shows that we may replace $\rho$ by $\delta$ in Proposition 3.

**Proposition 5.** *Let $f : (X,\Gamma_X) \to (Y,\Gamma_Y)$ . Then $f$ is uniformly continuous, if and only if $\forall U \in \Gamma_Y$, $\exists V \in \Gamma_X$, such that for all $x,y \in X$, if $\delta_X(x,y) \subseteq V$, then $\delta_Y(f(x), f(y)) \subseteq U$.*

*Proof.* Let $f : (X,\Gamma_X) \to (Y,\Gamma_Y)$ be uniformly continuous. Let $U \in \Gamma_Y$, then $\exists W \in \Gamma_X$ such that, if $\rho_X(x,y) \subseteq W$, then $\rho_Y(f(x), f(y)) \subseteq U$. Let $V$ be a proper subset of $W$. We want to show that $V$ is the required set. Let $x,y$ be such that $\delta_X(x,y) \subseteq V$, then $\delta_X(x,y) \subsetneqq W$. So by Proposition 3.2, $(ii)$ in [7] $\rho_X(x,y) \subseteq W$, so $\delta(f(x), f(y)) \subseteq \rho_Y(f(x), f(y)) \subseteq W$.

Conversely, let $U \in \Gamma_Y$, choose $W$ a proper subset of $U$, by assumption $\exists V \in \Gamma_X$, such that for all $x,y \in X$, if $\delta_X(x,y) \subseteq V$, then $\delta_Y(f(x), f(y)) \subseteq W \subsetneqq U$. Also by Proposition 3.2, $and(ii)$ in [7], $\delta_Y(f(x), f(y)) \subsetneqq U$, so $\rho_Y(f(x), f(y)) \subseteq U$. □

In [6], Tallafha gave an example of a space which is semi-linear uniform spaces, but not metrizable. Untill now, to define a function $f$ that satisfies Lipschitz condition, or to be a contraction, it should be defined on a metric space to another metric space. The main idea of this paper is to define such concepts without metric spaces, just we need a semi-linear uniform space, which is weaker as we mentioned before.

**Definition 8.** *Let $f:(X,\Gamma)\longrightarrow(X,\Gamma)$, then $f$ satisfied Lipschitz condition if there exist $m,n\in\mathbb{N}$ such that $m\delta(f(x),f(y))\subseteq n\delta(x,y)$. Moreover if $m>n$, then we call $f$ a contraction.*

**Remark 2.** *One may use the set valued function $\rho$, instead of $\delta$ in the above definition. But we use $\delta$, since $\delta$ satisfies Corollary* 2.

It is known that, every topological space $(X,\tau)$, whose topology induced by a metric or a norm on $X$, can be generated by a uniform space see [4]. In the following Theorem we shall show $(X,\tau)$, can be generated also by a semi-linear uniform space.

**Theorem 1.** *Every topological space whose topology induced by a metric or a norm on X, can be generated by a semi-linear uniform space.*

*Proof.* Let $(X,\tau)$ be a topological space whose topology induced by a metric or a norm on $X$. Let $\Gamma=\left\{V_\in,\in>0:V_\in=\bigcup\limits_{x\in X}\{x\}\times B(x,\in)\right\}$, clearly $\Gamma$ is a chain and, since $B\left(x,V_\in\right)=B(x,\in)$, then the topology induced by $\Gamma$ on $X$, is $\tau$. Moreover we have.

1. $\Delta\subseteq V_\in$, for all $\in>0$.
2. If $(s,t)\in V_\in$ then $(s,t)\in\{s\}\times B(s,\in)$, hence $(t,s)\in\{t\}\times B(t,\in)\subseteq V_\in$.
3. $V_{\frac{\in}{2}}\circ V_{\frac{\in}{2}}\subseteq V_\in$.
4. $\bigcap\limits_{V\in\Gamma}V=\bigcap\limits_{\in>0}V_\in=\Delta$.
5. For all $\in>0$, $\bigcup\limits_{n=1}^{\infty}nV_\in=\bigcup\limits_{n=1}^{\infty}V_{n\in}=X\times X$.

□

We known that, every uniformly continuous function is continuous, see [4]. Now we shall use our notation to prove the following Theorems, which is a generalization of a similar Theorems in metric spaces.

**Theorem 2.** *Let $(X,\Gamma_X)$, $\left(Y,\Gamma_Y\right)$ be two semi-linear uniform spaces, and $f:(X,\Gamma_X)\longrightarrow\left(Y,\Gamma_Y\right)$, then.*

1. *If $f$ is continuous at $x$ then $x_n\longrightarrow x$ implies $f\left(x_n\right)\longrightarrow f(x)$.*
2. *If $f$ is uniformly continuous, then $x_n$ is Cauchy implies $f\left(x_n\right)$ is Cauchy.*

*Proof.*

1. Let $f : (X, \Gamma_X) \longrightarrow (Y, \Gamma_Y)$ be continuous, and $x_n \longrightarrow x$. Let $U \in \Gamma_Y$, so $\exists V \in \Gamma_X$ such that, for all $x, y \in X$, if $\rho_X(x, y) \subseteq V$, then $\rho_Y(f(x), f(y)) \subseteq U$. Now since $x_n \longrightarrow x$, there exists $k$ such that $(x_n, x) \in V$ for every $n \geq k$, which implies that $(f(x_n), f(x)) \in U$ for every $n \geq k$.

2. Let $f : (X, \Gamma_X) \longrightarrow (Y, \Gamma_Y)$ be uniformly continuous, and $x_n$ is Cauchy. Let $U \in \Gamma_Y$, by uniformly continuity $\exists V \in \Gamma_X$, such that for all $x, y \in X$, if $\rho_X(x, y) \subseteq V$, then $\rho_Y(f(x), f(y)) \subseteq U$. Now since $x_n$ is Cauchy, there exists $k$ such that $(x_n, x_m) \in V$ for every $m, n \geq k$, which implies that $(f(x_n), f(x_m)) \in U$, for every $n, m \geq k$.

□

**Theorem 3.** *Let $(X, \Gamma_X)$ be any semi-linear uniform space, and $f : (X, \Gamma) \longrightarrow (X, \Gamma)$, then*

*1. If $f$ is a contraction then it satisfies Lipschitz condition.*

*2. If $f$ satisfies Lipschitz condition, then it is uniformly continuous.*

*Proof.* Since 1) is trivial, we shall prove 2). Let $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ satisfies Lipschitz condition, then there exist $m, n \in \mathbb{N}$ such that $m\delta(f(x), f(y)) \subseteq n\delta(x, y)$. Let $U \in \Gamma$, since $\exists V \in \Gamma$ such that, $nV \subseteq U$. Then for $x, y \in X$, if $\delta(x, y) \subseteq V$, then

$$\delta(f(x), f(y)) \subseteq m\delta(f(x), f(y)) \subseteq n\delta(x, y) \subseteq nV \subseteq U.$$

By Proposition 5, the result follows. □

**Definition 9** ([6]). *A semi-linear uniform space $(X, \Gamma)$ is called complete, if every Cauchy sequence is convergent.*

Fixed point theorems is one of the well known results in mathematics, and has a useful applications in many applied fields such as game theory, mathematical economics and the theory of quasi-variational inequalities. It states that every contraction from a complete metric space to it self has a unique fixed point. So the following question is natural.

**Question 2.** *Let $(X, \Gamma)$ be a complete semi-linear uniform space. And $f : (X, \Gamma) \rightarrow (X, \Gamma)$ be a contraction. Does $f$ has a unique fixed point.*

**Remark 3.** *All the results which was obtained using contraction on metric spaces can be consider as an open questions in semi-linear uniform space.*

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